Plancherel theory on real spherical spaces

Habilitationsschrift

von

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Universität Paderborn
2022

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Summary

This thesis comprises of the four articles

I. *The infinitesimal characters of discrete series for real spherical spaces* by B. Krötz, J.J. Kuit, E.M. Opdam and H. Schlichtkrull, [9].

II. *Ellipticity and discrete series* by B. Krötz, J.J. Kuit, E.M. Opdam and H. Schlichtkrull, [10].

III. *On the little Weyl group of a real spherical space* by J.J. Kuit and E. Sayag, [12].

IV. *The most continuous part of the Plancherel decomposition for a real spherical space* by J.J. Kuit and E. Sayag, [13].

The central theme of these articles is harmonic analysis, and in particular Plancherel theory, on real spherical spaces. In the following, we summarize the main results. The articles themselves are reprinted in the Chapters I – IV. The notation used in the articles is not fully consistent. The notation we use in this summary is therefore not in all cases matching the one in the articles.

Real spherical homogeneous spaces. Let G be the group of real points of an algebraic reductive group and H an algebraic subgroup of G. The homogeneous space $Z = G/H$ is called real spherical if a minimal parabolic subgroup of G admits an open orbit in Z. The class of real spherical homogeneous spaces is very rich. It includes the reductive groups G (considered as homogeneous spaces for $G \times G$), and reductive symmetric spaces. Whereas for a reductive symmetric space the subgroup H is reductive, for real spherical spaces H may be non-reductive. As an example one may consider $G = \mathrm{SL}(2,\mathbb{R})$ and H a connected 1-dimensional subgroup of G. Up to conjugation H is equal to $\mathrm{SO}(2)$, $\mathrm{SO}(1,1)_e$ or the unipotent subgroup of upper triangular matrices with diagonal entries equal to 1. The corresponding homogeneous spaces, namely the Poincare upper half-plane, the one sheeted hyperboloid and the punctured plane, are all real spherical. For the first two examples H is reductive, for the third H is not reductive.

Although the class of real spherical homogeneous spaces is very rich, these spaces still exhibit enough structure to develop interesting harmonic analysis on them. In particular it is feasible to give a precise description of the Plancherel decomposition for real spherical spaces. For reductive groups, and more generally for reductive symmetric spaces, such precise descriptions of the Plancherel decomposition have been given in the past.

In recent years harmonic analysis, and in particular Plancherel theory on real spherical homogeneous spaces has developed very rapidly. The methods differ substantially from those used previously for reductive groups and reductive symmetric spaces, and are inspired by the work [15] of Sakellaridis and Venkatesh for p-adic spherical spaces.

Abstract Plancherel decomposition. From now on we assume that $Z = G/H$ is a homogeneous real spherical space that admits a non-zero G-invariant Radon measure. The space $L^2(Z)$ of square integrable functions on Z carries a natural structure of a unitary representation of G. The Plancherel decomposition for Z is a decomposition of this representation into a direct integral of irreducible unitary representations. To be more precise, $L^2(Z)$ decomposes G-equivariantly as

$$L^2(Z) = \int_{\widehat{G}}^{\oplus} \pi \otimes \mathcal{M}_\pi \, d\mu_Z(\pi),$$

where $\widehat{G}$ is the unitary dual of G, and μ_Z is the Plancherel measure for Z, which is a Radon measure on $\widehat{G}$. Further, $\mathcal{M}_\pi$ is the multiplicity space for $\pi \in \widehat{G}$. An important property of real spherical spaces is that the multiplicity spaces are finite dimensional, see [8, Theorem C] and [11].

For general Z, the Plancherel decomposition neither has a purely discrete nature, as for homogeneous spaces of a compact group, nor a purely continuous nature, as for real vector spaces acting on themselves by translations. It is rather a mixture of discrete and continuous components.

The irreducible subrepresentations of $L^2(Z)$ occur discretely in the Plancherel decomposition and are therefore called discrete series representations. The other extreme is called the most continuous part of the Plancherel decomposition; it consists of the largest continuous families of representations.

Twisted discrete series representations. Not every real spherical homogeneous space Z admits non-trivial discrete series representations. An important obstruction lies in the normalizer of H. The normalizer $N_G(H)$ of a real spherical subgroup H has the property that

$$N_G(H)/H = \mathcal{M} \times \mathcal{A}$$

with $\mathcal{M}$ a compact group and $\mathcal{A} \simeq \mathbb{R}^n_{>0}$ for some $n \in \mathbb{N}_0$. The natural right action of $N_G(H)/H$ on Z commutes with the left action of G. If V is an irreducible subrepresentation of $L^2(Z)$, then one can find an equivalent subrepresentation V' of $L^2(Z)$ so that $\mathcal{A}$ acts from the right on the functions in V' by a character χ. By an application of Fubini's theorem it is easily seen that the non-zero functions in V' cannot be square integrable if $\mathcal{A}$ is not trivial. However, there is a simple generalization of the discrete series that removes at least this obstruction for the existence.

Given a unitary character λ of $\mathcal{A}$, one may consider the space $L^2(Z, \lambda)$ of square integrable sections of the line bundle over $G/\widehat{H}$ defined by λ (up to a normalizing character to make the right action of $\mathcal{A}$ unitary), where $\widehat{H}$ is the inverse image of $\mathcal{A}$ under the projection $N_G(H) \to N_G(H)/H$. The space $L^2(Z)$ then decomposes G-equivariantly as a direct integral

$$L^2(Z) \simeq \int_{\widehat{\mathcal{A}}}^{\oplus} L^2(Z, \lambda) \, d\lambda,$$

where $d\lambda$ is the Haar measure on the unitary character group $\widehat{\mathcal{A}}$ of $\mathcal{A}$. The irreducible subrepresentations of $L^2(Z,\lambda)$ for some $\lambda \in \widehat{\mathcal{A}}$ are said to belong to the twisted discrete series of representations for Z. For the group case and more generally for symmetric spaces full classifications of the (twisted) discrete series of representations exist. We mention here the explicit parametrization of the discrete series for a reductive group by Harish-Chandra [5] and the construction of all discrete series representations for reductive symmetric spaces by Flensted-Jensen [4] and Matsuki and Oshima [14]. For general real spherical spaces very little is known about the twisted discrete series of representations.

As an example of twisted discrete series representations we consider a minimal parabolic subgroup P with Langlands decomposition $P = MAN$. Then $Z = G/N$ is real spherical. In this case we may identify $\mathcal{M}$ with M and $\mathcal{A}$ with A. Now $L^2(G/N)$ decomposes as

$$L^2(G/N) \simeq \int_{\widehat{A}}^{\oplus} L^2(G/AN,\lambda)\, d\lambda \simeq \widehat{\bigoplus_{\xi\in\widehat{M}}} \int_{\widehat{A}}^{\oplus} \mathrm{Ind}_P^G(\xi\otimes\lambda\otimes\mathbf{1})\, d\chi.$$

The twisted discrete series of representations for G/N consists therefore of the unitary minimal principal series representations.

Not every real spherical homogeneous $Z = G/H$ space admits non-trivial twisted discrete series representations. If for example G is a simple group of the non-compact type and $H = K$ is a maximal compact subgroup, then the $L^2(G/K)$ admits no non-trivial irreducible subrepresentations, even though K is its own normalizer.

Infinitesimal characters of twisted discrete series representations. In the article [9] the infinitesimal characters of twisted discrete series representations are studied.

Let P be a minimal parabolic subgroup and $P = MAN$ a Langlands decomposition of P. Denote by $\mathfrak{m}$ and $\mathfrak{a}$ the Lie algebras of M and A respectively. Choose a maximal torus $\mathfrak{t} \subseteq \mathfrak{m}$ and define $\mathfrak{c} := \mathfrak{a} + i\mathfrak{t}$. Then $\mathfrak{c}_{\mathbb{C}}$ is Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $W_{\mathfrak{c}}$ be the Weyl group of the root system $\Sigma(\mathfrak{g}_{\mathbb{C}},\mathfrak{c})$ of $\mathfrak{g}_{\mathbb{C}}$ in $\mathfrak{c}$. For $\pi \in \widehat{G}$ we denote by $\chi_\pi \in \mathfrak{c}_{\mathbb{C}}^*/W_{\mathfrak{c}}$ the infinitesimal character of π.

Theorem 1 ([9, Theorem 1.1]). *There exists a $W_{\mathfrak{c}}$-invariant lattice $\Lambda_Z \subseteq \mathfrak{c}^*$, rational with respect to the root system in $\mathfrak{c}$, such that $\mathrm{Re}\,\chi_\pi \in \Lambda_Z/W_{\mathfrak{c}}$ for every twisted discrete series representation π for Z. Moreover, if π is a discrete series representation for Z, then χ_π is real, and hence $\chi_\pi \in \Lambda_Z/W_{\mathfrak{c}}$.*

Theorem 1 has the following corollary.

Corollary 2 ([9, Corollary 8.4]). *Let $K \subseteq G$ be a maximal compact subgroup and τ a K-type. Further, let $\lambda \in \widehat{\mathcal{A}}$. Then there exist only finitely many twisted discrete series representations (π, V) for Z with $\mathcal{A}$-character λ such that the τ-isotypical component $V[\tau]$ of π is non-zero.*

Theorem 1 implies a spectral gap for twisted discrete series representations. This is the important ingredient for the uniform constant term approximation for tempered eigenfunctions in [3].

Existence of twisted discrete series representations. For the group case the existence of discrete series representations can be characterized geometrically by the following theorem of Harish-Chandra.

Theorem 3 ([5, Theorem 13])**.** *The existence of a compact Cartan subalgebra of G is a necessary and sufficient condition for the existence of discrete series representations for G.*

For the more general class of reductive symmetric spaces Harish-Chandra's rank conditions generalizes: a reductive symmetric space $Z = G/H$ admits discrete series representations if and only if there exists a compact Cartan subspace in the Killing complement $\mathfrak{h}^\perp$ of $\mathfrak{h}$. Alternatively, this can be phrased as

$$\text{Z admits discrete series representations} \iff \operatorname{int}\{X \in \mathfrak{h}^\perp \mid X \text{ elliptic}\} \neq \emptyset\,, \tag{1}$$

where the interior int is taken in $\mathfrak{h}^\perp$. The equivalence (1) is conjectured to hold true for all algebraic homogeneous spaces Z. In [2] the existence of discrete series representations for a real spherical space $Z = G/H$ was proven under the condition that $\mathfrak{h}^\perp$ contains a relatively open subset of elliptic elements. This result was generalized in [6, Theorem 1.7] to general algebraic homogeneous spaces for G. The other implication is still an open problem. The existence of twisted discrete series representations for a real spherical spaces $Z = G/H$ has been conjectured to be equivalent to

$$\operatorname{int}\{X \in N_{\mathfrak{g}}(\mathfrak{h})^\perp \mid X \text{ weakly elliptic}\} \neq \emptyset,$$

where $N_{\mathfrak{g}}(\mathfrak{h})$ is the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$.

In [10] a new proof for the necessity in Theorem 3 is given. The proof is based on Theorem 1, namely on the fact that infinitesimal characters of discrete series representations are real. The following is a brief sketch of this new proof.

Let $\mathfrak{c} = \mathfrak{a} \oplus i\mathfrak{t}$ be as before. We show that the existence of a compact Cartan subalgebra is equivalent to the occurrence of the map

$$\theta : \mathfrak{c} = \mathfrak{a} \oplus i\mathfrak{t} \to \mathfrak{c}; \quad X + iY \mapsto -X + iY$$

in the Weyl group $W_{\mathfrak{c}}$ of the root system of $\mathfrak{c}$ in $\mathfrak{g}_{\mathbb{C}}$. By elementary means it is further shown that if $\chi \in \mathfrak{c}^*/W_{\mathfrak{c}}$ and occurs as the infinitesimal character of a unitary representation, then it satisfies

$$\theta\chi = \chi. \tag{2}$$

This holds in particular for the infinitesimal characters of discrete series representations. From a given discrete series representation we construct by using Zuckerman's translation functor another discrete series representation, with an infinitesimal character $\chi \in \mathfrak{c}^*/W_{\mathfrak{c}}$, so that the stabilizer in the extended Weyl group $\langle W_{\mathfrak{c}}, \theta\rangle$ of any point in the $W_{\mathfrak{c}}$-orbit χ is trivial. It then follows from (2) that θ is contained in the Weyl group $W_{\mathfrak{c}}$. We expect that this proof can be generalized to a proof of (1) for real spherical spaces.

The little Weyl group. Let $\mathrm{Gr}(\mathfrak{g}, n)$ be the Grassmannian of n-dimensional subspaces of $\mathfrak{g}$ with $n = \dim(\mathfrak{h})$. It is easy to see that each subspace in the closure of $\mathrm{Ad}(G)\mathfrak{h}$ in $\mathrm{Gr}(\mathfrak{g}, n)$ is a Lie subalgebra of $\mathfrak{g}$. It is more surprising that each of these subalgebras is again real spherical. If $X \in \mathfrak{g}$ is an hyperbolic element and $E \in \mathrm{Gr}(\mathfrak{g}, n)$, then the limit

$$E_X := \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)E$$

exists in $\mathrm{Gr}(\mathfrak{g}, n)$. For a point $z \in Z$ we write $\mathfrak{h}_z$ for its stabilizer subalgebra. We fix a minimal parabolic subgroup P of G and a Langlands decomposition $P = MAN$ of P. Given a direction $X \in \mathfrak{a} := \mathrm{Lie}(A)$ we consider the limit subalgebra

$$\mathfrak{h}_{z,X} := \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)\mathfrak{h}_z. \tag{3}$$

These limit subalgebras play an important role in [9]. In [12] and [13] the limits are used to analyse P-orbits in Z.

The goal of [12] is a new construction of an invariant of Z called the little Weyl group, which for real spherical spaces was first defined in [7, Section 9]. Our construction is in terms of the limit subalgebras $\mathfrak{h}_{z,X}$.

If X is contained in the negative Weyl chamber with respect to P, then the limit $\mathfrak{h}_{z,X}$ is up to M-conjugacy the same for all $z \in Z$ so that $P \cdot z$ is open. This limit $\mathfrak{h}_\emptyset$ is called the horospherical degeneration of $\mathfrak{h}_z$. We denote the M-conjugacy class of a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ by $[\mathfrak{s}]$ and define $\mathfrak{a}_\mathfrak{h} := \mathfrak{a} \cap \mathfrak{h}_\emptyset$. We define the subgroup of G

$$\mathcal{N}_\emptyset := \{v \in N_G(\mathfrak{a}) : \mathrm{Ad}(v)[\mathfrak{h}_\emptyset] = [\mathfrak{h}_\emptyset]\}.$$

In fact, $\mathcal{N}_\emptyset$ is a normal subgroup of $N_G(\mathfrak{a}) \cap N_G(\mathfrak{a}_\mathfrak{h})$. For $z \in Z$ we further define

$$\mathcal{V}_z := \{v \in N_G(\mathfrak{a}) : [\mathfrak{h}_{z,X}] = \mathrm{Ad}(v)[\mathfrak{h}_\emptyset] \text{ for some } X \in \mathfrak{a}\}.$$

This set is for suitable $z \in Z$ a subset of $N_G(\mathfrak{a}) \cap N_G(\mathfrak{a}_\mathfrak{h})$. For these z we set

$$\mathcal{W}_z := \mathcal{V}_z/\mathcal{N}_\emptyset \subseteq \big(N_G(\mathfrak{a}) \cap N_G(\mathfrak{a}_\mathfrak{h})\big)/\mathcal{N}_\emptyset.$$

The main result of the[12] is the following.

Theorem 4 ([12, Theorem 1.1]). *For a suitable choice of $z \in Z$ the set $\mathcal{W}_z$ is a subgroup of $\big(N_G(\mathfrak{a}) \cap N_G(\mathfrak{a}_\mathfrak{h})\big)/\mathcal{N}_\emptyset$ and acts on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$ as a finite crystallographic group. This crystallographic group is naturally identified with the little Weyl group of Z as defined in [7].*

Plancherel decomposition in terms of Bernstein morphisms. In order to summarize the contents of [13], we first recall the main result from [2].

Real spherical homogeneous spaces admit good compactifications. As one moves to the boundary of Z in a compactification, the space Z deforms into a real spherical homogeneous space $Z_I = G/H_I$. Such a space Z_I is called a boundary degeneration of Z. It may be viewed as the normal bundle of a G-orbit in the boundary of Z. The least deformed boundary degeneration is Z itself. The other extreme, the most deformed boundary degeneration is called the horospherical boundary degeneration $Z_\emptyset$.

In [2] it is shown that the Plancherel decomposition of Z can be described in terms of twisted discrete series of the boundary degenerations Z_I. To be more precise, there exists a canonical G-equivariant surjective map

$$B : \bigoplus_I L^2(Z_I)_{\mathrm{tds}} \to L^2(Z), \tag{4}$$

where $L^2(Z_I)_{\mathrm{tds}}$ is the closed subspace of $L^2(Z_I)$ that decomposes as a direct integral of twisted discrete series representations for Z_I. This map is called the Bernstein morphism. The Bernstein morphism is a sum of partial isometries.

The most continuous part of $L^2(Z)$. For general I little is known about the twisted discrete series for Z_I, not even existence. This is different for the most degenerate of the boundary degenerations, i.e., the boundary degeneration $Z_\emptyset$. For $Z_\emptyset$ the Plancherel decomposition can be computed rather easily. There exists a parabolic subgroup $\overline{Q}$ and a Langlands decomposition $\overline{Q} = M_Q A_Q \overline{N}_Q$, so that

$$H_\emptyset = (M_Q \cap H)(A_Q \cap H)\overline{N}_Q$$

and the homogeneous space $M_Q/(M_Q \cap H)$ is compact and $\mathfrak{a}_Q/\mathfrak{a}_Q \cap \mathfrak{h} \simeq \mathfrak{a}/\mathfrak{a}_\mathfrak{h}$. As a consequence $L^2(Z_\emptyset)$ decomposes as

$$L^2(Z_\emptyset) \simeq \widehat{\bigoplus_{\xi\in\widehat{M}_Q}} \int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+}^{\oplus} \mathcal{M}_{\emptyset,\xi} \otimes \mathrm{Ind}_{\overline{Q}}^G(\xi \otimes \lambda \otimes \mathbf{1})\, d\lambda. \tag{5}$$

Here $d\lambda$ is the Lebesgue measure on $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ and $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ is a fundamental domain for the action of the stabilizer of $\mathfrak{a}_\mathfrak{h}$ in the Weyl group on $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. The multiplicity spaces $\mathcal{M}_{\emptyset,\xi}$ are independent of λ and can only be non-zero for finite dimensional unitary representations ξ of M_Q. All representations contributing to the Plancherel decomposition of $Z_\emptyset$ belong to the twisted discrete series of representations for this space.

The closed subspace $L^2_{\mathrm{mc}}(Z) := B(L^2(Z_\emptyset))$ of $L^2(Z)$ is called the most-continuous part of $L^2(Z)$. The properties of the Bernstein morphism and the Plancherel decomposition (5) of $L^2(Z_\emptyset)$ guarantee that the most continuous part decomposes as

$$L^2_{\mathrm{mc}}(Z) \simeq \widehat{\bigoplus_{\xi\in\widehat{M}_{Q,\mathrm{fu}}}} \int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+}^{\oplus} \mathcal{M}_{\xi,\lambda} \otimes \mathrm{Ind}_{\overline{Q}}^G(\xi \otimes \lambda \otimes \mathbf{1})\, d\lambda. \tag{6}$$

Here $\widehat{M}_{Q,\mathrm{fu}}$ denotes the set of equivalence classes of finite dimensional unitary representation of M_Q and $\mathcal{M}_{\xi,\lambda}$ is the multiplicity space for the representation $\mathrm{Ind}_{\overline{Q}}^G(\xi \otimes \lambda \otimes \mathbf{1})$. In [13] the multiplicity spaces together with their inner products are determined, thus making the unitary equivalence (6) precise.

For $\xi \in \widehat{M}_{Q,\mathrm{fu}}$ and $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ let $\mathcal{H}_{\xi,\lambda}$ be the space of smooth vectors of the representation $\mathrm{Ind}_{\overline{Q}}^G(\xi\otimes\lambda\otimes\mathbf{1})$. Each multiplicity space $\mathcal{M}_{\xi,\lambda}$ can naturally be viewed as a subspace of the space $(\mathcal{H}'_{\xi,\lambda})^H$ of H-fixed functionals on $\mathcal{H}_{\xi,\lambda}$. We provide a construction for all functionals in $(\mathcal{H}'_{\xi,\lambda})^H$ for generic $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. The construction heavily relies on the theory developed around the limit subalgebras (3) developed in [9] and [12]. We use the theory of the constant term developed in [3] to prove that for generic $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ all H-fixed functionals on $\mathcal{H}_{\xi,\lambda}$ are tempered. Moreover, the multiplicity spaces are in fact given by

$$\mathcal{M}_{\xi,\lambda} = (\mathcal{H}'_{\xi,\lambda})^H.$$

Finally, we refine the Maaß-Selberg relations from [2, Theorem 9.6] to determine the inner products induced by the Plancherel decomposition on $(\mathcal{H}'_{\xi,\lambda})^H$. Thus we give a precise description of the Plancherel decomposition of the most continuous part of $L^2_{\rm mc}(Z)$.

Following Sakellaridis and Venkatesh [15] Delorme introduced scattering operators for real spherical spaces $Z = G/H$ with G split in [1]. Assuming a conjecture on the nature of twisted discrete series representation, he shows that the scattering operators determine the kernel of the Bernstein morphism (4). We give a concrete formula for the scattering operators for the most continuous part, also for non-split G. In this case the scattering operators form a representation of the little Weyl group.

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Zusammenfassung

Diese Habilitationsschrift besteht aus den vier Artikeln

I. *The infinitesimal characters of discrete series for real spherical spaces* von B. Krötz, J.J. Kuit, E.M. Opdam und H. Schlichtkrull, [9].

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III. *On the little Weyl group of a real spherical space* von J.J. Kuit und E. Sayag, [12].

IV. *The most continuous part of the Plancherel decomposition for a real spherical space* von J.J. Kuit und E. Sayag, [13].

Das zentrale Thema dieser Artikel ist die harmonische Analyse und insbesondere die Plancheireltheorie auf reell sphärischen Räumen. Im Folgenden fassen wir die wichtigsten Ergebnisse zusammen. Die Artikel selbst sind in den Kapiteln I – IV aufgenommen. Die in den Artikeln verwendete Notation ist nicht vollständig konsistent. Die Notation, die in dieser Zusammenfassung verwendet wird, stimmt daher nicht in allen Fällen mit der in den Artikeln überein.

Reell sphärische homogene Räume. Seien G die Gruppe der reellen Punkte einer algebraischen reduktiven Gruppe und H eine algebraische Untergruppe von G. Der homogene Raum $Z = G/H$ heißt reell sphärisch, wenn eine minimale parabolische Untergruppe von G eine offene Bahn in Z zulässt. Die Klasse der reell sphärischen homogenen Räume ist sehr groß. Sie umfasst die reduktiven Gruppen G (welche als homogene Räume für die Gruppen $G \times G$ angesehen werden) und reduktive symmetrische Räume. Obwohl für einen reduktiven symmetrischen Raum die Untergruppe H reduktiv ist, kann H für reelle sphärische Räume nicht reduktiv sein. Ein Beispiel dafür ist $G = \mathrm{SL}(2, \mathbb{R})$ und H eine zusammenhängende 1-dimensionale Untergruppe von G. Bis auf eine Konjugation ist H gleich $\mathrm{SO}(2)$, $\mathrm{SO}(1, 1)_e$ oder die unipotente Untergruppe der oberen Dreiecksmatrizen mit diagonalen Einträgen gleich 1. Die entsprechenden homogenen Räume, nämlich die obere Poincaré-Halbebene, das einschichtige Hyperboloid und die punktierte Ebene, sind alle reell sphärisch. Für die ersten beiden Beispiele ist H reduktiv, für das dritte ist H nicht reduktiv.

Obwohl die Klasse der reell sphärischen homogenen Räume sehr groß ist, weisen diese Räume immer noch genug Struktur auf, um interessante harmonische Analysis auf ihnen zu entwickeln. Insbesondere ist eine genaue Beschreibung der Plancherel-Zerlegung für reell sphärische Räume möglich. Für reduktive Gruppen und allgemeiner für reduktive symmetrische Räume wurden in der Vergangenheit solche genauen Beschreibungen der Plancherel-Zerlegung gegeben.

In den letzten Jahren hat sich die harmonische Analyse und insbesondere die Plancherel-Theorie reell sphärischer homogener Räume sehr schnell entwickelt. Die Methoden unterscheiden sich wesentlich von denen, die zuvor für reduktive Gruppen und reduktive symmetrische Räume verwendet wurden, und sind inspiriert von der Arbeit [15] von Sakellaridis und Venkatesh für p-adische sphärische Räume.

Abstrakte Plancherel-Zerlegung. Von nun an nehmen wir an, dass $Z = G/H$ ein homogener reell spärischer Raum ist, der ein G-invariantes Radonmaß zulässt. Der Raum $L^2(Z)$ der quadratisch integrierbarer Funktionen auf Z trägt eine natürliche Struktur einer unitären Darstellung von G. Die Plancherel-Zerlegung für Z ist eine Zerlegung dieser Darstellung in ein direktes Integral irreduzibler unitären Darstellungen. Genauer gesagt, zerfällt $L^2(Z)$ G-äquivariant als

$$L^2(Z) = \int_{\widehat{G}}^{\oplus} \pi \otimes \mathcal{M}_\pi \, d\mu_Z(\pi),$$

wobei $\widehat{G}$ das unitäre Dual von G und μ_Z das Plancherel-Maß für Z ist. Letzteres ist ein Radon-Maß auf $\widehat{G}$. Außerdem ist $\mathcal{M}_\pi$ der Multiplizitätsraum für $\pi \in \widehat{G}$. Eine wichtige Eigenschaft reell sphärischer Räume ist, dass die Multiplizitätsräume endlichdimensional sind, siehe [8, Theorem C] und [11].

Für allgemeine Z hat die Plancherel-Zerlegung weder einen rein diskreten Charakter, wie für homogene Räume kompakter Gruppe, noch einen rein kontinuierlichen Charakter, wie für reelle Vektorräume, die durch Translationen auf sich selbst wirken. Vielmehr handelt es sich um eine Mischung aus diskreten und kontinuierlichen Komponenten.

Die irreduziblen Unterdarstellungen von $L^2(Z)$ treten diskret in der Plancherel-Zerlegung auf und heißen daher Darstellungen der diskrete Reihe. Das andere Extrem wird als der kontinuierlichste Teil der Plancherel-Zerlegung bezeichnet; sie besteht aus den größten stetigen Familien von Darstellungen.

Darstellungen der getwisteten diskreten Reihe. Nicht jeder reell sphärische homogene Raum Z lässt Darstellungen der diskreten Reihe zu. Eine wichtige Beschränkung liegt im Normalisator von H. Der Normalisator $N_G(H)$ einer reell sphärischen Untergruppe H hat die Eigenschaft, dass

$$N_G(H)/H = \mathcal{M} \times \mathcal{A},$$

wobei $\mathcal{M}$ eine kompakte Gruppe und $\mathcal{A} \simeq \mathbb{R}^n_{>0}$ für ein $n \in \mathbb{N}_0$ ist. Die natürliche Rechtswirkung von $N_G(H)/H$ auf Z vertauscht mit der Linkswirkung von G. Wenn V eine irreduzible Unterdarstellung von $L^2(Z)$ ist, dann gibt es eine zu V äquivalente Unterdarstellung V' von $L^2(Z)$, sodass $\mathcal{A}$ von rechts auf die Funktionen in V' wirkt mittels einem Charakter χ. Durch Anwendung des Satzes von Fubini sieht man leicht, dass die von Null verschiedenen Funktionen in V' nicht quadratintegrierbar sein können, wenn $\mathcal{A}$ nicht trivial ist. Es gibt jedoch eine einfache Verallgemeinerung der diskreten Reihe, die zumindest diese Einschränkung für die Existenz beseitigt.

Für einen gegebenen Charakter λ von $\mathcal{A}$ kann man den Raum $L^2(Z, \lambda)$ von quadratisch integrierbaren Schnitten des Geradenbündels über $G/\widehat{H}$ definiert durch λ betrachten (bis auf einen normalisierenden Charakter, um die Rechtswirkung von $\mathcal{A}$ unitär zu machen), wobei $\widehat{H}$ das Urbild von $\mathcal{A}$ unter der Projektion $N_G(H) \to N_G(H)/H$ ist. Der Raum $L^2(Z)$ zerlegt G-äquivariant als ein direktes Integral

$$L^2(Z) \simeq \int_{\widehat{\mathcal{A}}}^{\oplus} L^2(Z, \lambda)\, d\lambda,$$

wobei $d\lambda$ das Haar-Maß auf der unitären Charaktergruppe $\widehat{\mathcal{A}}$ von $\mathcal{A}$ ist. Die Darstellungen der getwisteten diskreten Reihe für Z sind die irreduziblen Unterdarstellungen von $L^2(Z, \lambda)$ für einen Charakter $\lambda \in \widehat{\mathcal{A}}$. Für den Gruppenfall und allgemeiner für symmetrische Räume gibt es vollständige Klassifikationen der Darstellungen der (getwisteten) diskreten Reihe. Wir erwähnen hier die explizite Parametrisierung der diskreten Reihen für eine reduktive Gruppe durch Harish-Chandra [5] und die Konstruktion aller Darstellungen der diskreten Reihe für reduktive symmetrische Räume durch Flensted-Jensen [4] und Matsuki und Oshima [14]. Für allgemeine reell sphärische Räume ist sehr wenig über die Darstellungen der getwisteten diskreten Reihe bekannt.

Als Beispiel von Darstellungen der getwisteten diskreten Reihe betrachten wir eine minimale parabolische Untergruppe P mit Langlands-Zerlegung $P = MAN$. Dann ist $Z = G/N$ reell sphärisch. In diesem Fall können wir $\mathcal{M}$ mit M und $\mathcal{A}$ mit A identifizieren. Der Raum $L^2(G/N)$ zerfällt als

$$L^2(B/N) \simeq \int_{\widehat{A}}^{\oplus} L^2(G/AN, \lambda)\, d\lambda \simeq \widehat{\bigoplus_{\xi \in \widehat{M}}} \int_{\widehat{A}}^{\oplus} \operatorname{Ind}_P^G(\xi \otimes \lambda \otimes \mathbf{1})\, d\chi.$$

Die Darstellungen der getwisteten diskreten Reihe für G/N sind also die unitären minimalen Hauptreihendarstellungen.

Nicht jeder reell sphärische homogene Raum $Z = G/H$ lässt Darstellungen der getwisteten diskreten Reihe zu. Wenn beispielsweise G eine einfache Gruppe vom nichtkompakten Typ und $H = K$ eine maximal kompakte Untergruppe ist, dann hat $L^2(G/K)$ keine nicht-trivialen irreduziblen Unterdarstellungen, obwohl K sein eigener Normalisator ist.

Infinitesimale Charaktere von Darstellungen der getwisteten diskreten Reihe. Im Artikel [9] werden die infinitesimalen Charaktere von Darstellungen der getwisteten diskreten Reihe untersucht.

Seien P eine minimale parabolische Untergruppe und $P = MAN$ eine Langlands-Zerlegung von P. Wir bezeichnen mit $\mathfrak{m}$ und $\mathfrak{a}$ die Lie-Algebren von M beziehungsweise A. Weiter wählen wir einen maximalen Torus $\mathfrak{t} \subseteq \mathfrak{m}$ und definieren $\mathfrak{c} := \mathfrak{a} + i\mathfrak{t}$. Dann ist $\mathfrak{c}_\mathbb{C}$ eine Cartan-Unteralgebra von $\mathfrak{g}_\mathbb{C}$. Sei $W_\mathfrak{c}$ die Weylgruppe des Wurzelsystems $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{c})$ von $\mathfrak{g}_\mathbb{C}$ in $\mathfrak{c}$. Für $\pi \in \widehat{G}$ bezeichnen wir mit $\chi_\pi \in \mathfrak{c}_\mathbb{C}^*/W_\mathfrak{c}$ den infinitesimalen Charakter von π.

Satz 1 ([9, Theorem 1.1]). *Es gibt ein $W_\mathfrak{c}$-invariantes Gitter $\Lambda_Z \subseteq \mathfrak{c}^*$, welches rational bezüglich des Wurzelsystems in $\mathfrak{c}$ ist, sodass $\operatorname{Re}\chi_\pi \in \Lambda_Z/W_\mathfrak{c}$ für jede Darstellung π der* *getwisteten diskreten Reihe für Z gilt. Wenn außerdem π eine Darstellung der diskrete Reihe für Z ist, dann ist χ_π reell und damit $\chi_\pi \in \Lambda_Z/W_{\mathfrak{c}}$.*

Satz 1 hat das folgende Korollar.

Korollar 2 ([9, Corollary 8.4])**.** *Seien $K \subseteq G$ eine maximal kompakte Untergruppe und τ ein K-Typ. Weiterhin sei $\lambda \in \widehat{\mathcal{A}}$. Dann gibt es nur endlich viele Darstellungen (π, V) der getwisteten diskreten Reihe für Z mit $\mathcal{A}$-Charakter λ, sodass die τ-isotypische Komponente $V[\tau]$ von π nicht null ist.*

Satz 1 impliziert die Existenz einer spektralen Lücke für Darstellungen der getwisteten diskreten Reihe. Dies ist ein wichtiger Bestandteil für die gleichmäßige Abschätzung für den Rest in der konstanten Term-Approximierung für temperierte Eigenfunktionen in [3].

Existenz von Darstellungen der getwisteten diskreten Reihe für Z. Für den Gruppenfall lässt sich die Existenz von Darstellungen der diskreten Reihe durch den folgenden Satz von Harish-Chandra geometrisch charakterisieren.

Satz 3 ([5, Theorem 13])**.** *Die Existenz einer kompakten Cartan-Unteralgebra von G ist eine notwendige und hinreichende Bedingung für die Existenz von Darstellungen der diskreten Reihe für G.*

Für die allgemeinere Klasse der reduktiven symmetrischen Räume verallgemeinern die Rangbedingungen von Harish-Chandra: Für einen reduktiven symmetrischen Raum $Z = G/H$ existieren genau dann Darstellungen der diskreten Reihe, wenn es einen kompakten Cartan-Unterraum im Killing-Komplement $\mathfrak{h}^\perp$ von $\mathfrak{h}$ gibt. Alternativ kann dies auch so formuliert werden:

$$\begin{aligned}&\text{Es existieren Darstellungen der diskrete Reihe für } Z \\ \iff\ &\operatorname{int}\{X \in \mathfrak{h}^\perp \mid X \text{ elliptisch}\} \neq \emptyset\,,\end{aligned} \tag{7}$$

wobei das innere int in $\mathfrak{h}^\perp$ genommen wird. Es wird vermutet, dass die Äquivalenz (7) für alle algebraischen homogenen Räume Z gilt. In [2] wurde die Existenz von Darstellungen der diskreten Reihe für einen reell sphärischen Raum $Z = G/H$ unter der Bedingung bewiesen, dass $\operatorname{int}\{X \in \mathfrak{h}^\perp \mid X \text{ elliptisch}\} \neq \emptyset$. Dieses Ergebnis wurde in [6, Theorem 1.7] auf allgemeine algebraische homogene Räume für G verallgemeinert. Die andere Implikation ist noch ein offenes Problem. Es wird vermutet, dass die Existenz von Darstellungen der getwisteten diskreten Reihe für reell sphärische Räume $Z = G/H$ äquivalent zu

$$\operatorname{int}\{X \in N_{\mathfrak{g}}(\mathfrak{h})^\perp \mid X \text{ schwach elliptisch}\} \neq \emptyset$$

ist, wobei $N_{\mathfrak{g}}(\mathfrak{h})$ der Normalisator von $\mathfrak{h}$ in $\mathfrak{g}$ ist.

In [10] wird ein neuer Beweis für die Notwendigkeit der Existenz einer kompakten Cartan-Untergruppe in Satz 3 gegeben. Der Beweis basiert auf Theorem 1, nämlich darauf, dass infinitesimale Charaktere von Darstellungen der diskreten Reihe reell sind. Das Folgende ist eine kurze Skizze dieses neuen Beweises.

Sei $\mathfrak{c} = \mathfrak{a} \oplus i\mathfrak{t}$ wie zuvor. Wir zeigen, dass die Existenz einer kompakten Cartan-Unteralgebra dazu äquivalent ist, dass die Abbildung

$$\theta : \mathfrak{c} = \mathfrak{a} \oplus i\mathfrak{t} \to \mathfrak{c}; \quad X + iY \mapsto -X + iY$$

in der Weyl-Gruppe $W_\mathfrak{c}$ des Wurzelsystems von $\mathfrak{c}$ in $\mathfrak{g}_\mathbb{C}$ erhalten ist. Mit elementaren Mitteln zeigen wir weiter, dass wenn $\chi \in \mathfrak{c}^*/W_\mathfrak{c}$ reell ist und als infinitesimaler Charakter einer unitären Darstellung auftritt, der Parameter χ die Gleichung

$$\theta\chi = \chi \tag{8}$$

erfüllt. Dies gilt insbesondere für die infinitesimalen Charaktere von Darstellungen der diskreten Reihe. Aus einer gegebenen Darstellung der diskreten Reihe konstruieren wir unter Verwendung des Zuckerman'schen Verschiebungsfunktor eine weitere Darstellung der diskrete Reihe mit einem infinitesimalen Charakter $\chi \in \mathfrak{c}^*/W_\mathfrak{c}$, so dass der Stabilisator in der erweiterten Weyl-Gruppe $\langle W_\mathfrak{c}, \theta\rangle$ von jedem Punkt in den $W_\mathfrak{c}$-Orbit χ trivial ist.

Aus einer gegebenen Darstellung der diskreten Reihe konstruieren wir unter Verwendung des Zuckerman'schen Verschiebungsfunktors eine weitere Darstellung der diskreten Reihe mit einem infinitesimalen Charakter $\chi \in \mathfrak{c}^*/W_\mathfrak{c}$, sodass sein Stabilisator in der erweiterten Weyl-Gruppe $\langle W_\mathfrak{c}, \theta\rangle$ trivial ist. Aus (8) folgt dann, dass θ in der Weylgruppe $W_\mathfrak{c}$ enthalten ist. Wir erwarten, dass dieser Beweis zu einem Beweis von (7) für reell sphärischen Räume verallgemeinert werden kann.

Die kleine Weyl-Gruppe. Sei $\mathrm{Gr}(\mathfrak{g}, n)$ die Graßmann-Mannigfaltigkeit von Unterräumen von $\mathfrak{g}$ der Dimension n. Es ist leicht zu sehen, dass jeder Unterraum im Abschluss von $\mathrm{Ad}(G)\mathfrak{h}$ in $\mathrm{Gr}(\mathfrak{g}, n)$ eine Lie-Unteralgebra von $\mathfrak{g}$ ist. Überraschender ist, dass jede dieser Unteralgebren wieder reell sphärisch ist. Wenn $X \in \mathfrak{g}$ ein hyperbolisches Element und $E \in \mathrm{Gr}(\mathfrak{g}, n)$ ist, dann existiert der Limes

$$E_X := \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)E$$

in $\mathrm{Gr}(\mathfrak{g}, n)$. Für einen Punkt $z \in Z$ schreiben wir $\mathfrak{h}_z$ für seine Stabilisator-Unteralgebra. Seien P eine minimale parabolische Untergruppe von G und $P = MAN$ eine Langlands-Zerlegung von P. Bei gegebener Richtung $X \in \mathfrak{a} := \mathrm{Lie}(A)$ betrachten wir die Limesunteralgebra

$$\mathfrak{h}_{z,X} := \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)\mathfrak{h}_z. \tag{9}$$

Diese Limesunteralgebren spielen eine wichtige Rolle in [9]. In [12] und [13] werden die Limes verwendet, um P-Orbiten in Z zu analysieren.

Das Ziel von [12] ist eine neue Konstruktion einer Invarianten von Z, nämlich die kleine Weyl-Gruppe, die erstmals in [7, Section 9] für reell sphärische Räume definiert wurde. Unsere Beschreibung der kleinen Weyl-Gruppe basiert auf den Limesunteralgebren $\mathfrak{h}_{z,X}$.

Wenn X in der negativen Weyl-Kammer (bezüglich P) enthalten ist, dann ist der Limes $\mathfrak{h}_{z,X}$ bis auf M-Konjugation gleich für alle $z \in Z$ für die $P \cdot z$ offen ist. Dieser Limes $\mathfrak{h}_\emptyset$ wird die horosphärische Entartung von $\mathfrak{h}_z$ genannt. Wir schreiben $[\mathfrak{s}]$ für die

M-Konjugationsklasse einer Unteralgebra $\mathfrak{s}$ von $\mathfrak{g}$ und definieren $\mathfrak{a}_{\mathfrak{h}} := \mathfrak{a} \cap \mathfrak{h}_{\emptyset}$. Weiter definieren wir die Untergruppe $\mathcal{N}_{\emptyset}$ von G als

$$\mathcal{N}_{\emptyset} := \{v \in N_G(\mathfrak{a}) : \mathrm{Ad}(v)[\mathfrak{h}_{\emptyset}] = [\mathfrak{h}_{\emptyset}]\}.$$

Dann ist $\mathcal{N}_{\emptyset}$ ein Normalteiler von $N_G(\mathfrak{a}) \cap N_G(\mathfrak{a}_{\mathfrak{h}})$. Für $z \in Z$ definieren wir

$$\mathcal{V}_z := \{v \in N_G(\mathfrak{a}) : [\mathfrak{h}_{z,X}] = \mathrm{Ad}(v)[\mathfrak{h}_{\emptyset}] \text{ für einige } X \in \mathfrak{a}\}.$$

Für geeignete $z \in Z$ is diese Menge eine Teilmenge von $N_G(\mathfrak{a}) \cap N_G(\mathfrak{a}_{\mathfrak{h}})$. Für diese z definieren wir abschließend

$$\mathcal{W}_z := \mathcal{V}_z/\mathcal{N}_{\emptyset} \subseteq \big(N_G(\mathfrak{a}) \cap N_G(\mathfrak{a}_{\mathfrak{h}})\big)/\mathcal{N}_{\emptyset}.$$

Das Hauptergebnis von [12] ist das Folgende.

Satz 4 ([12, Theorem 1.1])**.** *Bei geeigneter Wahl von $z \in Z$ ist $\mathcal{W}_z$ eine Untergruppe von $\big(N_G(\mathfrak{a}) \cap N_G(\mathfrak{a}_{\mathfrak{h}})\big)/\mathcal{N}_{\emptyset}$ und wirkt als eine endliche kristallographische Gruppe auf $\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}$. Diese kristallographische Gruppe ist auf natürliche Weise isomorph zu der kleinen Weyl-Gruppe von Z aus [7].*

Bernstein-Morphismen. Um den Inhalt von [13] zusammenzufassen, schauen wir uns zunächst das Hauptergebnis aus [2] an.

Reell sphärische homogene Räume lassen gute Kompaktifizierungen zu. Bewegt man sich in einer Kompaktifizierung zum Rand von Z, verformt sich der Raum Z zu einem reell sphärischen homogenen Raum $Z_I = G/H_I$. Einen solchen Raum Z_I nennt man eine Randentartung von Z. Eine Randentartung kann als Normalenbündel eines G-Orbits im Rand von Z angesehen werden. Die am wenigsten entartete Randentartung ist Z selbst. Das andere Extrem, die am stärksten entartete Randentartung wird als horosphärische Randentartung $Z_{\emptyset}$ bezeichnet.

In [2] wird gezeigt, dass die Plancherel-Zerlegung von Z mit Darstellungen der getwisteten diskreten Reihe der Randentartung Z_I beschrieben werden kann. Genauer gesagt gibt es eine kanonische G-äquivariante surjektive Abbildung

$$B : \bigoplus_I L^2(Z_I)_{\mathrm{tds}} \to L^2(Z), \tag{10}$$

wobei $L^2(Z_I)_{\mathrm{tds}}$ der geschlossene Unterraum von $L^2(Z_I)$ ist, der sich als direktes Integral von Darstellungen der getwisteten diskreten Reihe für Z_I zerlegt. Diese Abbildung wird als Bernstein-Morphismus bezeichnet. Der Bernstein-Morphismus ist eine Summe von partiellen Isometrien.

Das kontinuierlichste Teil von $L^2(Z)$. Für allgemeine I ist wenig über die getwistete diskrete Reihe für Z_I bekannt, nicht einmal die Existenz von Darstellungen der getwisteten diskreten Reihe. Anders ist dies bei der am weitesten entarteten Randentartung, der Randentartung $Z_{\emptyset}$. Für diesen Raum kann die Plancherel-Zerlegung ziemlich einfach berechnet werden. Es gibt eine parabolische Untergruppe $\overline{Q}$ und eine Langlands-Zerlegung $\overline{Q} = M_Q A_Q \overline{N}_Q$, sodass

$$H_{\emptyset} = (M_Q \cap H)(A_Q \cap H)\overline{N}_Q.$$

Der homogene Raum $M_Q/(M_Q \cap H)$ ist kompakt und $\mathfrak{a}_Q/\mathfrak{a}_Q \cap \mathfrak{h} \simeq \mathfrak{a}/\mathfrak{a}_\mathfrak{h}$. Folglich zerfällt $L^2(Z_\emptyset)$ als

$$L^2(Z_\emptyset) \simeq \widehat{\bigoplus_{\xi \in \widehat{M}_Q}} \int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+}^{\oplus} \mathcal{M}_{\emptyset,\xi} \otimes \mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes \mathbf{1})\, d\lambda. \tag{11}$$

Hier ist $d\lambda$ das Lebesgue-Maß auf $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ und $i\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ ist ein Fundamentalbereich für die Wirkung des Stabilisators von $\mathfrak{a}_\mathfrak{h}$ in der Weylgruppe auf $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. Die Multiplizitätsräume $\mathcal{M}_{\emptyset,\xi}$ sind unabhängig von λ und können nur für die endlichdimensionale unitäre Darstellungen ξ von M_Q ungleich Null sein. Alle Darstellungen, die zur Plancherel-Zerlegung von $Z_\emptyset$ beitragen, gehören zu der getwisteten diskreten Reihe von Darstellungen für diesen Raum.

Der abgeschlossene Unterraum $L^2_{\mathrm{mc}}(Z) := B(L^2(Z_\emptyset))$ von $L^2(Z)$ wird das kontinuierlichste Teil von $L^2(Z)$ genannt. Die Eigenschaften des Bernstein-Morphismus und der Plancherel-Zerlegung (11) von $L^2(Z_\emptyset)$ garantieren, dass der stetigste Teil zerfällt wie

$$L^2_{\mathrm{mc}}(Z) \simeq \widehat{\bigoplus_{\xi \in \widehat{M}_{Q,\mathrm{fu}}}} \int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+}^{\oplus} \mathcal{M}_{\xi,\lambda} \otimes \mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes \mathbf{1})\, d\lambda. \tag{12}$$

Dabei bezeichnet $\widehat{M}_{Q,\mathrm{fu}}$ die Menge der Äquivalenzklassen der endlichdimensionalen unitären Darstellungen von M_Q und $\mathcal{M}_{\xi,\lambda}$ ist der Multiplizitätsraum für die Darstellung $\mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes \mathbf{1})$. In [13] werden die Multiplizitätsräume zusammen mit ihren Skalarprodukten bestimmt, womit die unitäre (12) Äquivalenz präzise gemacht wird.

Für $\xi \in \widehat{M}_{Q,\mathrm{fu}}$ und $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ sei $\mathcal{H}_{\xi,\lambda}$ der Raum glatter Vektoren der Darstellung $\mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes \mathbf{1})$. Jeder Multiplizitätsraum $\mathcal{M}_{\xi,\lambda}$ kann auf natürliche Weise als Unterraum des Raumes $(\mathcal{H}'_{\xi,\lambda})^H$ von H-festen Funktionalen auf $\mathcal{H}_{\xi,\lambda}$ gesehen werden. Wir geben eine Konstruktion für alle Funktionale in $(\mathcal{H}'_{\xi,\lambda})^H$ für generisches $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. Die Konstruktion basiert stark auf der in [9] und [12] entwickelten Theorie für Limesunteralgebren (9). Wir verwenden die in [3] entwickelte Theorie des konstanten Terms, um zu beweisen, dass für generische $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ alle H-festen Funktionale auf $\mathcal{H}_{\xi,\lambda}$ temperiert sind und die Multiplizitätsräume gegeben werden durch

$$\mathcal{M}_{\xi,\lambda} = (\mathcal{H}'_{\xi,\lambda})^H.$$

Zum Schluß verfeinern wir die Maaß-Selberg-Relationen aus [2, Theorem 9.6], um die durch die Plancherel-Zerlegung auf $(\mathcal{H}'_{\xi,\lambda})^H$ induzierten Skalarprodukte zu bestimmen. Damit geben wir eine genaue Beschreibung der Plancherel-Zerlegung des kontinuierlichsten Teils von $L^2_{\mathrm{mc}}(Z)$.

Nach Sakellaridis und Venkatesh [15] führte Delorme in [1] Streuoperatoren für reell sphärische Räume $Z = G/H$ ein, unter der Annahme, dass G spaltend ist. Unter einer weiteren Annahme einer Vermutung über die Natur der Darstellungen der getwisteten diskreten Reihe zeigt er, dass die Streuoperatoren den Kern des Bernstein-Morphismus (10) bestimmen. Wir geben, auch für nicht spaltende Gruppen G, eine konkrete Formel für die Streuoperatoren für den stetigsten Teil. In diesem Fall bilden die Streuoperatoren eine Darstellung der kleinen Weyl-Gruppe.

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Chapter I

The infinitesimal characters of discrete series for real spherical spaces [†]

Joint with Bernhard Krötz, Eric Opdam and Henrik Schlichtkrull.

Abstract

Let $Z = G/H$ be the homogeneous space of a real reductive group and a unimodular real spherical subgroup, and consider the regular representation of G on $L^2(Z)$. It is shown that all representations of the discrete series, that is, the irreducible subrepresentations of $L^2(Z)$, have infinitesimal characters which are real and belong to a lattice. Moreover, let K be a maximal compact subgroup of G. Then each irreducible representation of K occurs in a finite set of such discrete series representations only. Similar results are obtained for the twisted discrete series, that is, the discrete components of the space of square integrable sections of a line bundle, given by a unitary character on an abelian extension of H.

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[†]This article was published in Geom. Funct. Anal. **30** (2020), no. 3, 804–857.

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1 Introduction

Let $Z = G/H$ be a homogeneous space attached to a real reductive group G and a closed subgroup H. A principal objective in the harmonic analysis of Z is the understanding of the G-equivariant spectral decomposition of the space $L^2(Z)$ of square integrable half-densities. The irreducible components of $L^2(Z)$ are of particular interest, they comprise the discrete series for Z. We will assume that Z is unimodular, that is, it carries a positive G-invariant Radon measure. Then $L^2(Z)$ is identified as the space of square integrable functions with respect to this measure.

Later on we shall restrict ourselves to the case where Z is real spherical, that is, the action of a minimal parabolic subgroup $P \subseteq G$ on Z admits an open orbit. Symmetric spaces are real spherical, as well as real forms of complex spherical spaces. We mention that a classification of real spherical spaces G/H with H reductive became recently available, see [20] and [21].

For symmetric spaces it is known (see [5], [2]) that the spectral components of $L^2(Z)$ are built by means of induction from certain parabolic subgroups of G. The inducing representations belong to the discrete series of a symmetric space of the Levi subgroup, twisted by unitary characters on its center. For real spherical spaces the results on tempered representations obtained in [25] suggest similarly that the spectral decomposition of $L^2(Z)$ will be built from the twisted discrete spectrum of a certain finite set of satellites $Z_I = G/H_I$ of Z, which are again unimodular real spherical spaces. A first step towards obtaining a spectral decomposition is then to obtain key properties of the twisted discrete series for all unimodular real spherical spaces.

As usual we write $\widehat{G}$ for the unitary dual of G and disregard the distinction between equivalence classes $[\pi] \in \widehat{G}$ and their representatives π. Representations $\pi \in \widehat{G}$ which occur in $L^2(Z)$ discretely will be called representations of the *discrete series for Z*. This notion distinguishes a subset of $\widehat{G}$ which we denote by $\widehat{G}_{H,\mathrm{d}}$. We write $\widehat{G}_{\mathrm{d}}$ for the discrete series of G, i.e., $\widehat{G}_{\mathrm{d}} = \widehat{G}_{\{e\},\mathrm{d}}$. Note that in general there is no relation between the sets $\widehat{G}_{\mathrm{d}}$ and $\widehat{G}_{H,\mathrm{d}}$ if H is non-trivial.

To explain the notion of being twisted we recall the automorphism group $N_G(H)/H$ of Z, where $N_G(H)$ denotes the normalizer of H. It gives rise to a right action of $N_G(H)/H$ on $L^2(Z)$ commuting with the left regular action of G. For a real spherical space $N_G(H)/H$ is fairly well behaved: $N_G(H)/H$ is a product of a compact group and a non-compact torus [24]. It is easy to see that in this case there exists no discrete spectrum unless $N_G(H)/H$ is compact. Let $\mathcal{A}$ be a maximal non-compact torus in $N_G(H)/H$. Hence if $\mathcal{A}$ is non-trivial, there exist no discrete series representations for Z. In this case we generalize the notion of discrete series as follows. We have an equivariant disintegration into G-modules

$$L^2(Z) \simeq \int_{\widehat{\mathcal{A}}}^{\oplus} L^2(Z;\chi)\, d\chi\,.$$

Here $\widehat{\mathcal{A}}$ denotes the set of unitary characters χ of $\mathcal{A}$, and $L^2(Z;\chi)$ denotes the space of functions on Z, which transform by χ (times a modular character) and are square integrable modulo $\mathcal{A}$ (as half-densities, since in general $G/N_G(H)$ is not unimodular). The set of representations $\pi \in \widehat{G}$ which are in the discrete spectrum of $L^2(Z;\chi)$ is called the χ-*twisted discrete series* and is denoted $\widehat{G}_{H,\chi}$. The union $\widehat{G}_{H,\mathrm{td}}$ of these sets over all $\chi \in \widehat{\mathcal{A}}$ is referred to as the *twisted discrete series for* Z.

Let $P = MAN$ be a Langlands decomposition of the minimal parabolic subgroup P. Denote by $\mathfrak{m}$ and $\mathfrak{a}$ the Lie algebras of M and A respectively. Choose a maximal torus $\mathfrak{t} \subseteq \mathfrak{m}$ and set $\mathfrak{c} := \mathfrak{a} + i\mathfrak{t}$. We note that $\mathfrak{c}_\mathbb{C}$ is Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$ and denote by $W_\mathfrak{c}$ the Weyl group of the root system $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{c}) \subseteq \mathfrak{c}^*$. For every $\pi \in \widehat{G}$ we denote by $\chi_\pi \in \mathfrak{c}_\mathbb{C}^*/W_\mathfrak{c}$ its infinitesimal character and recall a theorem of Harish-Chandra ([10, Thm. 7]), which asserts that the map

$$\mathfrak{X} : \widehat{G} \to \mathfrak{c}_\mathbb{C}^*/W_\mathfrak{c}, \quad \pi \mapsto \chi_\pi \tag{1.1}$$

has uniformly finite fibers. Note that $\mathfrak{X}$ is continuous if $\widehat{G}$ is endowed with the Fell topology.

A priori it is not clear that $\mathfrak{X}(\widehat{G}_{H,\mathrm{d}})$ or $\mathfrak{X}(\widehat{G}_{H,\chi})$ is a discrete subset of $\mathfrak{c}_\mathbb{C}^*/W_\mathfrak{c}$. However, we believe this to be true for general real algebraic homogeneous spaces Z. For real spherical spaces Z it is a consequence of the main theorem, Theorem 8.3 below, which slightly simplified can be phrased as follows.

Theorem 1.1. *Let $Z = G/H$ be a unimodular real spherical space. Assume that the pair (G, H) is real algebraic. Then there exists a $W_\mathfrak{c}$-invariant lattice $\Lambda_Z \subseteq \mathfrak{c}^*$, rational with respect to the root system in $\mathfrak{c}$, such that:*

(i) $\mathfrak{X}(\widehat{G}_{H,\mathrm{d}}) \subseteq \Lambda_Z/W_\mathfrak{c}$,

(ii) $\operatorname{Re}\mathfrak{X}(\widehat{G}_{H,\mathrm{td}}) \subseteq \Lambda_Z/W_\mathfrak{c}$.

A few remarks related to this theorem are in order.

Remark 1.2.

(1) The statement in (i) implies that the infinitesimal characters χ_π are *real* and *discrete* for $\pi \in \widehat{G}_{H,\mathrm{d}}$. Furthermore (see Corollary 8.4 below), these properties of χ_π lead to the following. Let $K \subseteq G$ be a maximal compact subgroup. For all $\tau \in \widehat{K}$ and $\chi \in \widehat{\mathcal{A}}$ the set
$$\{\pi \in \widehat{G}_{H,\chi} \mid \operatorname{Hom}_K(\pi|_K, \tau) \neq 0\}$$

is finite. In other words, there are only finitely many χ-twisted discrete series representations containing a given K-type. For p-adic spherical spaces of wavefront type this was shown by Sakellaridis and Venkatesh in [37, Theorem 9.2.1].

(2) There is a simple relation between the leading exponents of generalized matrix coefficients attached to $\pi \in \widehat{G}_{H,\mathrm{td}}$ and the infinitesimal character χ_π of π (cf. Lemma 3.4). Further, twisted discrete series can be described by inequalities satisfied by the leading exponents (cf. [25] or (3.3)-(3.4) below). The inclusion $\operatorname{Re}\mathfrak{X}(\widehat{G}_{H,\mathrm{td}}) \subseteq \Lambda_Z/W_{\mathfrak{c}}$ then implies that all real parts of leading exponents are uniformly bounded away from "rho". Phrased differently, Theorem 1.1(ii) implies a spectral gap for twisted discrete series. In [37], Prop. 9.4.8, this is called "uniform boundedness of exponents" and is a key fact for establishing the Plancherel formula for p-adic spherical spaces of wavefront type.

(3) The lattice Λ_Z can be taken of the form $\frac{1}{N}\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{c})$, where N is an integer which only depends on $\mathfrak{g}$. (We may use the integer N from Theorem 8.3, which is the product of the integers from Theorem 7.4 and Proposition B.1. The latter two integers only depend on $\mathfrak{g}$.)

Theorem 1.1 is the crucial ingredient for the uniform constant term approximation for tempered eigenfunctions in [7]. Thus it lies at the heart of the Plancherel theorem for $L^2(Z)$ in terms of Bernstein-morphisms, established in [6] and motivated by [37], Section 11. Notice that the strategy of proof designed in [37] for the Plancherel theorem differs from the earlier approach where the discrete spectrum is classified first (see [11] for groups and [2], [5] for symmetric spaces). In [37] the discrete series is taken as a black box which features a spectral gap, and the Plancherel theorem is established without knowing the discrete spectrum explicitly.

For reductive groups an explicit parametrization of the discrete series $\widehat{G}_\mathrm{d}$ was obtained by Harish-Chandra [12]. More generally, for symmetric spaces G/H discrete series were constructed by Flensted-Jensen [8], and his work was completed by Matsuki and Oshima [35] to a full classification of $\widehat{G}_{H,\mathrm{d}}$. For a general real spherical space such an explicit parameter description appears currently to be out of reach and for non-symmetric spaces the existence or non-existence of discrete series is known only in a few cases. See [26, Corollary 5.6] and in [15, Corollary 4.5].

More importantly, the existence of discrete series can be characterized geometrically by the existence of a compact Cartan subalgebra in the group case, and of a compact Cartan subspace in $\mathfrak{h}^\perp$ in the more general case of symmetric spaces. One can phrase this uniformly as:

$$\widehat{G}_{H,\mathrm{d}} \neq \emptyset \iff \operatorname{int}\{X \in \mathfrak{h}^\perp \mid X \text{ elliptic}\} \neq \emptyset\,, \tag{1.2}$$

where the interior int is taken in $\mathfrak{h}^\perp$. We expect that (1.2) is true for all algebraic homogeneous spaces Z. A geometric characterization for the existence of twisted discrete series is less clear; in the real spherical case we expect

$$\widehat{G}_{H,\mathrm{td}} \neq \emptyset \iff \operatorname{int}\{X \in N_\mathfrak{g}(\mathfrak{h})^\perp \mid X \text{ weakly elliptic}\} \neq \emptyset \tag{1.3}$$

with $N_\mathfrak{g}(\mathfrak{h})$ the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$.

A combination of the Bernstein decomposition of $L^2(Z)$ in [6] with soft techniques from microlocal analysis [13] yields the implication "$\Leftarrow$" in (1.2), see [6, Th. 12.1]. Developing the techniques in [13] a bit further would yield the more general implication "$\Leftarrow$" in 1.3. Let us point out that we consider the implication "$\Rightarrow$" in 1.3 as one of the most interesting current problems in this area.

Representations of the discrete series feature interesting additional structures. For instance, for a reductive group Schmid realized the discrete spectrum in L^2-Dolbeault cohomology [38]. This was the first of series of realizations of the discrete series representations for reductive Lie groups. Vogan established that the representations of the discrete series on a symmetric space are cohomologically induced [41]. It would be interesting to know for non-symmetric spaces to which extent $\widehat{G}_{H,\mathrm{d}}$ consists of cohomologically induced representations.

1.1 Methods

We first describe the idea of proof for Theorem 1.1 in the case $Z = G$ is a semisimple group. Let $\pi \in \widehat{G}_{\mathrm{d}}$ be a discrete series. Let $\sigma \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be such that there is a quotient

$$\pi_{\lambda,\sigma} = \mathrm{Ind}_P^G(\lambda \otimes \sigma) \twoheadrightarrow \pi$$

of the principal series representation $\mathrm{Ind}_P^G(\lambda\otimes\sigma)$. Here induction is normalized and from the left. Such a quotient exists for every irreducible representation π by the subrepresentation theorem of Casselman.

Let now $v \in \pi_{\lambda,\sigma}^\infty$ be a smooth vector and let $\overline{v}$ be its image in π^∞. Further let $\overline{\eta}$ be any smooth vector in $(\pi^\vee)^\infty$ where $\pi^\vee$ is the dual representation of π. We view $\overline{\eta}$ as an element of $(\pi_{\lambda,\sigma}^\vee)^\infty = \pi_{-\lambda,\sigma^\vee}^\infty$, denote it then by η, and record the relation

$$m_{\overline{v},\overline{\eta}}(g) := \langle \overline{\eta}, \pi(g)^{-1}\overline{v}\rangle = \langle \eta, \pi_{\lambda,\sigma}(g^{-1})v\rangle =: m_{v,\eta}(g) \qquad (g \in G)\,.$$

We now use the non-compact model for $\pi_{\lambda,\sigma}$, i.e. σ-valued functions on $\overline{N}$ (the opposite of N), and let v be a σ-valued a test function on $\overline{N}$. Let $g = a \in A$. As v is compactly supported on $\overline{N}$, the functions $\overline{n} \mapsto a^{-2\rho}v(a\overline{n}a^{-1})$ form a Dirac sequence on $\overline{N}$ for $a \in A^-$ tending to infinity along a regular ray, and a partial Dirac sequence in case of a semi-regular ray. Here $A^- = \exp(\mathfrak{a}^-)$ with $\mathfrak{a}^- \subseteq \mathfrak{a}$ the closure of the negative Weyl chamber determined by N. Dirac approximation and appropriate choices of $\overline{v}$ and $\overline{\eta}$ then give a constant $c = c(\overline{v},\overline{\eta}) \neq 0$ and the asymptotic behavior:

$$m_{\overline{v},\overline{\eta}}(a) \sim c \cdot a^{-\lambda+\rho} \qquad (a \in A^-, a \to \infty)\,. \tag{1.4}$$

Strictly speaking, the constant c above also depends on the ray along which we go to infinity, in case it is not regular. The asymptotics (1.4) are motivated by a lemma of Langlands [33, Lemma 3.12] which is at the core of the Langlands classification. This lemma asserts for K-finite vectors v and η, and for λ in the range of absolute convergence of the long intertwining operator, say I, that

$$c(\overline{v},\overline{\eta}) = \langle I(v)(e), \eta(e)\rangle_\sigma\,.$$

As our v is compactly supported on $\overline{N}$ the integral defining $I(v)$ is in fact absolutely convergent for every parameter λ.

As π belongs to the discrete series, $m_{\overline{v},\overline{\eta}}$ is square integrable on G. One then derives from (1.4) and the integral formula for the Cartan decomposition $G = KA^-K$ that the parameter λ has to satisfy the strict inequality

$$\operatorname{Re}\lambda\big|_{\mathfrak{a}^-\setminus\{0\}} > 0\,. \tag{1.5}$$

There exists a number $N(G) \in \mathbb{N}$ such that every rank one standard intertwiner

$$I_\alpha : \operatorname{Ind}_P^G(s_\alpha\lambda \otimes s_\alpha\sigma) \to \operatorname{Ind}_P^G(\lambda \otimes \sigma)$$

is an isomorphism for $\lambda(\alpha^\vee) \notin \frac{1}{N(G)}\mathbb{Z}$ (see Proposition B.1 below). Suppose that $\lambda(\alpha^\vee) \notin \frac{1}{N(G)}\mathbb{Z}$ for some simple root $\alpha \in \Sigma(\mathfrak{n}, \mathfrak{a})$. Then we obtain an additional quotient morphism $\pi_{s_\alpha\lambda, s_\alpha\sigma} \twoheadrightarrow \pi$. As above this implies

$$\operatorname{Re} s_\alpha\lambda\big|_{\mathfrak{a}^-\setminus\{0\}} > 0\,. \tag{1.6}$$

Motivated by (1.6) we define an equivalence relation on $\mathfrak{a}_\mathbb{C}^*$ in Section 7.1 as follows: $\lambda \sim \mu$ provided μ is obtained from λ by a sequence $\lambda = \mu_0, \mu_1, \dots, \mu_l = \mu$ such that

(a) $\mu_{i+1} = s_i(\mu_i)$ for $s_i = s_{\alpha_i}$ a simple reflection,

(b) $\mu_i(\alpha_i^\vee) \notin \frac{1}{N(G)}\mathbb{Z}$.

The equivalence class of $\lambda \in \mathfrak{a}_\mathbb{C}^*$ is denoted $[\lambda]$ and (by slight abuse of terminology introduced in Section 7.2) we say that λ is strictly integral-negative provided all elements of $[\lambda]$ satisfy (1.6). In particular we see that any parameter λ, for which there exists a discrete series representation (π, V) and a quotient $\pi_{\lambda,\sigma} \twoheadrightarrow V$, is strictly integral-negative.

Using the geometry of the Euclidean apartment of the Weyl group we show in Section 7 (Corollary 7.5) that there exists an $N = N(\mathfrak{g}) \in \mathbb{N}$ such that for strictly integral-negative parameters $\lambda \in \mathfrak{a}_\mathbb{C}^*$ one has

$$\lambda(\alpha^\vee) \in \frac{1}{N}\mathbb{Z} \qquad (\alpha \in \Sigma)\,.$$

In particular strictly integral-negative parameters are real and discrete.

For a general real spherical space $Z = G/H$ we start with a twisted discrete series representation π and consider it as a quotient $\pi_{\lambda,\sigma} = \operatorname{Ind}_P^G(\lambda \otimes \sigma) \twoheadrightarrow \pi$ of a principal series representation. The role of $\overline{\eta} \in (\pi^\vee)^\infty$ above is now played by an element $\overline{\eta} \in (\pi^{-\infty})^H$ where $\pi^{-\infty}$ refers to the dual of π^∞. We let η be the lift of $\overline{\eta}$ to an element of $(\pi_{\lambda,\sigma}^{-\infty})^H$.

The function

$$m_{\overline{v},\overline{\eta}}(g) := \overline{\eta}(\pi(g^{-1})\overline{v}) = \eta(\pi_{\lambda,\sigma}(g^{-1})v) =: m_{v,\eta}(g)$$

descends to a smooth function on $Z = G/H$ and is referred to as a generalized matrix coefficient.

Now η is supported on various H-orbits on $P\backslash G$ and we pick one with maximal dimension, say PxH for some $x \in G$. Here one meets the first serious technical obstruction: Unlike in the symmetric case (Matsuki [34], Rossmann [36]), there is no explicit description of the $P \times H$ double cosets, but merely the information that the number of double cosets is finite [29]. However, for computational purposes related to asymptotic analysis it turns out that one can replace the unknown isotropy algebra $\mathfrak{h}_x := \mathrm{Ad}(x)\mathfrak{h}$ by its deformation

$$\mathfrak{h}_{x,X} := \lim_{t\to\infty} e^{t \operatorname{ad} X}\mathfrak{h}_x \qquad (X \in \mathfrak{a}^- \text{ regular})\,.$$

There are only finitely many of those for regular X and they are all $\mathfrak{a}$-stable, i.e. nicely lined up for arguments related to Dirac-compression. One is then interested in the asymptotics of $t \mapsto m_{v,\eta}(\exp(tX)x)$ for appropriately compactly supported v. The main technical result of this paper is a generalization of (1.4) in terms of natural geometric data related to $\mathfrak{h}_{x,X}$, see Theorem 5.1 and Corollary 5.3. As above it leads to a variant of (1.5) in Corollary 6.2 and the final conclusion is derived via our Weyl group techniques from Section 7.

Acknowledgement: We would like to thank Patrick Delorme for posing the question about the spectral gap for twisted discrete series representations, and for explaining to us how to adapt the work of Sakellaridis and Venkatesh for p-adic spherical spaces to real spherical spaces.

2 Notions and Generalities

We write $\mathbb{N} = \{1, 2, 3 \ldots\}$ for the set of natural numbers and put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Throughout this paper we use upper case Latin letters $A, B, C \ldots$ to denote Lie groups and write $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ for their corresponding Lie algebras. If $A, B \subseteq G$ are Lie groups, then we write $N_A(B) := \{a \in A \mid aBa^{-1} = B\}$ for the normalizer of B in A and likewise we denote by $Z_A(B)$ the centralizer of B in A. Correspondingly if $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{g}$ are subalgebras, then we write $N_{\mathfrak{a}}(\mathfrak{b})$ for the normalizer of $\mathfrak{b}$ in $\mathfrak{a}$.

For a real vector space V we write $V_{\mathbb{C}}$ for the complexification $V \otimes \mathbb{C}$ of V.

If L is a real reductive Lie group, then we denote by L_{n} the normal subgroup generated by all unipotent elements of L, or, phrased equivalently, L_{n} is the connected subgroup with Lie algebra equal to the direct sum of all non-compact simple ideals of $\mathfrak{l}$.

Let G be an open subgroup of the real points $\mathbf{G}(\mathbb{R})$ of a reductive algebraic group $\mathbf{G}$ defined over $\mathbb{R}$. Let $\mathbf{H}$ be an algebraic subgroup of $\mathbf{G}$ defined over $\mathbb{R}$ and let H be an open subgroup of $\mathbf{H}(\mathbb{R}) \cap G$. Define the homogeneous space $Z := G/H$. We assume that Z is unimodular, i.e., carries a G-invariant positive Radon measure. Let $z_0 := e \cdot H \in Z$ be the standard base point.

Let $P \subseteq G$ be a minimal parabolic subgroup. We assume that Z is *real spherical*, that is, the action of P on Z admits an open orbit. After replacing P by a conjugate we will assume that $P \cdot z_0$ is open in Z. The local structure theorem (see [24]) asserts the existence of a parabolic subgroup $Q \supseteq P$ with Levi-decomposition $Q = L \ltimes U$ such that:

$$\begin{aligned} P \cdot z_0 &= Q \cdot z_0\,, \\ Q \cap H &= L \cap H, \\ L_{\mathrm{n}} &\subseteq L \cap H\,. \end{aligned} \tag{2.1}$$

We emphasize that the choice of L has to be taken in accordance with the local structure theorem, see [6, Remark 2.2].

Let now $L = K_L A N_L$ be any Iwasawa-decomposition of L and set $A_H := A \cap H$ and $A_Z := A/A_H$. We note that A_H is connected. The number $\operatorname{rank}_{\mathbb{R}} Z := \dim A_Z$ is an invariant of Z and referred to as the *real rank of* Z.

We inflate K_L to a maximal compact subgroup $K \subseteq G$ and set $M := Z_K(\mathfrak{a})$. We denote by θ the Cartan involution on $\mathfrak{g}$ defined by K and set $\overline{\mathfrak{u}} := \theta(\mathfrak{u})$. We may and will assume that $A \subseteq P$. Let $P = MAN$ be the corresponding Langlands decomposition of P and define $\overline{\mathfrak{n}} := \theta(\mathfrak{n})$.

2.1 Spherical roots and the compression cone

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system for the pair $(\mathfrak{g}, \mathfrak{a})$ and

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha$$

be the attached root space decomposition. Write $(\mathfrak{l} \cap \mathfrak{h})^{\perp_{\mathfrak{l}}} \subseteq \mathfrak{l}$ for the orthogonal complement of $\mathfrak{l} \cap \mathfrak{h}$ in $\mathfrak{l}$ with respect to a non-degenerate $\operatorname{Ad}(G)$-invariant bilinear form on $\mathfrak{g}$ restricted to $\mathfrak{l}$. From $\mathfrak{g} = \mathfrak{q} + \mathfrak{h} = \mathfrak{u} \oplus (\mathfrak{l} \cap \mathfrak{h})^{\perp_{\mathfrak{l}}} \oplus \mathfrak{h}$ and $\mathfrak{g} = \mathfrak{q} \oplus \overline{\mathfrak{u}}$ we infer the existence of a linear map $T : \overline{\mathfrak{u}} \to \mathfrak{u} \oplus (\mathfrak{l} \cap \mathfrak{h})^{\perp_{\mathfrak{l}}}$ such that $\mathfrak{h} = \mathfrak{l} \cap \mathfrak{h} \oplus \mathcal{G}(T)$ with $\mathcal{G}(T) \subseteq \overline{\mathfrak{u}} \oplus \mathfrak{u} \oplus (\mathfrak{l} \cap \mathfrak{h})^{\perp_{\mathfrak{l}}}$ the graph of T.

Set $\Sigma_{\mathfrak{u}} := \Sigma(\mathfrak{u}, \mathfrak{a}) \subseteq \Sigma$. For $\alpha \in \Sigma_{\mathfrak{u}}$ and $\beta \in \Sigma_{\mathfrak{u}} \cup \{0\}$ we denote by $T_{\alpha,\beta} : \mathfrak{g}^{-\alpha} \to \mathfrak{g}^\beta$ the map obtained by restriction of T to $\mathfrak{g}^{-\alpha}$ and projection to $\mathfrak{g}^\beta$. Then

$$T\big|_{\mathfrak{g}^{-\alpha}} = \sum_{\beta \in \Sigma_{\mathfrak{u}} \cup \{0\}} T_{\alpha,\beta}\,.$$

Let $\mathcal{M} \subseteq \mathfrak{a}^* \backslash \{0\}$ be the additive semi-group generated by

$$\{\alpha + \beta \mid \alpha \in \Sigma_{\mathfrak{u}}, \beta \in \Sigma_{\mathfrak{u}} \cup \{0\} \text{ such that } T_{\alpha,\beta} \neq 0\}\,.$$

We recall from [19], Cor. 12.5 and Cor. 10.9, that the cone generated by $\mathcal{M}$ is simplicial. We fix a set of generators S of this cone with the property $\mathcal{M} \subseteq \mathbb{N}_0[S]$ and refer to S as a set of *(real) spherical roots*. Note that all elements of $\mathcal{M}$ vanish on $\mathfrak{a}_H$ so that we can view $\mathcal{M}$ and S as subsets of $\mathfrak{a}_Z^*$.

We define the *compression cone* by

$$\mathfrak{a}_Z^- := \{X \in \mathfrak{a}_Z \mid (\forall \alpha \in S)\alpha(X) \leq 0\}$$

and write $\mathfrak{a}_{Z,E} := \mathfrak{a}_Z^- \cap (-\mathfrak{a}_Z^-)$ for its edge. We note that

$$\#S = \dim \mathfrak{a}_Z/\mathfrak{a}_{Z,E} \,.$$

For an $\mathfrak{a}$-fixed subspace $\mathfrak{s}$ of $\mathfrak{g}$, we define

$$\rho(\mathfrak{s})(X) := \frac{1}{2}\operatorname{tr}(\operatorname{ad}(X)\big|_{\mathfrak{s}}) \qquad (X \in \mathfrak{a})\,.$$

We write ρ_P for $\rho(\mathfrak{p})$ and ρ_Q for $\rho(\mathfrak{q})$. Recall that the unimodularity of Z implies that $\rho_Q|_{\mathfrak{a}_H} = 0$, see [23, Lemma 4.2].

Let $\Pi \subseteq \Sigma^+$ be the set of simple roots. We let

$$\mathfrak{a}^{\pm} := \{X \in \mathfrak{a} \mid (\forall \alpha \in \Pi) \ \pm\alpha(X) \geq 0\}$$

and write $\mathfrak{a}^{--}$ for the interior (Weyl chamber) of $\mathfrak{a}^-$.

We write $p : \mathfrak{a} \to \mathfrak{a}_Z$ for the projection and set $\mathfrak{a}_E := p^{-1}(\mathfrak{a}_{Z,E})$ and $A_E = \exp(\mathfrak{a}_E)$. Set $\widehat{H} = HA_E$ and note that $\widehat{H}$ normalizes H. Obviously $\widehat{H}$ is real spherical as well. Finally, we define $\widehat{Z} := G/\widehat{H}$.

2.2 The normalizer of a real spherical subalgebra

Lemma 2.1. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a real spherical subalgebra. Then the following assertions hold:*

(i) $N_{\mathfrak{g}}(\mathfrak{h}) = \widehat{\mathfrak{h}} + \widehat{\mathfrak{m}}$ *with* $\widehat{\mathfrak{m}} \subseteq \mathfrak{m}$, *the sum not necessarily being direct.*

(ii) $\widehat{\widehat{\mathfrak{h}}} = \widehat{\mathfrak{h}}$.

(iii) $[N_{\mathfrak{g}}(\mathfrak{h})]_{\mathrm{n}} = \mathfrak{h}_{\mathrm{n}}$, *i.e. every* $\operatorname{ad}_{\mathfrak{g}}$*-nilpotent element in* $N_{\mathfrak{g}}(\mathfrak{h})$ *is contained in* $\mathfrak{h}$.

Proof. For (i) see [22, (5.10)]. Lemma 4.1 in [24] implies (ii). Finally, (iii) follows from (i). □

3 Twisted discrete series as quotients of principal series

3.1 The spherical subrepresentation theorem

For a Harish-Chandra module V, we denote by V^∞ the unique smooth moderate growth Fréchet globalization and by $V^{-\infty}$ the continuous dual of V^∞. If $\eta \in (V^{-\infty})^H \setminus \{0\}$, then the pair (V, η) is called a spherical pair.

For a Harish-Chandra module V we denote by $\widetilde{V}$ its contragredient or dual Harish-Chandra module, that is, $\widetilde{V}$ consist of the K-finite vectors in the algebraic dual V^* of V. Further we denote by $\overline{V}$ the conjugate Harish-Chandra module, that is, $\overline{V} = V$ as $\mathbb{R}$-vector space but with the conjugate complex multiplication. We recall that $\widetilde{V} = \overline{V}$ in case V is unitarizable. In particular if (V, η) is a spherical pair with V unitarizable, then so is $(\overline{V}, \overline{\eta})$ with $\overline{\eta}(v) := \overline{\eta(v)}$.

Associated to $\eta \in (V^{-\infty})^H$ and $v \in V^\infty$ we find the generalized matrix coefficient on Z

$$m_{v,\eta}(z) := \eta(g^{-1}v) \qquad (z = gH \in Z),$$

which defines a smooth function on Z. If $v \in V$ then $m_{v,\eta}$ admits a convergent power series expansion (cf. [25], Sect. 6):

$$m_{v,\eta}(ma \cdot z_0) = \sum_{\mu \in \mathcal{E}} \sum_{\alpha \in \mathbb{N}_0[S]} c^\alpha_{\mu,v}(m; \log a) a^{\mu+\alpha} \qquad (a \in A_Z^-, m \in M).$$

Here $\mathcal{E} \subseteq \mathfrak{a}^*_{Z,\mathbb{C}}$ is a finite set of leading exponents only depending on (V, η); the term "leading" refers to the following relation: for all $\mu, \mu' \in \mathcal{E}$, $\mu \neq \mu'$ one has $\mu \notin \mu' + \mathbb{N}_0[S]$. Further, for each $\mu \in \mathcal{E}$, $\alpha \in \mathbb{N}_0[S]$ and $v \in V$, the assignment

$$c^\alpha_{\mu,v} : M \times \mathfrak{a}_Z \to \mathbb{C}, \ \ (m, X) \to c^\alpha_{\mu,v}(m; X)$$

is polynomial in X and M-finite. Moreover, for each $\mu \in \mathcal{E}$ there exists a $v \in V$ such that $c^0_{\mu,v} \neq 0$. The M-types which can occur are those obtained from branching the K-module $\mathrm{span}_\mathbb{C}\{K \cdot v\}$ to M. The degrees of the polynomials are uniformly bounded and we set $d_\mu := \max_{v \in V} \deg c^0_{\mu,v} \in \mathbb{N}_0$.

Let us set $A_L := Z(L) \cap A$. Then $L = M_L A_L$ for a complementary reductive subgroup $M_L \subseteq L$. For a unitary representation (σ, V_σ) of M_L and $\lambda \in \mathfrak{a}^*_{L,\mathbb{C}}$ we denote by $\mathrm{Ind}_{\overline{Q}}^G(\lambda \otimes \sigma)$ the normalized left induced representation. Note that the elements $v \in \mathrm{Ind}_{\overline{Q}}^G(\lambda \otimes \sigma)$ are K-finite functions $v : G \to V_\sigma$ which satisfy

$$v(\overline{u}mag) = a^{\lambda - \rho_Q} \sigma(m) v(g)$$

for all $g \in G$, $a \in A_L$, $u \in \overline{U}$ and $m \in M_L$.

Note that $A_L A_H = A$ and that therefore there exists a natural inclusion $\mathfrak{a}^*_Z \hookrightarrow \mathfrak{a}^*_L$. The representations $\mathrm{Ind}_{\overline{Q}}^G(\lambda \otimes \sigma)$ are related to spherical representation theory as follows.

Lemma 3.1. *Let (V, η) be a spherical pair with V irreducible and $\mu \in \mathfrak{a}^*_Z \subseteq \mathfrak{a}^*_L$ a leading exponent. Then there exist an irreducible finite dimensional representation σ of M_L with a $(M_L \cap H)$-fixed vector, and an embedding of Harish-Chandra modules:*

$$V \hookrightarrow \mathrm{Ind}_{\overline{Q}}^G((-\mu + \rho_Q) \otimes \sigma). \tag{3.1}$$

Proof. This is implicitly contained in [29], Section 4. We confine ourselves with a sketch of the argument.

Recall d_μ and fix a basis $X_1, \ldots, X_n$ of $\mathfrak{a}_Z$. For $\mathbf{m} \in \mathbb{N}_0^n$, $X = \sum_{j=1}^n x_j X_j \in \mathfrak{a}_Z$ we set $X^\mathbf{m} := x_1^{m_1} \cdot \ldots \cdot x_n^{m_n}$. Then

$$c^0_{\mu,v}(m; X) = \sum_{|\mathbf{m}| \leq d_\mu} c^\mathbf{m}_{\mu,v}(m) X^\mathbf{m} \qquad (m \in M)$$

where $c^\mathbf{m}_{\mu,v}$ is an M-finite function. Fix now $\sigma \in \widehat{M}$ and $\mathbf{m} \in \mathbb{N}_0^n$ with $|\mathbf{m}| = d_\mu$ such that the σ-isotypical part of $c^\mathbf{m}_{\mu,v}(m)$ is non-zero. This gives rise to a non-trivial M-equivariant map

$$V \to V_\sigma, \ \ v \mapsto c^\mathbf{m}_{\mu,v}[\sigma].$$

It is easy to see that $(\mathfrak{l}\cap\mathfrak{h}+\overline{\mathfrak{u}})V$ is in the kernel of this map. Note that $M\cap M_{L,n}$ is a normal subgroup of M that is contained in $M\cap H$. From the fact that σ admits a non-zero $M\cap H$-fixed vector it follows that $\sigma\big|_{M\cap M_{L,n}}$ is trivial. We may thus extend σ to a representation of $M_L\simeq M\underset{M\cap M_{L,n}}{\ltimes}M_{L,n}$ by setting $\sigma\big|_{M_{L,n}}=1$. The assertion now follows from Frobenius reciprocity. □

3.2 Discrete series and twisted discrete series

For $\chi\in(\widehat{\mathfrak{h}}/\mathfrak{h})^*_\mathbb{C}\simeq\mathfrak{a}^*_{Z,E,\mathbb{C}}$ we define the space of functions

$$C_c(\widehat{Z};\chi):=\left\{\phi\in C_c(G):\phi(\,\cdot\,ha)=a^{-\chi}\phi\text{ for all }a\in A_E,h\in H\right\}.$$

We call $\chi\in(\widehat{\mathfrak{h}}/\mathfrak{h})^*_\mathbb{C}$ *normalized unitary* if

$$\operatorname{Re}\chi\big|_{\mathfrak{a}_E}=-\rho_Q\big|_{\mathfrak{a}_E}.$$

Let $\Delta_{\widehat{Z}}$ be the modular function of $\widehat{Z}$. By [25, Lemma 8.4] we have

$$\Delta_{\widehat{Z}}(ha)=a^{-2\rho_Q}\qquad(h\in H,a\in A_E).\tag{3.2}$$

For $g\in G$, let l_g denote left multiplication by g. Let $\Omega\in\bigwedge^{\dim Z}(\mathfrak{g}/\mathfrak{h})^*\setminus\{0\}$. If $\chi\in(\widehat{\mathfrak{h}}/\mathfrak{h})^*_\mathbb{C}$ is normalized unitary, then it follows that for all $\phi,\psi\in C_c(\widehat{Z};\chi)$ the density

$$|\Omega|_{\phi,\psi}:G\ni g\mapsto\phi(g)\overline{\psi(g)}\big(T_gl_{g^{-1}}\big)^*|\Omega|$$

factors to a smooth density on $\widehat{Z}$, and the bilinear form

$$C_c(\widehat{Z};\chi)\times C_c(\widehat{Z};\chi)\to\mathbb{C};\qquad(\phi,\psi)\mapsto\int_{\widehat{Z}}|\Omega|_{\phi,\psi}$$

is an inner product. We write $L^2(\widehat{Z};\chi)$ for the Hilbert completion of $C_c(\widehat{Z};\chi)$ with respect to this inner product. Note that the inner product is invariant under the left regular action of G and thus $L^2(\widehat{Z};\chi)$ equipped with the left-regular representation is a unitary representation of G.

Definition 3.2. If $\chi\in(\widehat{\mathfrak{h}}/\mathfrak{h})^*_\mathbb{C}$ is normalized unitary, then we say that the spherical pair (V,η) belongs to the *χ-twisted discrete series for Z* provided that V is irreducible, $\pi^\vee(Y)\eta=-\chi(Y)\eta$ for all $Y\in\widehat{\mathfrak{h}}$, and $m_{v,\eta}\in L^2(\widehat{Z};\chi)$ for all $v\in V^\infty$. Furthermore, we say that (V,η) belongs to the *twisted discrete series* for Z if (V,η) belongs to the χ-twisted discrete series for some normalized unitary χ. Finally we say that (V,η) belongs to the *discrete series* for Z provided that V is irreducible and $m_{v,\eta}\in L^2(Z)$ for all $v\in V^\infty$.

Lemma 3.3. *If there exits a spherical pair (V,η) belonging to the discrete series for Z, then $H=\widehat{H}=HA_E$. Hence $\widehat{\mathfrak{h}}/\mathfrak{h}=0$ and therefore the discrete series for Z coincide with the 0-twisted discrete series for Z.*

Proof. Let (V, η) be a spherical pair belonging to the discrete series for Z. The right-action of A_E commutes with the left-action of G on $L^2(Z)$, and thus induces a natural action of A_E on $(V^{-\infty})^H$. By [27] and [30] the space $(V^{-\infty})^H$ is finite dimensional. We may therefore assume that η is a joint-eigenvector for the right-action of A_E, i.e., the generalized matrix coefficients of V satisfy

$$m_{v,\eta}(gha) = a^{-\chi} m_{v,\eta}(g) \qquad (g \in G, h \in H, a \in A_E)$$

for some normalized unitary $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})^*_{\mathbb{C}}$. Let A_0 be a subgroup of A such that $A_0 \times A_E \simeq A$. If $g \in G$ and $m_{v,\eta}(g \cdot z_0) \neq 0$, then, if the Haar measures are properly normalized,

$$\begin{aligned}
\int_Z |m_{v,\eta}(z)|^2 \, dz &\geq \int_{gQ\cdot z_0} |m_{v,\eta}(z)|^2 \, dz \\
&= \int_U \int_M \int_{A_0} \int_{A_E/(A\cap H)} (a_0 a_E)^{-2\rho_Q} |m_{v,\eta}(gnma_0a_E)|^2 \, da_E \, da_0 \, dm \, dn \\
&= \int_U \int_M \int_{A_0} \int_{A_E/(A\cap H)} a_0^{-2\rho_Q} |m_{v,\eta}(gnma_0)|^2 \, da_E \, da_0 \, dm \, dn \,.
\end{aligned}$$

Clearly the last repeated integral can only be absolutely convergent if $A_E/(A \cap H)$ has finite volume, or equivalently if $A_E = A \cap H$. □

We recall from Section 8 in [25] that (V, η) belongs to the twisted discrete series for Z only if the conditions

$$(\operatorname{Re}\mu - \rho_Q)|_{\mathfrak{a}_Z^- \setminus \mathfrak{a}_{Z,E}} < 0 \,, \tag{3.3}$$

$$(\operatorname{Re}\mu - \rho_Q)|_{\mathfrak{a}_{Z,E}} = 0 \tag{3.4}$$

hold for all leading exponents μ. Moreover,

$$\mu|_{\mathfrak{a}_{Z,E}} = -\chi \tag{3.5}$$

when (V, η) belongs to the χ-twisted discrete series. Note that (3.3) implies (3.4) unless $\mathfrak{a}_{Z,E} = \mathfrak{a}_Z$.

3.3 Quotient morphisms

It is technically easier to work with representations induced from the minimal parabolic P. Set $\rho_P^Q = \rho_P - \rho_Q$ and observe that there is a natural inclusion

$$\operatorname{Ind}_Q^G(\lambda \otimes \sigma) \to \operatorname{Ind}_P^G((\lambda + \rho_P^Q) \otimes \sigma|_M) \,.$$

In particular (3.1) yields

$$V \hookrightarrow \operatorname{Ind}_P^G((-\mu + \rho_P) \otimes \sigma) \,, \tag{3.6}$$

where we allowed ourselves to write σ for $\sigma|_M$.

For general $\mathrm{Ind}_{\overline{P}}^{G}(\lambda\otimes\sigma)$ we record that its dual representation is $\mathrm{Ind}_{\overline{P}}^{G}(-\lambda\otimes\sigma^\vee)$. The natural pairing between these two representations is given as follows in the non-compact picture:

$$v^\vee(v)=\int_N v^\vee(n)\big(v(n)\big)\,dn$$

for $v^\vee\in\mathrm{Ind}_{\overline{P}}^{G}(-\lambda\otimes\sigma^\vee)$ and $v\in\mathrm{Ind}_{\overline{P}}^{G}(\lambda\otimes\sigma)$.

Let now (V,η) be an irreducible spherical pair belonging to the twisted discrete series. Then $\overline{\mu}$ is a leading exponent for the dual pair $(\overline{V},\overline{\eta})$. By applying (3.6) to $\overline{V}$ we embed

$$\overline{V}\hookrightarrow\mathrm{Ind}_{\overline{P}}^{G}((-\overline{\mu}+\rho_P)\otimes\sigma^\vee)\,.$$

Dualizing this inclusion we obtain the quotient morphism

$$\mathrm{Ind}_{\overline{P}}^{G}((\overline{\mu}-\rho_P)\otimes\sigma)\twoheadrightarrow V\,. \tag{3.7}$$

In view of the $P\times H$-geometry of G it is a bit inconvenient to work with representations induced from the left by the opposite parabolic $\overline{P}$. We can correct this by employing the long Weyl group element $w_0\in W=W(\mathfrak{g},\mathfrak{a})$, which maps $\overline{P}$ to P. This gives us for every $\lambda\in\mathfrak{a}_\mathbb{C}^*$ and $\sigma\in\widehat{M}$ an isomorphism

$$\mathrm{Ind}_{P}^{G}(\lambda\otimes\sigma)\to\mathrm{Ind}_{\overline{P}}^{G}(w_0\lambda\otimes w_0\sigma);\ \ v\mapsto v(w_0\,\cdot\,)\,, \tag{3.8}$$

where $w_0\sigma:=\sigma\circ w_0\in\widehat{M}$. With proper choices of λ and σ we obtain from (3.8) and (3.7) a quotient morphism of $\mathrm{Ind}_{P}^{G}(\lambda\otimes\sigma)$ onto V.

We write now $\pi_{\lambda,\sigma}$ for $\mathrm{Ind}_{P}^{G}(\lambda\otimes\sigma)$ and record that functions $v\in\pi_{\lambda,\sigma}$ feature the transformation property

$$v(mang)=a^{\lambda+\rho_P}\sigma(m)v(g)\,. \tag{3.9}$$

To summarize our discussion so far:

Lemma 3.4. *Let (V,η) be a twisted discrete series representation for Z and $\mu\in\mathfrak{a}_\mathbb{C}^*$ a leading exponent. Then there exists a $\sigma\in\widehat{M}$ and a surjective quotient morphism $\pi_{\lambda,\sigma}\twoheadrightarrow V$ with $\lambda=w_0\overline{\mu}+\rho_P$.*

We write $\pi_{\lambda,\sigma}^\infty$ for the smooth Fréchet globalization of moderate growth. In the sequel we will model $\pi_{\lambda,\sigma}^\infty$ on all smooth functions which satisfy (3.9).

4 Generalized volume growth

4.1 Limiting subalgebras

Define *order-regular* elements in $\mathfrak{a}^{--}$ by

$$\mathfrak{a}^{--}_{\mathrm{o-reg}}:=\{X\in\mathfrak{a}^{--}\mid\alpha(X)\neq\beta(X),\alpha,\beta\in\Sigma,\alpha\neq\beta\}\,.$$

In this and the next section we will make heavy use of certain limits of subspaces of $\mathfrak{g}$ in the Grassmannian. In the following lemma we collect the important properties of such limits.

Lemma 4.1. *Let E be a subspace of $\mathfrak{g}$ and let $X \in \mathfrak{a}$. Then the limit*

$$E_X := \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)E\,,$$

exists in the Grassmannian. If $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ are the eigenvalues and $p_1, \ldots, p_n$ the corresponding projections onto the eigenspaces V_i of $\mathrm{ad}(X)$, then E_X is given by

$$E_X = \bigoplus_{i=1}^{n} p_i\big(E \cap \bigoplus_{j=1}^{i} V_j\big). \tag{4.1}$$

The following hold.

(i) If E is a Lie subalgebra of $\mathfrak{g}$, then E_X is a Lie subalgebra of $\mathfrak{g}$.

(ii) If $X \in \mathfrak{a}^{--}$, then $\big(\mathrm{Ad}(man)E\big)_X = \mathrm{Ad}(ma)\big(E_X\big)$ for all $m \in M$, $a \in A$ and $n \in N$. Moreover, if X is order-regular, then E_X is A-stable.

(iii) Let $\mathcal{C}$ be a connected component of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$. Then $\big(E_X\big)_Y = E_Y$ for all $X \in \overline{\mathcal{C}}$ and $Y \in \mathcal{C}$. In particular, if $X, Y \in \mathcal{C}$, then $E_X = E_Y$.

(iv) If $X, X' \in \mathfrak{a}^{--}$, then $\mathfrak{a} \cap E_X = \mathfrak{a} \cap E_{X'}$.

Proof. Let $k = \dim(E)$ and let $\iota : \mathrm{Gr}(\mathfrak{g}, k) \hookrightarrow \mathbb{P}(\bigwedge^k \mathfrak{g})$ be the Plücker embedding, i.e., ι is the map given by

$$\iota\big(\mathrm{span}(v_1, \ldots, v_k)\big) = \mathbb{R}(v_1 \wedge \cdots \wedge v_k). \tag{4.2}$$

The map ι is a diffeomorphism onto a compact submanifold of $\mathbb{P}(\bigwedge^k \mathfrak{g})$. The map $\mathrm{ad}(X)$ acts diagonalizably on $\bigwedge^k \mathfrak{g}$, say with eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_m$. Let $\xi \in \bigwedge^k \mathfrak{g} \setminus \{0\}$ be so that $\iota(E) = \mathbb{R}\xi$. We decompose ξ into eigenvectors for $\mathrm{ad}(X)$ as

$$\xi = \sum_{i=1}^{m} \xi_i\,,$$

where ξ_i is an eigenvector of $\mathrm{ad}(X)$ with eigenvalue μ_i. Now

$$\mathrm{Ad}\big(\exp(tX)\big)(\mathbb{R}\xi) = \mathbb{R}\Big(\sum_{i=1}^{m} e^{t\mu_i}\xi_i\Big)\,.$$

Let $1 \leq k \leq m$ be the largest number so that $\xi_k \neq 0$. Then $\mathrm{Ad}\big(\exp(tX)\big)(\mathbb{R}\xi)$ converges for $t \to \infty$ to $\mathbb{R}\xi_k$. Let $E_X = \iota^{-1}(\mathbb{R}\xi_k)$. Since ι is a diffeomorphism, $\mathrm{Ad}\big(\exp(tX)\big)E$ converges to E_X for $t \to \infty$.

We move on to prove (4.1). For $1 \leq i \leq n$ we define

$$E_i := E \cap \bigoplus_{j=1}^{i} V_j\,.$$

We will prove with induction that for every $1 \leq i \leq n$

$$\big(E_i\big)_X = \bigoplus_{j=1}^{i} p_j\big(E_j\big)\,. \tag{4.3}$$

Clearly $E_1 = E \cap V_1$ is stable under the adjoint action of X, and hence $(E_1)_X = E_1$. This proves (4.3) for $i = 1$. Assume that (4.3) holds for some i. We claim that

$$\bigoplus_{j=1}^{i+1} p_j\big(E_j\big) \subseteq \big(E_{i+1}\big)_X\,. \tag{4.4}$$

In view of the induction hypothesis it suffices to prove that $p_{i+1}(Y) \in \big(E_{i+1}\big)_X$ for every $Y \in E_{i+1} \setminus E_i$. We decompose Y as

$$Y = \sum_{j=1}^{i+1} p_j(Y)\,.$$

Then $p_{i+1}(Y) \neq 0$ and thus

$$(\mathbb{R}Y)_X = \lim_{t\to\infty} \mathbb{R}\Big(\sum_{j=1}^{i+1} e^{t\lambda_i} p_j(Y)\Big) = \mathbb{R}p_{i+1}(Y)\,.$$

This shows that $p_{i+1}(Y) \in (E_{i+1})_X$. Therefore, the inclusion (4.4) holds. In fact, equality holds because the dimensions agree. This proves (4.1).

Observe that $[E, E] \subseteq E$ is a closed condition in the Grassmannian. Therefore, the set of Lie subalgebras in the Grassmannian is a closed set. It follows that E_X is a Lie subalgebra if E is a Lie subalgebra. This proves (i).

Assume that $X \in \mathfrak{a}^{--}$. If $n \in N$, then $\exp(tX)n\exp(tX)^{-1}$ converges to e for $t \to \infty$. Now

$$\begin{aligned}\big(\operatorname{Ad}(man)E\big)_X &= \lim_{t\to\infty} \operatorname{Ad}(ma)\operatorname{Ad}\big(\exp(tX)n\exp(tX)^{-1}\big)\operatorname{Ad}\big(\exp(tX)\big)E \\ &= \operatorname{Ad}(ma)\big(E_X\big).\end{aligned}$$

If $X \in \mathfrak{a}^{--}_{\mathrm{o-reg}}$, then the eigenvalues $\{\alpha(X) : \alpha \in \Sigma \cup \{0\}\}$ of $\operatorname{ad}(X)$ are in bijection with $\Sigma \cup \{0\}$. Therefore, all projections p_i in (4.1) are projections onto $\mathfrak{a}$-eigenspaces, namely the root spaces and $\mathfrak{m} \oplus \mathfrak{a}$. This implies that E_X is A-stable. This proves (ii).

We move on to prove (iii). It follows from (4.1) that for every $X \in \mathfrak{a}^-$ the limit E_X is spanned by the limits L_X of the lines L in E. Hence we may assume that E is 1-dimensional. Let $X \in \overline{\mathcal{C}}$ and $Y \in \mathcal{C}$. For $\alpha \in \Sigma \cup \{0\}$ we define p_α to be the projection $\mathfrak{g} \to \mathfrak{g}_\alpha$ along the root space decomposition. Let $\alpha_0 \in \Sigma \cup \{0\}$ be so that $\alpha_0(Y)$ is maximal among the numbers $\alpha(Y)$ with $\alpha \in \Sigma \cup \{0\}$ for which $p_\alpha(E) \neq \{0\}$. By (4.1) we have $E_Y = p_{\alpha_0}(E)$. Since $Y \in \mathcal{C}$ and $X \in \overline{\mathcal{C}}$ we have $\alpha(X) \geq \beta(X)$ if $\alpha(Y) > \beta(Y)$. In particular the largest eigenvalue of $\operatorname{ad}(X)$ that appears in E is equal to $\alpha_0(X)$. The projection onto the eigenspace of $\operatorname{ad}(X)$ with eigenvalue $\alpha_0(X)$ is given by

$$\sum_{\substack{\alpha\in\Sigma\cup\{0\}\\ \alpha(X)=\alpha_0(X)}} p_\alpha\,.$$

Therefore,

$$E_X = \Big(\sum_{\substack{\alpha\in\Sigma\cup\{0\}\\ \alpha(X)=\alpha_0(X)}} p_\alpha\Big)(E)\,,$$

and hence

$$(E_X)_Y = p_{\alpha_0}\bigg(\Big(\sum_{\substack{\alpha\in\Sigma\cup\{0\}\\ \alpha(X)=\alpha_0(X)}} p_\alpha\Big)(E)\bigg) = p_{\alpha_0}(E) = E_Y\,.$$

If $X, Y \in \mathcal{C}$, then by (ii) the space E_X is $\mathfrak{a}$-stable and therefore $(E_X)_Y = E_X$. This proves (iii).

Finally we prove (iv). Let $X \in \mathfrak{a}^{--}$. Let $p_\mathfrak{m}$, $p_\mathfrak{a}$ be the projections $\mathfrak{g} \to \mathfrak{m}$ and $\mathfrak{g} \to \mathfrak{a}$, respectively, along the Bruhat decomposition. Since X is regular, it follows from (4.1) that

$$(\mathfrak{m}\oplus\mathfrak{a})\cap E_X = (p_\mathfrak{m}+p_\mathfrak{a})\big((\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n})\cap E\big)\,.$$

Clearly $p_\mathfrak{a}\big((\mathfrak{a}\oplus\mathfrak{n})\cap E\big) \subseteq \mathfrak{a}\cap E_X$. Moreover, if $Y \in \mathfrak{a}\cap E_X$ and $Y' \in (\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n})\cap E$ is so that $(p_\mathfrak{m}+p_\mathfrak{a})(Y') = Y$, then $p_\mathfrak{m}(Y') = 0$. Hence $Y \in p_\mathfrak{a}\big((\mathfrak{a}\oplus\mathfrak{n})\cap E\big)$. It follows that

$$p_\mathfrak{a}\big((\mathfrak{a}\oplus\mathfrak{n})\cap E\big) = \mathfrak{a}\cap E_X\,.$$

The left-hand side is independent of X. □

Let $\mathcal{C}$ be a connected component of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$. If $X \in \mathcal{C}$, then in view of (iii) in Lemma 4.1 the space E_X does not depend on the specific choice of X. Therefore for every subspace E of $\mathfrak{g}$ we may define

$$E_\mathcal{C} := E_X \qquad (X\in\mathcal{C})\,.$$

Let $x \in G$. We define the following spaces. First set

$$\mathfrak{h}_{\mathcal{C},x} := \big(\mathrm{Ad}(x)\mathfrak{h}\big)_\mathcal{C}\,.$$

Observe that by (ii) in Lemma 4.1

$$\mathfrak{h}_{\mathcal{C},manxh} = \mathrm{Ad}(m)\mathfrak{h}_{\mathcal{C},x} \qquad \big(m\in M, a\in A, n\in N, h\in H\big)\,. \tag{4.5}$$

We define

$$\mathfrak{a}_x := \mathfrak{h}_{\mathcal{C},x}\cap\mathfrak{a}\,.$$

In view of Lemma 4.1(iv) this space does not depend on $\mathcal{C}$. Note that (4.5) implies that $\mathfrak{a}_x$ only depends on the double coset $PxH \in P\backslash G/H$, not on the representative $x \in G$ for that coset. We further define the $\mathfrak{a}$-stable subalgebras

$$\overline{\mathfrak{n}}_{\mathcal{C},x} := \mathfrak{h}_{\mathcal{C},x}\cap\overline{\mathfrak{n}}, \qquad \mathfrak{u}_{\mathcal{C},x} := \mathfrak{h}_{\mathcal{C},x}\cap\mathfrak{n}\,.$$

Since $\mathfrak{h}_{\mathcal{C},x}$ is $\mathfrak{a}$-stable, it follows that

$$\mathfrak{h}_{\mathcal{C},x} = \overline{\mathfrak{n}}_{\mathcal{C},x}\oplus\big((\mathfrak{m}\oplus\mathfrak{a})\cap\mathfrak{h}_{\mathcal{C},x}\big)\oplus\mathfrak{u}_{\mathcal{C},x}\,. \tag{4.6}$$

Finally we choose $\overline{\mathfrak{n}}^x_\mathcal{C}$ and $\mathfrak{u}^x_\mathcal{C}$ to be $\mathfrak{a}$-stable complementary subspaces to $\overline{\mathfrak{n}}_{\mathcal{C},x}$ in $\overline{\mathfrak{n}}$ and $\mathfrak{u}_{\mathcal{C},x}$ in $\mathfrak{n}$, respectively, so that

$$\overline{\mathfrak{n}} = \overline{\mathfrak{n}}_{\mathcal{C},x}\oplus\overline{\mathfrak{n}}^x_\mathcal{C}, \qquad \mathfrak{n} = \mathfrak{u}_{\mathcal{C},x}\oplus\mathfrak{u}^x_\mathcal{C}\,. \tag{4.7}$$

Lemma 4.2. *For every $x \in G$*

$$\mathfrak{g} = (\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p}) \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x .$$

Proof. Let $X \in \mathcal{C}$. In view of (4.7) and (4.6) we have

$$\mathfrak{g} = \overline{\mathfrak{n}}_{\mathcal{C},x} \oplus \mathfrak{p} \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x = (\mathfrak{h}_{\mathcal{C},x} + \mathfrak{p}) \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x . \tag{4.8}$$

If $\mathfrak{g} \neq (\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p}) + \overline{\mathfrak{n}}_{\mathcal{C}}^x$, then also

$$\mathfrak{g} \neq \mathrm{Ad}(a)\big(\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p} + \overline{\mathfrak{n}}_{\mathcal{C}}^x\big) = (\mathrm{Ad}(ax)\mathfrak{h} + \mathfrak{p}) + \overline{\mathfrak{n}}_{\mathcal{C}}^x$$

for every $a \in A$. This would imply that the limit of $(\mathrm{Ad}(\exp(tX)x)\mathfrak{h} + \mathfrak{p}) + \overline{\mathfrak{n}}_{\mathcal{C}}^x$ for $t \to \infty$ is a proper subspace of $\mathfrak{g}$. This in turn would contradict (4.8). Therefore, $\mathfrak{g} = (\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p}) + \overline{\mathfrak{n}}_{\mathcal{C}}^x$. Moreover, it follows from (4.1) that $\mathfrak{p} \cap \mathfrak{h}_{\mathcal{C},x} = \mathfrak{p} \cap \mathrm{Ad}(x)\mathfrak{h}$, and hence

$$\dim(\mathfrak{h}_{\mathcal{C},x} + \mathfrak{p}) = \dim(\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p}) .$$

Therefore, by comparing with (4.8) we see that the sum $(\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p}) + \overline{\mathfrak{n}}_{\mathcal{C}}^x$ is direct. □

4.2 Volume-weights

We recall the volume-weight function on Z

$$\mathbf{v}(z) := \mathrm{vol}_Z(Bz) \qquad (z \in Z) ,$$

where B is some compact neighborhood of e in G. We refer to Appendix A for the properties of volume-weights. The volume weight naturally shows up in the treatment of twisted discrete series representations.

The following proposition is a direct corollary of the invariant Sobolev lemma in Appendix A.

Proposition 4.3. *Let (V, η) be a spherical pair corresponding to a twisted discrete series representation. Then*

$$\sup_{z \in Z} |m_{v,\eta}(z)| \mathbf{v}(z)^{\frac{1}{2}} < \infty . \tag{4.9}$$

Moreover, if $(z_n)_{n \in \mathbb{N}}$ is a sequence in Z such that its image in $\widehat{Z}$ tends to infinity, then

$$\lim_{n \to \infty} |m_{v,\eta}(z_n)| \mathbf{v}(z_n)^{\frac{1}{2}} = 0 . \tag{4.10}$$

The basic asymptotic behavior of $\mathbf{v}$ on the compression cone is

$$\mathbf{v}(a \cdot z_0) \asymp a^{-2\rho_Q} \quad (a \in A_Z^-) . \tag{4.11}$$

See [23, Proposition 4.3]. We investigate now the growth of $\mathbf{v}$ with the base point z_0 shifted by an element $x \in G$, i.e., we investigate how $\mathbf{v}(ax \cdot z_0)$ grows for $a \in A^-$.

Recall the parabolic subgroup $Q = LU$ from (2.1). For $x = e$, we have $\mathfrak{h}_{\mathcal{C},e} = (\mathfrak{l} \cap \mathfrak{h}) \oplus \overline{\mathfrak{u}}$, and thus

$$\rho(\mathfrak{h}_{\mathcal{C},e}) = -\rho_Q .$$

Hence the following proposition is a partial generalization of the lower bound in (4.11) for shifted base points.

Proposition 4.4. *Let $x \in G$ and $X \in \mathfrak{a}^-$. Let $\mathcal{C}$ be a connected component of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$ such that $X \in \overline{\mathcal{C}}$. Then there exists a $C > 0$ such that*

$$\mathbf{v}(\exp(tX)x \cdot z_0) \geq Ce^{2t\rho(\mathfrak{h}_{\mathcal{C},x})(X)} \qquad (t \geq 0)\,.$$

Proof. Set $\overline{N}^x = \exp(\overline{\mathfrak{n}}^x_{\mathcal{C}})$. Since $\exp : \overline{\mathfrak{n}} \to \overline{N}$ is a polynomial isomorphism the group $\overline{N}^x$ is an affine subvariety of $\overline{N}$. Define an affine subvariety of N by $U^x := \exp(\mathfrak{u}^x_{\mathcal{C}})$. Let $\mathfrak{a}^x$ be the orthogonal complement of $\mathfrak{a}_x$ in $\mathfrak{a}$ and set $A^x := \exp(\mathfrak{a}^x)$. Further let $X_1, \ldots, X_k$ be a basis of a subspace in $\mathfrak{m}$ which is complementary to $p_{\mathfrak{m}}(\mathfrak{h}_{\mathcal{C},x})$ in $\mathfrak{m}$, where $p_{\mathfrak{m}}$ is the projection $\mathfrak{g} \to \mathfrak{m}$ along the Bruhat decomposition. We may assume in addition that the X_j are so that $M_j := \exp(\mathbb{R}X_j) \simeq \mathbb{R}/\mathbb{Z}$. Now

$$\mathfrak{a} \oplus \mathfrak{m} = \big((\mathfrak{m} \oplus \mathfrak{a}) \cap \mathfrak{h}_{\mathcal{C},x}\big) \oplus \mathfrak{a}^x \oplus \bigoplus_{j=1}^k \mathbb{R}X_j\,.$$

Further, we define the affine variety $\mathcal{M} := M_1 \times \ldots \times M_k$. For $m = (m_1, \ldots, m_k) \in \mathcal{M}$ we set $\phi(m) := m_1 \cdot \ldots \cdot m_k \in M$.

For $t \in \mathbb{R}$ define $a_t := \exp(tX)$ and consider the algebraic map

$$\Phi_t :\ U^x \times \overline{N}^x \times A^x \times \mathcal{M} \times H \to G; \qquad (u, \overline{n}, a, m, h) \mapsto u\overline{n}a\phi(m)a_t xh\,.$$

We have

$$\mathfrak{g} = \mathfrak{u}^x_{\mathcal{C}} \oplus \overline{\mathfrak{n}}^x_{\mathcal{C}} \oplus \mathfrak{a}^x \oplus \bigoplus_{j=1}^k \mathbb{R}X_j \oplus \mathfrak{h}_{\mathcal{C},x}\,.$$

Note that if $\mathrm{Ad}(ax)\mathfrak{h}$ would not be transversal to $V := \mathfrak{u}^x_{\mathcal{C}} \oplus \overline{\mathfrak{n}}^x_{\mathcal{C}} \oplus \mathfrak{a}^x \oplus \bigoplus_{j=1}^k \mathbb{R}X_j$ for some $a \in A$, then it would not be transversal for any $a \in A$ since V is A-invariant. This would contradict the fact that $\mathfrak{h}_{\mathcal{C},x}$ is transversal to V. We thus conclude that for every $a \in A$

$$\mathfrak{g} = \mathfrak{u}^x_{\mathcal{C}} \oplus \overline{\mathfrak{n}}^x_{\mathcal{C}} \oplus \mathfrak{a}^x \oplus \bigoplus_{j=1}^k \mathbb{R}X_j \oplus \mathrm{Ad}(ax)\mathfrak{h}\,.$$

In particular this holds for $a = a_t$. This implies for generic t, and hence in particular for $t \gg 0$, that the map Φ_t is dominant and as such has generically finite fibers, with a fiber bound independent of t. See [9, Prop. 15.5.1(i)].

Let U^x_B, $\overline{N}^x_B$, A^x_B, $\mathcal{M}_B$ and H_B be relatively compact, open neighborhoods of e in U^x, $\overline{N}^x$, A^x, $\phi(\mathcal{M})$ and H respectively. We choose these sets small enough so that $U^x_B\overline{N}^x_B A^x_B \mathcal{M}_B \subseteq B$. Then

$$\mathbf{v}(a_t x \cdot z_0) \geq \int_Z \mathbf{1}_{U^x_B\overline{N}^x_B A^x_B \mathcal{M}_B a_t x \cdot z_0}(z)\, dz\,. \tag{4.12}$$

For $y \in G$, let F_y be the projection onto H of $\Phi_t^{-1}(\{y\})$. If $yh \in U^x_B\overline{N}^x_B A^x_B \mathcal{M}_B a_t x H_B$ then $y \in U^x_B\overline{N}^x_B A^x_B \mathcal{M}_B a_t x H_B h^{-1}$. Hence $H_B h^{-1}$ contains an element from F_y and h belongs to $(F_y)^{-1}H_B$. Therefore,

$$\begin{aligned}\int_H \mathbf{1}_{U^x_B\overline{N}^x_B A^x_B \mathcal{M}_B a_t x H_B}(yh)\, dh &\leq \int_H \mathbf{1}_{(F_y)^{-1}H_B}(h)dh\, \mathbf{1}_{U^x_B\overline{N}^x_B A^x_B \mathcal{M}_B a_t x \cdot z_0}(y \cdot z_0)\\ &\leq \#\Phi_t^{-1}(\{y\})\,\mathrm{vol}_H(H_B)\mathbf{1}_{U^x_B\overline{N}^x_B A^x_B \mathcal{M}_B a_t x \cdot z_0}(y \cdot z_0)\,.\end{aligned}$$

Let $c = \big(n \operatorname{vol}_H(H_B)\big)^{-1}$, where n is the generic fiber bound. Then for generic $y \in G$ we have

$$\mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x \cdot z_0}(y \cdot z_0) \geq c \int_H \mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x H_B}(yh)\, dh.$$

By inserting this inequality into (4.12) we obtain

$$\begin{aligned}
\mathbf{v}(a_t x \cdot z_0) &\geq \int_Z c \int_H \mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x H_B}(yh)\, dh\, dyH \\
&= c \int_G \mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x H_B}(y)\, dy \\
&= c \int_G \mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x H_B x^{-1} a_{-t}}(y)\, dy\,.
\end{aligned}$$

For the last equality we used the invariance of the Haar measure on G.

We define $\Xi := U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B$ and set

$$\Psi_t :\ \Xi \times x H_B x^{-1} \to G; \qquad (\xi, y) \mapsto \xi a_t y a_{-t}\,.$$

The fibers of Ψ_t are bounded by the fibers of Φ_t, and hence are generically finite with fiber bound independent of t for $t \gg 0$. Let ω_G be the section of $\bigwedge^{\dim G} T^*G$ corresponding to the Haar measure on G. Then

$$\mathbf{v}(a_t x \cdot z_0) \geq \frac{c}{k} \int_\Xi \int_{x H_B x^{-1}} \Psi_t^* \omega_G\,,$$

where k is the fiber bound of Ψ_t.

We finish the proof by estimating $\Psi_t^*\omega_G$. For $g \in G$, let $l_g : G \to G$ and $r_g : G \to G$ be left and right-multiplication by g, respectively. Let

$$\xi \in \Xi, \quad y \in x H_B x^{-1}, \quad Y_1 \in T_\xi \Xi \quad \text{and} \quad Y_2 \in T_y(x H x^{-1})\,.$$

Let $\gamma : \mathbb{R} \to \xi^{-1}\Xi$ and $\delta : \mathbb{R} \to x H_B x^{-1} y^{-1}$ be smooth paths so that

$$\gamma(0) = \delta(0) = e, \quad \gamma'(0) = (T_e l_\xi)^{-1} Y_1 \quad \text{and} \quad \delta'(0) = T_y r_{y^{-1}} Y_2\,.$$

Then

$$\frac{d}{ds}\gamma(s) a_t \delta(s) a_{-t}\big|_{s=0} = \gamma'(0) + \operatorname{Ad}(a_t)\delta'(0) = (T_e l_\xi)^{-1} Y_1 + \operatorname{Ad}(a_t)\Big(T_y r_{y^{-1}} Y_2\Big).$$

Now $\xi\gamma$ is a smooth path in Ξ with $(\xi\gamma)(0) = \xi$ and $(\xi\gamma)'(0) = Y_1$. Likewise, δy is a smooth path in $x H_B x^{-1}$ satisfying $(\delta y)(0) = y$ and $(\delta y)'(0) = Y_2$.

The tangent map of Ψ_t is determined by the following identity of elements in $T_\xi G$

$$\begin{aligned}
T_{(\xi,y)}\big(r_{a_t y^{-1} a_{-t}} \circ \Psi_t\big)\big(Y_1, Y_2\big) &= \frac{d}{ds}\, \Psi_t\Big(\xi\gamma(s), \delta(s) y\Big) a_t y^{-1} a_{-t}\,\big|_{s=0} \\
&= \frac{d}{ds}\, \xi\gamma(s) a_t \delta(s) a_{-t}\,\big|_{s=0} = T_e l_\xi\Big(\frac{d}{ds}\gamma(s) a_t \delta(s) a_{-t}\,\big|_{s=0}\Big) \\
&= Y_1 + T_e l_\xi \operatorname{Ad}(a_t)\Big(T_y r_{y^{-1}} Y_2\Big)\,. \qquad (4.13)
\end{aligned}$$

We write $\mathfrak{h}_{X,x}$ for the limit for $t\to\infty$ of $\mathrm{Ad}(a_t)\,\mathrm{Ad}(x)\mathfrak{h}$ in the Grassmannian. Let Y be a non-zero eigenvector of $\mathrm{ad}(X)$ in $\mathfrak{h}_{X,x}$ and let $\alpha\in\Sigma\cup\{0\}$ be such that $\alpha(X)$ is the eigenvalue. It follows from (4.1) with $E=\mathrm{Ad}(x)\mathfrak{h}$, that there exists an element

$$Y'\in\Big(Y+\sum_{\substack{\beta\in\Sigma\cup\{0\}\\ \beta(X)<\alpha(X)}}\mathfrak{g}^{\beta}\Big)\cap\mathrm{Ad}(x)\mathfrak{h}.$$

Let $\widetilde{Y}$ be a right-invariant vector field on xHx^{-1} such that $\widetilde{Y}(e)=Y'$. Then

$$\begin{aligned}\lim_{t\to\infty}e^{-t\alpha(X)}T_{(\xi,y)}\big(r_{a_ty^{-1}a_{-t}}\circ\Psi_t\big)\big(0,\widetilde{Y}(y)\big)&=\lim_{t\to\infty}e^{-t\alpha(X)}\Big(T_el_\xi\circ\mathrm{Ad}(a_t)\Big)\big(T_yr_{y^{-1}}\widetilde{Y}(y)\big)\\&=\lim_{t\to\infty}e^{-t\alpha(X)}\Big(T_el_\xi\circ\mathrm{Ad}(a_t)\Big)(Y').\end{aligned}$$

For $\beta\in\Sigma\cup\{0\}$ with $\beta(X)<\alpha(X)$, let $Y'_\beta\in\mathfrak{g}^\beta$ be so that

$$Y'=Y+\sum_{\substack{\beta\in\Sigma\cup\{0\}\\ \beta(X)<\alpha(X)}}Y'_\beta.$$

Then

$$e^{-t\alpha(X)}\,\mathrm{Ad}(a_t)Y'=Y+\sum_{\substack{\beta\in\Sigma\cup\{0\}\\ \beta(X)<\alpha(X)}}a_t^{\beta-\alpha}Y'_\beta.$$

Therefore,

$$\begin{aligned}&\lim_{t\to\infty}e^{-t\alpha(X)}T_{(\xi,y)}\big(r_{a_ty^{-1}a_{-t}}\circ\Psi_t\big)\big(0,\widetilde{Y}(y)\big)\\&\quad=T_el_\xi Y+\lim_{t\to\infty}\sum_{\substack{\beta\in\Sigma\cup\{0\}\\ \beta(X)<\alpha(X)}}a_t^{\beta-\alpha}T_el_\xi Y'_\beta=T_el_\xi Y\,.\end{aligned}$$

The convergence is uniform in y and uniform on compact sets in ξ. Combining this with (4.13) yields that for every $Y_1\in T_\xi\Xi$ and Y as before we have

$$\lim_{t\to\infty}e^{-t\alpha(X)}T_{(\xi,y)}\big(r_{a_ty^{-1}a_{-t}}\circ\Psi_t\big)\big(Y_1,\widetilde{Y}(y)\big)=Y_1+T_el_\xi Y\,,$$

where again the convergence is uniform in y and uniform on compact sets in ξ. Define $\rho_{X,x}:=\frac{1}{2}\,\mathrm{tr}\,\big(\mathrm{ad}(X)\big|_{\mathfrak{h}_{X,x}}\big)$. It follows that

$$e^{-2t\rho_{X,x}}\Psi_t^*\omega_G=e^{-2t\rho_{X,x}}\big(r_{a_ty^{-1}a_{-t}}\circ\Psi_t\big)^*\omega_G$$

converges for $t\to\infty$ to a nowhere vanishing continuous section of the vector bundle $\bigwedge^{\dim G}T^*\big(\Xi\times xH_Bx^{-1}\big)$. The proposition now follows from the facts that Ξ and xH_Bx^{-1} are relatively compact and that $\rho(\mathfrak{h}_{\mathcal{C},x})(X)=\rho_{X,x}$. $\square$

4.3 Escaping to infinity on $\widehat{Z}$

Recall $\widehat{\mathfrak{h}} = \mathfrak{h} + \mathfrak{a}_E$. For a connected component $\mathcal{C}$ of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$ and $Y \in \mathcal{C}$, define $\widehat{\mathfrak{h}}_{\mathcal{C},x} = \lim_{t\to\infty} \mathrm{Ad}(\exp(tY)x)\widehat{\mathfrak{h}}$. Obviously we have $\mathfrak{h}_{\mathcal{C},x} \lhd \widehat{\mathfrak{h}}_{\mathcal{C},x}$ and that $\widehat{\mathfrak{h}}_{\mathcal{C},x}$ is $\mathfrak{a}$-invariant. We define

$$\mathfrak{a}_x^E := \widehat{\mathfrak{h}}_{\mathcal{C},x} \cap \mathfrak{a} \supseteq \mathfrak{a}_x \,.$$

It follows from Lemma 4.1(iv) that the space $\mathfrak{a}_x^E$ does not depend on the connected component $\mathcal{C}$ of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$. Furthermore, it is independent of the representative $x \in G$ of the double coset $PxH \in P\backslash G/H$, cf. (4.5). Note that $\mathfrak{a}_e^E = \mathfrak{a}_E$.

Proposition 4.5. *Let $X \in \mathfrak{a}^- \setminus \mathfrak{a}_x^E$. Then $\{\exp(tX)x\widehat{H} \mid t \geq 0\}$ is unbounded in $\widehat{Z}$.*

Proof. Set $a_t := \exp(tX)$. We argue by contradiction and assume that $\{a_t x\widehat{H} \mid t \geq 0\}$ is relatively compact in $\widehat{Z}$. Then there exists a compact set $C \subseteq G$ such that

$$a_t x \in Cx\widehat{H} \qquad (t \geq 0)\,. \tag{4.14}$$

Let

$$\widehat{\mathfrak{h}}^1 := \big(\mathrm{Ad}(x)\widehat{\mathfrak{h}}\big)_X \,.$$

With $\widehat{d} := \dim \widehat{\mathfrak{h}}$ we notice that the natural map

$$\widehat{Z} \to \mathrm{Gr}_{\widehat{d}}(\mathfrak{g}), \;\; g\widehat{H} \mapsto \mathrm{Ad}(g)\widehat{\mathfrak{h}}$$

is continuous and thus (4.14) implies that there exists a $c \in C$ such that $\widehat{\mathfrak{h}}^1 = \mathrm{Ad}(cx)\widehat{\mathfrak{h}}$. Since $\mathrm{Ad}(a_t)\widehat{\mathfrak{h}}^1 = \widehat{\mathfrak{h}}^1$ for all $t \in \mathbb{R}$ we thus obtain that $\mathrm{Ad}(c^{-1}a_t cx)\widehat{\mathfrak{h}} = \mathrm{Ad}(x)\widehat{\mathfrak{h}}$ and in particular $\mathrm{Ad}(c)^{-1}X \in N_{\mathfrak{g}}(\mathrm{Ad}(x)\widehat{\mathfrak{h}}) = \mathrm{Ad}(x)N_{\mathfrak{g}}(\widehat{\mathfrak{h}})$. Recall from Lemma 2.1 (i, ii) that $N_{\mathfrak{g}}(\widehat{\mathfrak{h}}) = \widehat{\mathfrak{h}} + \widehat{\mathfrak{m}}$ for some subalgebra $\widehat{\mathfrak{m}} \subseteq \mathfrak{m}$. Hence it follows that

$$X \in \widehat{\mathfrak{h}}^1 + \mathrm{Ad}(cx)\widehat{\mathfrak{m}} =: \widetilde{\mathfrak{h}}^1 \,. \tag{4.15}$$

We claim that $X \in \widehat{\mathfrak{h}}^1$. To see this, assume that $X \notin \widehat{\mathfrak{h}}^1$. Since X is hyperbolic and the elements in $\mathrm{Ad}(cx)\widehat{\mathfrak{m}}$ are elliptic, $X \notin \mathrm{Ad}(cx)\widehat{\mathfrak{m}}$. Let $X_{\mathfrak{m}} \in \mathrm{Ad}(cx)\widehat{\mathfrak{m}}$ be so that $X \in \widehat{\mathfrak{h}}^1 + X_{\mathfrak{m}}$. Let $\widetilde{H}^1$ and $\widehat{H}^1$ be the connected algebraic subgroups with Lie algebra equal to $\widetilde{\mathfrak{h}}^1$ and $\widehat{\mathfrak{h}}^1$, respectively. The map $\mathbb{R}X \to \mathbb{R}X_{\mathfrak{m}}$; $tX \mapsto tX_{\mathfrak{m}}$ induces a non-trivial algebraic homomorphism from $\mathbb{R}^\times$ to the compact group $\widetilde{H}^1/\widehat{H}^1$. This leads to a contradiction as such algebraic homomorphisms do not exist. This proves the claim.

Let $\mathcal{C}$ be a connected component of $\mathfrak{a}^{--}_{o-reg}$ so that $X \in \overline{\mathcal{C}}$ and let $Y \in \mathcal{C}$. Then by Lemma 4.1 (iii)

$$(\widehat{\mathfrak{h}}^1)_Y = (\mathrm{Ad}(x)\widehat{\mathfrak{h}})_Y = \widehat{\mathfrak{h}}_{\mathcal{C},x}.$$

Therefore,

$$X \in \widehat{\mathfrak{h}}^1 \cap \mathfrak{a} = \big(\widehat{\mathfrak{h}}^1 \cap \mathfrak{a}\big)_Y \subseteq (\widehat{\mathfrak{h}}^1)_Y \cap \mathfrak{a} = \widehat{\mathfrak{h}}_{\mathcal{C},x} \cap \mathfrak{a} = \mathfrak{a}_x^E \,,$$

which is the desired contradiction. □

5 Principal asymptotics

In this section we analyze the asymptotic behavior of generalized matrix coefficients $m_{v,\eta}$ where $\eta \in (\pi_{\lambda,\sigma}^{-\infty})^H$. Before we state the main theorem, we introduce some notation.

Let $\sigma \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. We identify $\pi_{\lambda,\sigma}^\infty$ with the space of smooth sections of the vector-bundle $V_\sigma \otimes \mathbb{C}_\lambda \times_P G \to P\backslash G$. The support of a section or a functional is defined in the usual way as a closed subset of $P\backslash G$. For an open subset U of $P\backslash G$ we define $\pi_{\lambda,\sigma}^\infty(U)$ to be the space of all $v \in \pi_{\lambda,\sigma}^\infty$ with compact support contained in U. We write $\pi_{\lambda,\sigma}^{-\infty}(U)$ for the continuous dual of $\pi_{\lambda,\sigma}^\infty(U)$.

For $x \in G$ we define $[x] \in P\backslash G$ to be the coset Px.

It follows from Lemma 2.1(iii) that $\widehat{\mathfrak{h}} \cap \mathrm{Ad}(x^{-1})\mathfrak{n} \subseteq \mathfrak{h}$. Moreover, for every $Y \in \mathfrak{a}_x^E$ there exists a $Y_\mathfrak{n} \in \mathfrak{n}$ such that $Y + Y_\mathfrak{n} \in \mathrm{Ad}(x)\widehat{\mathfrak{h}}$. (See equation (4.1) in Lemma 4.1.) Therefore for $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})_{\mathbb{C}}^*$ and $x \in G$ we may define $\chi_x \in (\mathfrak{a}_x^E)_{\mathbb{C}}^*$ to be given by the singleton

$$\{\chi_x(Y)\} = \chi\Big([\mathrm{Ad}(x^{-1})(Y + \mathfrak{n})] \cap \widehat{\mathfrak{h}}\Big) \qquad (Y \in \mathfrak{a}_x^E)\,. \tag{5.1}$$

Note that $\chi_x\big|_{\mathfrak{a}_x} = 0$ and that χ_x only depends on the H-orbit $P\backslash PxH$, not on the representative $x \in G$ of the orbit.

Theorem 5.1. *Let $\eta \in (\pi_{\lambda,\sigma}^{-\infty})^H$ and let $x \in G$. Assume that there exists an open neighborhood Υ of $[x]$ in $P\backslash G$ such that*

$$\mathrm{supp}\,\eta \cap \Upsilon = P\backslash PxH \cap \Upsilon\,. \tag{5.2}$$

Let $\mathcal{C}$ be a connected component of $\mathfrak{a}_{\mathrm{o-reg}}^{--}$. For every $X \in \overline{\mathcal{C}}$ there exists a neighborhood Ω of $[e]$ in Υx^{-1} and a unique pair of a constant $r_X \geq 0$ and a non-zero functional $\eta_{X,x} \in \pi_{\lambda,\sigma}^{-\infty}(\Omega)$, satisfying

$$\lim_{t\to\infty} e^{t\big(\lambda(X)+\rho_P(X)+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})(X)-r_X\big)}\pi_{\lambda,\sigma}^\vee\big(\exp(tX)x\big)\eta = \eta_{X,x}\,. \tag{5.3}$$

Here the limit is with respect to weak-$*$ topology on $\pi_{\lambda,\sigma}^{-\infty}(\Omega)$.

For $X \in \mathcal{C}$ outside of a finite set of hyperplanes $\mathcal{H}_\mathcal{C}$, there exists a $\omega \in -\mathbb{N}_0[\Pi]$, so that $\omega(X) = r_X$, and so that $\eta_{X,x}$ satisfies

$$\pi_{\lambda,\sigma}^\vee(\mathfrak{h}_{\mathcal{C},x})\eta_{X,x} = \{0\}\,, \tag{5.4}$$

$$\pi_{\lambda,\sigma}^\vee(Y)\eta_{X,x} = \big(-\lambda-\rho_P-2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})+\omega\big)(Y)\eta_{X,x} \qquad (Y \in \mathfrak{a})\,. \tag{5.5}$$

Moreover, if $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})_{\mathbb{C}}^*$ and η satisfy

$$\pi_{\lambda,\sigma}^\vee(Y)\eta = -\chi(Y)\eta \qquad (Y \in \widehat{\mathfrak{h}})\,, \tag{5.6}$$

then

$$\pi_{\lambda,\sigma}^\vee(Y)\eta_{X,x} = -\chi_x(Y)\eta_{X,x} \qquad (Y \in \mathfrak{a}_x^E)\,. \tag{5.7}$$

Remark 5.2. For every non-zero H-invariant functional $\eta \in \pi_{\lambda,\sigma}^{-\infty}$ there exist an $x \in G$ and an open neighborhood Υ of $[x]$ in $P\backslash G$ such that (5.2) holds. Indeed, let $\mathcal{O}_0$ be an H-orbit in $\operatorname{supp}(\eta)$ of maximal dimension and let $x \in \mathcal{O}_0$. The action of H on $P\backslash G$ admits finitely many orbits. (See [4] and [29].) Since H is a real algebraic group, and $P\backslash G$ is a real algebraic variety, and the action of H on $P\backslash G$ is real algebraic, the closure of any H-orbit $\mathcal{O}$ in $P\backslash G$ consists of $\mathcal{O}$ and H-orbits of strictly smaller dimension. See [16, Proposition 8.3]. Therefore,

$$\Upsilon := \mathcal{O}_0 \cup \bigcup_{\substack{\mathcal{O}\in P\backslash G/H \\ \dim(\mathcal{O})>\dim(\mathcal{O}_0)}} \mathcal{O}\,.$$

is an open neighborhood of $[x]$ and $\operatorname{supp}(\eta)\cap\Upsilon = \mathcal{O}_0$.

Before we prove the theorem we list some direct implications, which will be crucial in the following sections.

Corollary 5.3. *Let $\eta \in \big(\pi_{\lambda,\sigma}^{-\infty}\big)^H$ and let $x \in G$. Assume that there exists an open neighborhood Υ of $[x]$ in $P\backslash G$ such that (5.2) holds. Let $\mathcal{C}$ be a connected component of $\mathfrak{a}_{\mathrm{o-reg}}^{--}$.*

(i) For every $X \in \overline{\mathcal{C}}$ there exists a $r_X \geq 0$ and a $v \in \pi_{\lambda,\sigma}^{\infty}$ such that

$$m_{v,\eta}\big(\exp(tX)x\cdot z_0\big) \sim e^{t\big(-\lambda(X)-\rho_P(X)-2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})(X)+r_X\big)} \qquad (t\to\infty)\,.$$

(ii) There exists a $\omega \in -\mathbb{N}_0[\Pi]$ such that

$$\lambda\big|_{\mathfrak{a}_x} = \big(-\rho_P - 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x}) + \omega\big)\big|_{\mathfrak{a}_x}\,.$$

(iii) Let $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})^*_{\mathbb{C}}$ and assume that (5.6) is satisfied. Then there exists a $\omega \in -\mathbb{N}_0[\Pi]$ such that

$$\lambda\big|_{\mathfrak{a}_x^E} = \big(-\rho_P - 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x}) + \omega\big)\big|_{\mathfrak{a}_x^E} + \chi_x\,.$$

Here χ_x is given by (5.1).

Proof. Ad (i): The functional $\eta_{X,x}$ is non-zero, hence there exists a $v \in \pi_{\lambda,\sigma}^{\infty}(\Omega)$ for which $\eta_{X,x}(v) = 1$. The claim now follows from (5.3).
Ad (iii): Let $X \in \mathcal{C}\setminus\mathcal{H}_{\mathcal{C}}$. Since $\mathfrak{a}_x^E = \widehat{\mathfrak{h}}_{\mathcal{C},x}\cap\mathfrak{a}$, the identity follows from (5.5) and (5.7).
Ad (ii): The identity follows from (iii) since $\chi_x\big|_{\mathfrak{a}_x} = 0$. □

In the remainder of this section we give the proof of Theorem 5.1.

We fix an element $x \in G$ and a connected component $\mathcal{C}$ of $\mathfrak{a}_{\mathrm{o-reg}}^{--}$. Recall that $\overline{\mathfrak{n}}_{\mathcal{C}}^x \subseteq \overline{\mathfrak{n}}$ is an $\mathfrak{a}$-invariant vector complement of $\overline{\mathfrak{n}}_{\mathcal{C},x}$, so that $\overline{\mathfrak{n}} = \overline{\mathfrak{n}}_{\mathcal{C},x}\oplus\overline{\mathfrak{n}}_{\mathcal{C}}^x$. By Lemma 4.2 we have

$$\mathfrak{g} = (\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p}) \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x\,.$$

Choose a subspace $\mathfrak{p}'$ of $\mathfrak{p}$ so that $\mathfrak{g} = \mathrm{Ad}(x)\mathfrak{h} \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x \oplus \mathfrak{p}'$. Let

$$\psi : \overline{\mathfrak{n}}_{\mathcal{C},x} \to \overline{\mathfrak{n}}_{\mathcal{C}}^x + \mathfrak{p}$$

be minus the restriction of the projection $\mathfrak{g} \to \overline{\mathfrak{n}}_{\mathcal{C}}^x \oplus \mathfrak{p}'$ along this decomposition. Then

$$Y + \psi(Y) \in \mathrm{Ad}(x)\mathfrak{h} \qquad (Y \in \overline{\mathfrak{n}}_{\mathcal{C},x})\,.$$

For every $Y \in \overline{\mathfrak{n}}_{\mathcal{C},x}$

$$Y = (1+\psi)(Y) - \psi(Y) \in \mathrm{Im}(1+\psi) + \overline{\mathfrak{n}}_{\mathcal{C}}^x + \mathfrak{p}.$$

Combining this with a dimension count yields

$$\mathfrak{g} = \mathrm{Im}(1+\psi) \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x \oplus \mathfrak{p}. \tag{5.8}$$

For the proof of Theorem 5.1 we need the following lemma.

Lemma 5.4. *Let $X \in \overline{\mathcal{C}}$ and let $\psi : \overline{\mathfrak{n}}_{\mathcal{C},x} \to \overline{\mathfrak{n}}_{\mathcal{C}}^x + \mathfrak{p}$ as above. The limit*

$$\psi_X := \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big) \circ \psi \circ \mathrm{Ad}\big(\exp(-tX)\big)$$

exists in the space of linear maps $\overline{\mathfrak{n}}_{\mathcal{C},x} \to \overline{\mathfrak{n}}_{\mathcal{C}}^x + \mathfrak{p}$. Moreover, if $X \in \mathcal{C}$, then $\psi_X = 0$.

Proof. Let $X_0 \in \mathcal{C}$. If E is a line in the set $\mathrm{Ad}(x)\mathfrak{h} \setminus (\mathrm{Ad}(x)\mathfrak{h} \cap \mathfrak{p})$, then in view of (4.1) in Lemma 4.1, the limit E_{X_0} is a line in $\overline{\mathfrak{n}}$. Since this limit is also contained in $\mathfrak{h}_{\mathcal{C},x}$, it is in fact contained in $\mathfrak{h}_{\mathcal{C},x} \cap \overline{\mathfrak{n}} = \overline{\mathfrak{n}}_{\mathcal{C},x}$. In particular, if $Y \in \overline{\mathfrak{n}}_{\mathcal{C},x} \setminus \{0\}$, then $Y + \psi(Y) \in \mathrm{Ad}(x)\mathfrak{h} \setminus (\mathrm{Ad}(x)\mathfrak{h} \cap \mathfrak{p})$ by (5.8), and hence the limit of $\mathrm{Ad}\big(\exp(tX_0)\big)\mathbb{R}\big(Y + \psi(Y)\big)$ is a line in $\overline{\mathfrak{n}}_{\mathcal{C},x}$. Since $\overline{\mathfrak{n}}_{\mathcal{C}}^x \oplus \mathfrak{p}$ is stable under the adjoint action of A, the eigenvalues of $\mathrm{ad}(X_0)$ occurring in the decomposition of $\psi(Y)$ into eigenvectors must be smaller than the largest eigenvalue occurring in the decomposition of Y into eigenvectors. Therefore, it follows that

$$\lim_{t\to\infty} \frac{\|\,\mathrm{Ad}\big(\exp(tX_0)\big)\psi(Y)\|}{\|\,\mathrm{Ad}\big(\exp(tX_0)\big)Y\|} = 0 \qquad (Y \in \overline{\mathfrak{n}}_{\mathcal{C},x} \setminus \{0\})\,. \tag{5.9}$$

For $\alpha \in \Sigma \cup \{0\}$ let p_α be the projection onto $\mathfrak{g}^\alpha$ with respect to the root space decomposition. Here $\mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}$. Let $\alpha, \beta \in \Sigma \cup \{0\}$. It follows from (5.9) that $p_\beta \circ \psi \circ p_\alpha \neq 0$ implies that $\alpha(X_0) - \beta(X_0) > 0$. Since this holds for every $X_0 \in \mathcal{C}$, it follows that

$$\psi = \sum_{\substack{\alpha,\beta\in\Sigma\cup\{0\}\\ (\alpha-\beta)|_{\mathcal{C}}>0}} p_\beta \circ \psi \circ p_\alpha\,.$$

Now

$$\mathrm{Ad}\big(\exp(tX)\big) \circ \psi \circ \mathrm{Ad}\big(\exp(-tX)\big) = \sum_{\substack{\alpha,\beta\in\Sigma\cup\{0\}\\ (\alpha-\beta)|_{\mathcal{C}}>0}} e^{t\big(\beta(X)-\alpha(X)\big)} p_\beta \circ \psi \circ p_\alpha$$

If $X \in \overline{\mathcal{C}}$, then $(\alpha-\beta)\big|_{\mathcal{C}} > 0$ implies that $\alpha(X) \geq \beta(X)$. Therefore,

$$\lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big) \circ \psi \circ \mathrm{Ad}\big(\exp(-tX)\big) = \sum_{\substack{\alpha,\beta\in\Sigma\cup\{0\}\\ (\alpha-\beta)|_{\mathcal{C}}>0\\ \alpha(X)=\beta(X)}} p_\beta \circ \psi \circ p_\alpha\,.$$

The first claim in the lemma now follows with

$$\psi_X = \sum_{\substack{\alpha,\beta\in\Sigma\cup\{0\}\\ (\alpha-\beta)|_{\mathcal{C}}>0\\ \alpha(X)=\beta(X)}} p_\beta \circ \psi \circ p_\alpha\,. \tag{5.10}$$

If $X \in \mathcal{C}$ then the sum in (5.10) is over the empty set, and hence $\psi_X = 0$. This proves the second assertion in the lemma. $\square$

It follows from (5.8) and the inverse function theorem that for sufficiently small neighborhoods V_1 of 0 in $\overline{\mathfrak{n}}_{\mathcal{C},x}$ and V_2 of 0 in $\overline{\mathfrak{n}}_{\mathcal{C}}^x$, the map

$$\Phi : V_1 \times V_2 \to P\backslash G; \qquad (Y_1, Y_2) \mapsto P\exp(Y_2)\exp\big(Y_1 + \psi(Y_1)\big)x$$

is a diffeomorphism onto an open neighborhood of $[x]$. Moreover,

$$V_1 \ni Y \mapsto \Phi(Y, 0)$$

is a diffeomorphism onto a submanifold of $P\backslash G$ contained in $P\backslash PxH$. Because the dimension of the image equals the dimension of $P\backslash PxH$, it in fact covers an open neighborhood of $[x]$ in $P\backslash PxH$.

We view $\pi^\infty_{-\lambda,\sigma^\vee}$ and $C^\infty(V_1 \times V_2, V_\sigma)$ as spaces of smooth sections of vector-bundles and write Φ^* for the pull-back along Φ, i.e., Φ^* is the map $\pi^\infty_{-\lambda,\sigma^\vee} \to C^\infty(V_1 \times V_2, V_\sigma^*)$ given by $\Phi^* v = v \circ \Phi$. This map has a continuous extension to a map

$$\Phi^* : \pi^{-\infty}_{\lambda,\sigma} \to \mathcal{D}'(V_1 \times V_2) \otimes V_\sigma^*.$$

Similarly we have a pull-back map $\pi^\infty_{\lambda,\sigma} \to C^\infty(V_1 \times V_2, V_\sigma)$ which we also denote by Φ^*. We note that there exists a strictly positive smooth function J on $V_1 \times V_2$ such that

$$\varphi(\phi) = \Phi^*\varphi(J\Phi^*\phi) \tag{5.11}$$

for every $\varphi \in \pi^{-\infty}_{\lambda,\sigma}$ and $\phi \in \pi^\infty_{\lambda,\sigma}$ with $\operatorname{supp}\phi \subseteq \Phi(V_1 \times V_2)$.

Let $n = \dim(V_2)$ and let $e_1, \ldots, e_n$ a basis of $\overline{\mathfrak{n}}_{\mathcal{C}}^x$ of joint eigenvectors for the action of $\mathrm{ad}(\mathfrak{a})$. We write ∂_i for the partial derivative in the direction e_i, and whenever μ is an n-dimensional multi-index we write ∂^μ for $\partial_1^{\mu_1}\ldots\partial_n^{\mu_n}$.

Now $\Phi^*\eta$ is a V_σ^*-valued distribution on $V_1 \times V_2$. From the condition (5.2) on the support of η it follows that the support of $\Phi^*\eta$ is contained in $V_1 \times \{0\}$. It follows from [39, p. 102] that there exist a minimal $k \in \mathbb{N}$ and for every multi-index μ with $|\mu| \leq k$ a V_σ^*-valued distribution η_μ on V_1 such that

$$\Phi^*\eta = \sum_{|\mu|\leq k} \eta_\mu \otimes \partial^\mu\delta\,. \tag{5.12}$$

Here δ is the Dirac delta distribution at 0 on $\overline{\mathfrak{n}}_{\mathcal{C}}^x$. Note that this decomposition of $\Phi^*\eta$ is unique.

Lemma 5.5.

(i) For each multi-index μ, the distribution η_μ is given by a real analytic function

$$f_\mu : V_1 \to V_\sigma^*,$$

i.e. $\eta_\mu = f_\mu \, dY_1$ where dY_1 is the Lebesgue measure on V_1.

(ii) For each $Y_1 \in V_1$ there exists a μ of length $|\mu| = k$ so that $f_\mu(Y_1) \neq 0$.

Proof. In the first part of the proof we follow the analysis of Bruhat as it is described in [42, Section 5.2.3]. For $h \in H$ we write $U_h = \Phi^{-1}\big(\Phi(V_1 \times V_2)h^{-1}\big)$ and define the real analytic map

$$\rho_h : U_h \to V_1 \times V_2; \qquad v \mapsto \Phi^{-1}\big(\Phi(v)h\big)\,.$$

Note that ρ_h maps $U_h \cap (V_1 \times \{0\})$ to $V_1 \times \{0\}$. We further write

$$U_{h,1} := \{v \in V_1 : (v, 0) \in U_h\}$$

and we define the map $\xi_h : U_{h,1} \to V_1$ to be given by $\rho_h(v, 0) = (\xi_h(v), 0)$ for $v \in V_1$.

For all multi-indices μ and ν with $|\mu|, |\nu| \leq k$ there exists a real analytic function

$$\lambda_{\mu,\nu} : \{(h, v) \in H \times V_1 : v \in U_{h,1}\} \to \mathbb{R}$$

such that

$$\rho_h^*\big(\mathbf{1}_{V_1} \otimes \partial^\mu \delta\big) = \sum_{|\nu| \leq k} \lambda_{\nu,\mu}(h, \cdot\,) \otimes \partial^\nu \delta \qquad (h \in H)\,.$$

(The domain of definition of $\lambda_{\mu,\nu}$ is equal to the inverse image of $\Phi(V_1, V_2)$ under the smooth map $V_1 \times H \to P\backslash G$; $(v, h) \mapsto \Phi(v, 0)h^{-1}$, and hence it is open.) Note that pulling back along ρ_h does not increase the order of the transversal derivatives, hence $\lambda_{\nu,\mu} = 0$ whenever $|\nu| > |\mu|$. We apply this identity to (5.12) and obtain

$$\begin{aligned} \rho_h^*(\Phi^*\eta) &= \sum_{|\mu| \leq k} \sum_{|\nu| \leq |\mu|} \lambda_{\nu,\mu}(h, \cdot\,) \xi_h^* \eta_\mu \otimes \partial^\nu \delta \\ &= \sum_{|\mu| \leq k} \Big(\sum_{k \geq |\nu| \geq |\mu|} \lambda_{\mu,\nu}(h, \cdot\,) \xi_h^* \eta_\nu \Big) \otimes \partial^\mu \delta\,. \end{aligned}$$

Since η is an H-invariant functional we have $\rho_h^*(\Phi^*\eta) = \Phi^*\eta$ on U_h. Together with the uniqueness of the decomposition (5.12) this implies for each μ that

$$\eta_\mu\big|_{U_{h,1}} = \sum_{|\nu| \geq |\mu|} \lambda_{\mu,\nu}(h, \cdot\,) \xi_h^* \eta_\nu \qquad (h \in H)\,.$$

We now apply the pull-back along ξ_h to this identity with h replaced by h^{-1} and thus obtain

$$\xi_h^* \eta_\mu = \sum_{|\nu| \geq |\mu|} \lambda_{\mu,\nu}\big(h^{-1}, \xi_h(\,\cdot\,)\big) \eta_\nu\big|_{U_{h,1}}. \tag{5.13}$$

Here we used that $\xi_h^{-1}(U_{h^{-1},1}) = U_{h,1}$.

Let $n = \dim(\overline{\mathfrak{n}}_C^x)$ and let S be the set of multi-indices $\mu \in \mathbb{N}_0^n$ with $|\mu| \leq k$. We write p_μ for the projection of $(V_\sigma^*)^S$ onto the μ^{th} component and define ζ to be the $(V_\sigma^*)^S$-valued distribution on V_1 which for a multi-index μ is given by

$$p_\mu \zeta = \eta_\mu \, .$$

For $h \in H$ and $v \in U_{h,1}$, let $\Lambda(h,v) \in \operatorname{End}\big((V_\sigma^*)^S\big)$ be given by

$$p_\mu \circ \Lambda(h,v) \circ p_\nu = \lambda_{\mu,\nu}\big(h^{-1}, \xi_h(v)\big) \, .$$

Then

$$\xi_h^* \zeta = \Lambda(h,\,\cdot\,)\zeta\big|_{U_{h,1}} \, .$$

We will finish the proof of the lemma by invoking the elliptic regularity theorem to show that ζ is locally given by a real analytic $(V_\sigma^*)^S$-valued function. To this end, let D be a real analytic elliptic differential operator of order $d > 0$ on the trivial vector bundle $V_1 \times (V_\sigma^*)^S \to V_1$. (Such differential operators exist, e.g. $\Delta \otimes \mathbf{1}$ where Δ is the Laplacian on V_1 and $\mathbf{1}$ the identity operator on $(V_\sigma^*)^S$.) Let $u_1, \dots, u_l$ be a basis of $\mathcal{U}_d(\mathfrak{h})$. Since H acts transitively on $P\backslash PxH$, there exist real analytic functions $c_j : V_1 \to \operatorname{End}\big((V_\sigma^*)^S\big)$ such that for $\phi \in C^\infty\big(V_1, (V_\sigma^*)^S\big)$

$$D\phi(v) = \sum_{j=1}^{l} c_j(v) u_j (\xi_h^* \phi)(v)\big|_{h=e} \qquad (v \in V_1) \, .$$

Let $v_0 \in V_1$. Since D is elliptic of order $d > 0$, there exists a neighborhood U of v_0 such that the operator D', which for $\phi \in C^\infty\big(U, (V_\sigma^*)^S\big)$ is given by

$$D'\phi(v) = \sum_{j=1}^{l} c_j(v_0) u_j \big(\xi_h^* \phi - \Lambda(h,\,\cdot\,)\phi\big)(v)\big|_{h=e} \qquad (v \in U) \, ,$$

is a real analytic elliptic differential operator on the vector bundle $U \times (V_\sigma^*)^S \to U$. Note that $D'\zeta = 0$ on U. By the elliptic regularity theorem, there exists a real analytic function $f : U \to (V_\sigma^*)^s$ such that $\zeta = f \, dY_1$ on U. (See for example [43, Theorem IV.4.9] for the smoothness of the solutions and [17, p. 144] for the analyticity.) Since v_0 was chosen arbitrarily, it follows that f extends to an analytic function on V_1 and that $\zeta = f \, dY_1$ on V_1. Let $f_\mu = p_\mu f$. Then f_μ is real analytic and $\eta_\mu = f_\mu dY_1$. This proves (i).

By (5.13) we have for every μ of length $|\mu| = k$

$$f_\mu\big(\xi_h(Y_1)\big) = \sum_{|\nu|=k} \lambda_{\mu,\nu}\big(h^{-1}, \xi_h(Y_1)\big) f_\nu(Y_1) \qquad (h \in H, Y_1 \in U_{h,1}).$$

Let $Y_1 \in V_1$ be such that $f_\nu(Y_1) = 0$ for all ν of length $|\nu| = k$, then the right-hand side vanishes at the point Y_1 for all $h \in H$ such that $Y_1 \in U_{h,1}$. This implies that the left-hand side vanishes on an open neighborhood of Y_1. Since the f_μ are analytic, it follows that all f_μ for μ of length $|\mu| = k$ vanish on V_1. Assertion (ii) now follows from the definition of k. $\square$

Proof of Theorem 5.1. Let Φ be as before. Recall that $\pi^\infty_{\lambda,\sigma}\big(\operatorname{Im}(\Phi)\big)$ is the space of all $v\in\pi^\infty_{\lambda,\sigma}$ with compact support contained in the image $\operatorname{Im}(\Phi)$ of Φ. Let $v\in\pi^\infty_{\lambda,\sigma}\big(\operatorname{Im}(\Phi)\big)$. It follows from (5.11), (5.12) and Lemma 5.5(i) that

$$\begin{aligned}\eta(v) &= \Phi^*\eta\big(J\Phi^*(v)\big)\\ &= \sum_{|\mu|\leq k}(-1)^{|\mu|}\int_{V_1}\partial^\mu_{Y_2}\Big[J(Y_1,Y_2)f_\mu(Y_1)\Big(v\Big(\exp(Y_2)\exp\big(Y_1+\psi(Y_1)\big)x\Big)\Big)\Big]\Big|_{Y_2=0}\,dY_1\,.\end{aligned}$$

By the Leibniz rule the integrand on the right-hand side is equal to

$$\sum_{\nu\leq\mu}\binom{\mu}{\nu}\big[\partial^{\mu-\nu}_{Y_2}J(Y_1,Y_2)\big]\Big|_{Y_2=0}f_\mu(Y_1)\Big[\partial^\nu_{Y_2}v\Big(\exp(Y_2)\exp\big(Y_1+\psi(Y_1)\big)x\Big)\Big]\Big|_{Y_2=0}.$$

Note that the Jacobian J is a real analytic function. By Lemma 5.5(i) also the functions f_μ are real analytic. Let $\epsilon_1,\dots,\epsilon_m$ be a basis of $\overline{\mathfrak{n}}_{\mathcal{C},x}$ consisting of joint eigenvectors for the action of $\operatorname{ad}(\mathfrak{a})$ on $\overline{\mathfrak{n}}_{\mathcal{C},x}$. For a multi-index κ and $Y\in\overline{\mathfrak{n}}_{\mathcal{C},x}$ define $Y^\kappa\in\mathbb{R}$ in the usual manner with respect to the basis $\epsilon_1,\dots,\epsilon_m$. By shrinking V_1 and V_2 we may assume that the Taylor series of J and the f_μ are absolutely convergent on $V_1\times V_2$ and V_1, respectively. Let

$$(-1)^{|\mu|}\binom{\mu}{\nu}\big[\partial^{\mu-\nu}_{Y_2}J(Y_1,Y_2)\big]\big|_{Y_2=0}f_\mu(Y_1)=\sum_\kappa Y_1^\kappa c^\kappa_{\mu,\nu}\tag{5.14}$$

be the Taylor expansion of the function on the left-hand side. Here for every multi-index κ the coefficient $c^\kappa_{\mu,\nu}$ is an element of V^*_σ. Since the series on the right-hand side of (5.14) is absolutely convergent on V_1 and since v has compact support in $\operatorname{Im}(\Phi)$, we can apply Lebesgue's dominated convergence theorem to interchange the integral and the sums, and obtain

$$\eta(v)=\sum_{|\nu|\leq k}\sum_\kappa\int_{V_1}Y_1^\kappa C^\kappa_\nu\Big[\partial^\nu_{Y_2}v\Big(\exp(Y_2)\exp\big(Y_1+\psi(Y_1)\big)x\Big)\Big]\Big|_{Y_2=0}dY_1\,,\tag{5.15}$$

where

$$C^\kappa_\nu:=\sum_{|\mu|\leq k,\mu\geq\nu}c^\kappa_{\mu,\nu}\in V^*_\sigma\,.$$

Recall that $e_1,\dots,e_n$ is a basis of $\overline{\mathfrak{n}}^x_{\mathcal{C}}$ consisting of joint eigenvectors for the action of $\operatorname{ad}(\mathfrak{a})$ on $\overline{\mathfrak{n}}^x_{\mathcal{C}}$. For a multi-index ν, let $\omega_{2,\nu}\in-\mathbb{N}_0[\Pi]$ be the $\mathfrak{a}$-weight of $e_1^{\nu_1}\cdots e_n^{\nu_n}\in\mathcal{U}(\overline{\mathfrak{n}})$, where $\mathcal{U}(\overline{\mathfrak{n}})$ denotes the universal enveloping algebra of $\overline{\mathfrak{n}}$. Further, for a multi-index κ we define $\omega_{1,\kappa}\in-\mathbb{N}_0[\Pi]$ to be the $\mathfrak{a}$-weight of $\epsilon_1^{\kappa_1}\cdots\epsilon_m^{\kappa_n}\in\mathcal{U}(\overline{\mathfrak{n}})$. Define

$$\Xi:=\{(\nu,\kappa):C^\kappa_\nu\neq0\}\,.$$

Let $X\in\overline{\mathcal{C}}$ be fixed. The set $\{\omega_{2,\nu}(X)-\omega_{1,\kappa}(X):(\nu,\kappa)\in\Xi\}$ is discrete. Moreover, it is bounded from above as there exists only finitely many multi-indices ν of length at most k and $\omega_{1,\kappa}(X)\geq0$ for every κ. Define

$$r_X:=\max\{\omega_{2,\nu}(X)-\omega_{1,\kappa}(X):(\nu,\kappa)\in\Xi\}\tag{5.16}$$

and

$$\Xi_X := \{(\nu,\kappa) \in \Xi : \omega_{2,\nu}(X) - \omega_{1,\kappa}(X) = r_X\}\,.$$

By Lemma 5.5(ii) there exists a multi-index μ_0 of length k such that $f_{\mu_0}(0) \neq 0$. If we take $\mu = \nu = \mu_0$ then the left-hand side of (5.14) is non-zero in $Y_1 = 0$. Therefore, the coefficient $C^0_{\mu_0} = c^0_{\mu_0,\mu_0} \neq 0$, and hence $(\mu_0, 0) \in \Xi$. Since $\omega_{1,0} = 0$, we have $r_X \geq \omega_{2,\mu_0}(X) \geq 0$.

We will now specify the domain Ω that appears in the theorem. For this we first introduce a family of diffeomorphisms. For $t \in \mathbb{R}$, let $a_t := \exp(tX)$. We define

$$\Psi_t : \mathrm{Ad}(a_t)V_1 \times \mathrm{Ad}(a_t)V_2 \to P\backslash G; \quad (Y_1, Y_2) \mapsto \Phi\Big(\mathrm{Ad}(a_t^{-1})Y_1, \mathrm{Ad}(a_t^{-1})Y_2\Big)\, x^{-1}a_t^{-1}\,.$$

Observe that Ψ_t is a diffeomorphism onto its image for every $t \in \mathbb{R}$. For every $(Y_1, Y_2) \in \mathrm{Ad}(a_t)V_1 \times \mathrm{Ad}(a_t)V_2$ we have

$$\begin{aligned}\Psi_t(Y_1,Y_2) &= P\exp\Big(\mathrm{Ad}(a_t^{-1})Y_2\Big)\exp\Big(\mathrm{Ad}(a_t^{-1})Y_1 + \psi\big(\mathrm{Ad}(a_t^{-1})Y_1\big)\Big)a_t^{-1}\\ &= P\exp(Y_2)\exp\Big(Y_1 + \mathrm{Ad}(a_t)\psi\big(\mathrm{Ad}(a_t^{-1})Y_1\big)\Big)\,.\end{aligned}$$

Let $\mathcal{G}_X$ be the graph of ψ_X. Then $\mathfrak{g} = \mathfrak{p} \oplus \mathcal{G}_X \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x$, and thus there exist open neighborhoods W_1 and W_2 of 0 in $\overline{\mathfrak{n}}_{\mathcal{C},x}$ and $\overline{\mathfrak{n}}_{\mathcal{C}}^x$ respectively such that the map

$$\Psi_\infty : W_1 \times W_2 \to P\backslash G\,,$$

given by

$$\Psi_\infty(Y_1,Y_2) = P\exp(Y_2)\exp\big(Y_1 + \psi_X(Y_1)\big)\,,$$

is a diffeomorphism onto an open neighborhood of $[e]$ in $P\backslash G$. The map Ψ_∞ is a limit of the maps Ψ_t in the following sense. Since $\mathrm{Ad}(a_t)$ acts with eigenvalues larger or equal than 1 on $\overline{\mathfrak{n}}_{\mathcal{C},x}$ and $\overline{\mathfrak{n}}_{\mathcal{C}}^x$, there exist bounded open neighborhoods U_1 and U_2 of 0 in $\overline{\mathfrak{n}}_{\mathcal{C},x}$ and $\overline{\mathfrak{n}}_{\mathcal{C}}^x$, respectively, satisfying

$$\overline{U_1} \subseteq W_1 \cap \bigcap_{t\geq 0}\mathrm{Ad}(a_t)V_1 \quad\text{and}\quad \overline{U_2} \subseteq W_2 \cap \bigcap_{t\geq 0}\mathrm{Ad}(a_t)V_2\,.$$

It follows from Lemma 5.4 that

$$\lim_{t\to\infty}\Psi_t(Y_1,Y_2) = \Psi_\infty(Y_1,Y_2) \qquad \big((Y_1,Y_2) \in U_1 \times U_2\big)\,, \tag{5.17}$$

where the limit takes place in the space of smooth maps $U_1 \times U_2 \to P\backslash G$. We claim that for sufficiently large $R > 0$ there exists an open neighborhood Ω of $[e]$ in $P\backslash G$ such that

$$\Omega \subseteq \Psi_\infty(U_1 \times U_2) \cap \bigcap_{t>R}\Psi_t(U_1 \times U_2)\,. \tag{5.18}$$

Indeed, the constructive proof of the inverse function theorem (see for example Lemma 1.3 in [32]) gives a lower bound on the size of the open neighborhood of $[e] \in P\backslash G$

that is contained in $\Psi_t(U_1 \times U_2)$ in terms of the tangent map of Ψ_t at $(0,0)$. The claim therefore follows immediately from (5.17).

For $(\nu, \kappa) \in \Xi$, let $\eta_X^{\nu,\kappa} \in \pi_{\lambda,\sigma}^{-\infty}(\Omega)$ be the functional which for $v \in \pi_{\lambda,\sigma}^{\infty}(\Omega)$ is given by

$$\eta_X^{\nu,\kappa}(v) := \int_{U_1} Y_1^\kappa C_\nu^\kappa \Big[\partial_{Y_2}^\nu v\Big(\exp(Y_2) \exp\big(Y_1 + \psi_X(Y_1)\big)\Big)\Big]\Big|_{Y_2=0} dY_1 \,. \tag{5.19}$$

We claim that (5.3) holds with r_X given by (5.16) and $\eta_{X,x}$ by the sum

$$\eta_{X,x} := \sum_{(\nu,\kappa)\in\Xi_X} \eta_X^{\nu,\kappa}\,, \tag{5.20}$$

where the sum is convergent in $\pi_{\lambda,\sigma}^{-\infty}(\Omega)$ with respect to the weak-$*$-topology.

To prove the claim, let $t > R$ and consider $v \in \pi_{\lambda,\sigma}^{\infty}(\Omega)$. For every $Y_2 \in \overline{\mathfrak{n}}_{\mathcal{C}}^x$ and $Y \in \mathfrak{g}$.

$$[\pi_{\lambda,\sigma}(x^{-1}a_t^{-1})v]\big(\exp(Y_2)\exp(Y)x\big) = a_t^{-\lambda-\rho_P} v\Big(\exp\big(\operatorname{Ad}(a_t)Y_2\big)\exp\big(\operatorname{Ad}(a_t)Y\big)\Big)\,.$$

From (5.15) it follows that

$$\begin{aligned} &a_t^{\lambda+\rho_P}\eta\big(\pi_{\lambda,\sigma}(x^{-1}a_t^{-1})v\big) \\ &= \sum_{|\nu|\leq k}\sum_{\kappa}\int_{U_1} Y_1^\kappa C_\nu^\kappa\Big[\partial_{Y_2}^\nu v\Big(\exp\big(\operatorname{Ad}(a_t)Y_2\big)\exp\Big(\operatorname{Ad}(a_t)\big(Y_1+\psi(Y_1)\big)\Big)\Big)\Big]\Big|_{Y_2=0} dY_1\,. \end{aligned} \tag{5.21}$$

If $1 \leq i \leq n$ and α is the root so that $e_i \in \mathfrak{g}^\alpha$, then $\operatorname{Ad}(a_t)e_i = a_t^\alpha e_i$, and hence

$$\frac{d}{ds} v\Big(\exp\big(\operatorname{Ad}(a_t)(se_i)\big)\exp\big(\operatorname{Ad}(a_t)Y\big)\Big) = a_t^\alpha \frac{d}{ds} v\Big(\exp(se_i)\exp\big(\operatorname{Ad}(a_t)Y\big)\Big)\,.$$

Applying the previous identity repeatedly yields

$$\begin{aligned} &\partial_{Y_2}^\nu v\Big(\exp\big(\operatorname{Ad}(a_t)Y_2\big)\exp\big(\operatorname{Ad}(a_t)Y\big)\Big)\Big|_{Y_2=0} \\ &\qquad = a_t^{\omega_{2,\nu}}\partial_{Y_2}^\nu v\Big(\exp(Y_2)\exp\big(\operatorname{Ad}(a_t)Y\big)\Big)\Big|_{Y_2=0}\,. \end{aligned}$$

Combining this identity with (5.21), we obtain

$$\begin{aligned} &a_t^{\lambda+\rho_P}\eta\big(\pi_{\lambda,\sigma}(x^{-1}a_t^{-1})v\big) \\ &= \sum_{|\nu|\leq k}\sum_{\kappa} a_t^{\omega_{2,\nu}}\int_{U_1} Y_1^\kappa C_\nu^\kappa\Big[\partial_{Y_2}^\nu v\Big(\exp(Y_2)\exp\Big(\operatorname{Ad}(a_t)\big(Y_1+\psi(Y_1)\big)\Big)\Big)\Big]\Big|_{Y_2=0} dY_1\,. \end{aligned}$$

By definition of $\omega_{1,\kappa}$

$$\big(\operatorname{Ad}(a_t^{-1})Y_1\big)^\kappa = a_t^{-\omega_{1,\kappa}}Y_1^\kappa \qquad (Y_1 \in \overline{\mathfrak{n}}_{\mathcal{C},x})\,.$$

We now perform a substitution of variables and obtain that

$$\int_{U_1} Y_1^\kappa C_\nu^\kappa \Big[\partial_{Y_2}^\nu v\Big(\exp(Y_2)\exp\Big(\operatorname{Ad}(a_t)\big(Y_1+\psi(Y_1)\big)\Big)\Big)\Big]\Big|_{Y_2=0} dY_1$$
$$= a_t^{-2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})-\omega_{1,\kappa}} \int_{\operatorname{Ad}(a_t)U_1} Y_1^\kappa C_\nu^\kappa\big[v_{\nu,t}(Y_1)\big]\, dY_1\,,$$

where

$$v_{\nu,t}(Y_1) := \partial_{Y_2}^\nu v\Big(\exp(Y_2)\exp\Big(Y_1+\operatorname{Ad}(a_t)\psi\big(\operatorname{Ad}(a_t^{-1})Y_1\big)\Big)\Big)\Big|_{Y_2=0}$$

for $Y_1 \in \operatorname{Ad}(a_t)U_1$. It follows from (5.18) and the fact that v is supported in Ω, that

$$\operatorname{supp}(v_{\nu,t}) \subseteq U_1.$$

Now

$$e^{t\big(\lambda(X)+\rho_P(X)+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})(X)-r_X\big)}\eta\big(\pi_{\lambda,\sigma}(x^{-1}a_t^{-1})v\big)$$
$$= \sum_{|\nu|\le k}\sum_\kappa e^{t\big(\omega_{2,\nu}(X)-\omega_{1,\kappa}(X)-r_X\big)} \int_{U_1} Y_1^\kappa C_\nu^\kappa[v_{\nu,t}(Y_1)]\, dY_1\,. \tag{5.22}$$

Since U_1 is bounded, the support of the functions $v_{\nu,t}$ is bounded uniformly in $t>0$. Therefore, $v_{\nu,t}$ converges for $t\to\infty$ in the space $C_c^\infty(U_1,V_\sigma)$ to the function

$$Y_1 \mapsto \partial_{Y_2}^\nu v\Big(\exp(Y_2)\exp\big(Y_1+\psi_X(Y_1)\big)\Big)\Big|_{Y_2=0},$$

and thus we obtain,

$$\lim_{t\to\infty}\int_{U_1} Y_1^\kappa C_\nu^\kappa[v_{\nu,t}(Y_1)]\, dY_1$$
$$= \int_{U_1} Y_1^\kappa C_\nu^\kappa\Big[\partial_{Y_2}^\nu v\Big(\exp(Y_2)\exp\big(Y_1+\psi_X(Y_1)\big)\Big)\Big]\Big|_{Y_2=0} dY_1 = \eta_X^{\nu,\kappa}(v)\,.$$

For the last equality we used (5.19).

Let $r = \sup_{Y_1\in U_1}\|Y_1\|$. Since U_1 is bounded, we have $r<\infty$. Moreover, since $\overline{U_1}\subseteq V_1$, we also have that r is strictly smaller than the convergency radius of the Taylor series in (5.14), and hence

$$\sum_\kappa r^{|\kappa|}\|C_\nu^\kappa\| < \infty\,. \tag{5.23}$$

As $v_{\nu,t}$ is bounded uniformly in $t>0$ and ν, and $e^{t\big(\omega_{2,\nu}(X)-\omega_{1,\kappa}(X)-r_X\big)} \le 1$ for all $t>0$ and $(\nu,\kappa)\in\Xi$, it follows from (5.23) that the series in (5.22) is absolutely convergent uniformly in $t>0$. Therefore,

$$\lim_{t\to\infty} e^{t\big(\lambda(X)+\rho_P(X)+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})(X)-r_X\big)}\eta\big(\pi_{\lambda,\sigma}(x^{-1}a_t^{-1})v\big)$$
$$= \sum_{|\nu|\le k}\sum_\kappa \lim_{t\to\infty}\Big(e^{t\big(\omega_{2,\nu}(X)-\omega_{1,\kappa}(X)-r_X\big)}\int_{U_1} Y_1^\kappa C_\nu^\kappa[v_{\nu,t}(Y_1)]\, dY_1\Big)$$
$$= \sum_{(\nu,\kappa)\in\Xi_X} \eta_X^{\nu,\kappa}(v)\,.$$

This proves the claim that (5.3) holds with r_X given by (5.16) and $\eta_{X,x}$ by the convergent sum (5.20).

We claim that $\eta_{X,x} \neq 0$. Let $v \in \pi^\infty_{\lambda,\sigma}(\Omega)$. Since v is compactly supported, it follows from (5.23) and Lebesgue's dominated convergence theorem, that we may interchange the sum and the integral, so that

$$\begin{aligned}\eta_{X,x}(v) &= \sum_{(\nu,\kappa)\in\Xi_X} \int_{U_1} Y_1^\kappa C_\nu^\kappa \Big[\partial^\nu_{Y_2} v\Big(\exp(Y_2)\exp\big(Y_1+\psi_X(Y_1)\big)\Big)\Big]\Big|_{Y_2=0}\, dY_1\\ &= \sum_{|\nu|\leq k} \int_{U_1} F_{\nu,X}(Y_1)\Big[\partial^\nu_{Y_2} v\Big(\exp(Y_2)\exp\big(Y_1+\psi_X(Y_1)\big)\Big)\Big]\Big|_{Y_2=0}\, dY_1\,, \end{aligned}\tag{5.24}$$

where $F_{\nu,X} : U_1 \to V_\sigma^*$ is given by the absolutely convergent series

$$F_{\nu,X}(Y_1) := \sum_{\{\kappa:(\nu,\kappa)\in\Xi_X\}} Y_1^\kappa C_\nu^\kappa\,. \tag{5.25}$$

If $\{\kappa : (\nu,\kappa) \in \Xi_X\} \neq \emptyset$, then $F_{\nu,X}$ is not identically equal to 0 since it is given by an absolutely convergent power series with at least one non-zero coefficient. Since $\Xi_X \neq \emptyset$ there exists at least one multi-index ν_0 so that $F_{\nu_0,X}$ is not identically equal to 0.

Let $v_\sigma \in V_\sigma$ and let $\phi_1 \in C_c^\infty(U_1)$ and $\phi_2 \in C_c^\infty(U_2)$. We now take v to be the element of $\pi^\infty_{\lambda,\sigma}(\Omega)$ that is determined by

$$v\Big(\exp(Y_2)\exp\big(Y_1+\psi_X(Y_1)\big)\Big) = \phi_1(Y_1)\phi_2(Y_2)v_\sigma \qquad (Y_1\in U_1, Y_2\in U_2)\,.$$

(Recall that Ψ_∞ is a diffeomorphism, and hence v is well defined.) Then

$$\eta_{X,x}(v) = \sum_{|\nu|<k} \partial^\nu\phi_2(0)\int_{U_1}\big(F_{\nu,X}(Y_1)(v_\sigma)\big)\phi_1(Y_1)\,dY_1\,.$$

We assume that v_σ, ϕ_1 and ϕ_2 satisfy

(a) $\partial^{\nu_0}\phi_2(0) = 1$,

(b) If $\nu \neq \nu_0$, then $\partial^\nu\phi_2(0) = 0$,

(c) $Y_1 \mapsto F_{\nu_0,X}(Y_1)(v_\sigma)$ is not identically equal to 0,

(d) $\displaystyle\int_{U_1}\big(F_{\nu,X}(Y_1)(v_\sigma)\big)\phi_1(Y_1)\,dY_1 = 1.$

Under these assumptions we have $\eta_{X,x}(v) = 1$, and hence $\eta_{X,x} \neq 0$.

We move on to show (5.4) for $X \in \mathcal{C}$. Let $\alpha \in \Sigma \cup \{0\}$ and let $Y \in (\mathfrak{h}_{\mathcal{C},x} \cap \mathfrak{g}^\alpha) \setminus \{0\}$. For $Y' \in \mathfrak{g}$, we write

$$\mathrm{Ad}(x)Y' = \sum_{\beta\in\Sigma\cup\{0\}} Y'_{x,\beta}\,,$$

with $Y'_{x,\beta} \in \mathfrak{g}^\beta$ for every $\beta \in \Sigma \cup \{0\}$. In view of (4.1) in Lemma 4.1 there exists an element $Y' \in \mathfrak{h}$ such that $Y'_{x,\alpha} = Y$ and α is the unique element of $\Sigma \cup \{0\}$ satisfying

$$\alpha(X) = \max\{\beta(X) : \beta \in \Sigma \cup \{0\}, Y'_{x,\beta} \neq 0\}\,.$$

For every $v \in \pi^\infty_{\lambda,\sigma}$ we have

$$\begin{aligned}
&e^{-t\alpha(X)} e^{t(\lambda+\rho_P+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})-\omega_X)(X)} \pi^\vee_{\lambda,\sigma}\big(\exp(tX)x\big)\pi^\vee_{\lambda,\sigma}(Y')\eta \\
&\quad = \sum_{\beta\in\Sigma\cup\{0\}} e^{t(\beta-\alpha)(X)} \pi^\vee_{\lambda,\sigma}(Y'_{x,\beta})\big[e^{t(\lambda+\rho_P+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})-\omega_X)(X)} \pi^\vee_{\lambda,\sigma}\big(\exp(tX)x\big)\eta\big] \\
&\quad \longrightarrow \pi^\vee_{\lambda,\sigma}(Y'_{x,\alpha})\eta_{X,x} = \pi^\vee_{\lambda,\sigma}(Y)\eta_{X,x} \qquad (t \to \infty)\,.
\end{aligned}$$

Here the limit is taken with respect to the weak-$*$ topology. Since $Y' \in \mathfrak{h}$, we have $\pi^\vee_{\lambda,\sigma}(Y')\eta = 0$, hence $\pi^\vee_{\lambda,\sigma}(Y)\eta_{X,x} = 0$. This proves (5.4).

For $X \in \mathcal{C}$, we have $\psi_X = 0$ by Lemma 5.4. Let $\overline{N}_{\mathcal{C},x}$ be the connected subgroup of G with Lie algebra $\overline{\mathfrak{n}}_{\mathcal{C},x}$ and let $d\overline{n}$ denote the Haar measure on $\overline{N}_{\mathcal{C},x}$. Then the expression (5.24) for $\eta_{X,x}$ simplifies to

$$\begin{aligned}
\eta_{X,x}(v) &= \sum_{|\nu|\leq k} \int_{\overline{\mathfrak{n}}_{\mathcal{C},x}} F_{\nu,X}(Y_1)\Big[\partial^\nu_{Y_2} v\Big(\exp(Y_2)\exp(Y_1)\Big)\Big]\Big|_{Y_2=0}\, dY_1 \\
&= \sum_{|\nu|\leq k} \int_{\overline{N}_{\mathcal{C},x}} F_{\nu,X}\big(\log(\overline{n})\big)\Big[\partial^\nu_{Y_2} v\Big(\exp(Y_2)\overline{n}\Big)\Big]\Big|_{Y_2=0}\, d\overline{n} \qquad \big(v \in \pi^\infty_{\lambda,\sigma}(\Omega)\big)\,.
\end{aligned}$$

Since $\overline{\mathfrak{n}}_{\mathcal{C},x} \subseteq \mathfrak{h}_{\mathcal{C},x}$, it follows from (5.4) that $\pi^\vee_{\lambda,\sigma}(\overline{\mathfrak{n}}_{\mathcal{C},x})\eta_{X,x} = \{0\}$. Because of the invariance of the Haar measure on $\overline{N}_{\mathcal{C},x}$, this implies that $F_{\nu,X}$ is constant for every ν. Therefore, only terms with $\kappa = 0$ can contribute to $F_{\nu,X}$ in the series in (5.25). In particular it follows that $(\nu,\kappa) \in \Xi_X$ implies that $\kappa = 0$. Moreover, r_X in (5.16) is equal to $\omega_{2,\mu_0}(X)$ for some multi-index μ_0 with the property that $f_{\mu_0}(0) \neq 0$ and $f_\mu(0) = 0$ for every $\mu > \mu_0$. Let $\omega := \omega_{2,\mu_0} \in -\mathbb{N}_0[\Pi]$. Then Ξ_X consists of pairs $(\nu, 0)$ with $\omega_{2,\nu}(X) = \omega(X)$. The formula for $\eta_{X,x}$ simplifies further to

$$\eta_{X,x}(v) = \sum_{\substack{|\mu|\leq k \\ \omega_{2,\mu}(X)=\omega(X)}} \int_{\overline{N}_{\mathcal{C},x}} c_\mu \Big[\partial^\mu_{Y_2} v\big(\exp(Y_2)\overline{n}\big)\Big]\Big|_{Y_2=0}\, d\overline{n}, \qquad \big(v \in \pi^\infty_{\lambda,\sigma}(\Omega)\big)\,, \tag{5.26}$$

with $c_\mu := (-1)^{|\mu|} J(0,0) f_\mu(0) \in V^*_\sigma \setminus \{0\}$.

If we further impose on $X \in \mathcal{C}$ the condition that $\chi(X) \neq \chi'(X)$ whenever $\chi, \chi' \in -\mathbb{N}_0[\Pi]$ are two different elements, each of which being a sum of at most k roots in $-\Sigma$, then $\omega_{2,\mu}(X) = \omega(X)$ if and only if $\omega_{2,\mu} = \omega$. Equation (5.5) then follows directly from (5.26).

It remains to prove that (5.6) implies (5.7). Let $Y \in \mathfrak{a}^E_x$. Then $\mathrm{Ad}(x^{-1})(Y + \mathfrak{n}) \cap \widehat{\mathfrak{h}}$ is non-empty, see (4.1) in Lemma 4.1. Let $Y' \in \mathrm{Ad}(x^{-1})(Y + \mathfrak{n}) \cap \widehat{\mathfrak{h}}$. Then for every $v \in \pi^\infty_{\lambda,\sigma}$

$$\begin{aligned}
&e^{t(\lambda+\rho_P+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})-\omega_X)(X)} \pi^\vee_{\lambda,\sigma}\big(\exp(tX)x\big)\pi^\vee_{\lambda,\sigma}(Y')\eta \\
&\quad = e^{t(\lambda+\rho_P+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})-\omega_X)(X)} \pi^\vee_{\lambda,\sigma}\big(\mathrm{Ad}(\exp(tX)x)Y'\big)\pi^\vee_{\lambda,\sigma}\big(\exp(tX)x\big)\eta
\end{aligned}$$

converges to $\pi^\vee_{\lambda,\sigma}(Y)\eta_{X,x}$ for $t \to \infty$. Moreover, it follows from (5.6) that it also converges to $-\chi(Y')\eta_{X,x}$ for $t \to \infty$. Here again the limits are taken with respect to the weak-$*$ topology. It thus follows that

$$\pi^\vee_{\lambda,\sigma}(Y)\eta_{X,x} = -\chi(Y')\eta_{X,x}\,.$$

Now (5.7) follows as $\chi(Y') = \chi_x(Y)$. □

6 Integrality and negativity conditions

Let us denote by $(\cdot,\cdot)$ the Euclidean structure on $\mathfrak{a}^*$. For $\alpha \in \mathfrak{a}^*\backslash\{0\}$ we define $\alpha^\vee \in \mathfrak{a}$ by $\alpha^\vee := 2\frac{(\cdot,\alpha)}{(\alpha,\alpha)} \in (\mathfrak{a}^*)^* = \mathfrak{a}$. Recall that if $\alpha \in \Sigma$ then $\alpha^\vee$ is called the co-root of α and $\Sigma^\vee := \{\alpha^\vee \mid \alpha \in \Sigma\}$ is a root system on $\mathfrak{a}$, called the dual root system.

For a connected component $\mathcal{C}$ of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$ and $x \in G$, define $\mathfrak{l}_{\mathcal{C},x} := \mathfrak{h}_{\mathcal{C},x} \cap \theta\mathfrak{h}_{\mathcal{C},x}$. Note that $\mathfrak{l}_{\mathcal{C},x}$ is a reductive $\mathfrak{a}$-stable subalgebra of $\mathfrak{h}_{\mathcal{C},x}$ and $\mathfrak{l}_{\mathcal{C},x} \cap \mathfrak{a} = \mathfrak{a}_x$. Moreover, it follows from (4.5) that $\mathfrak{l}_{\mathcal{C},manxh} = \mathrm{Ad}(m)\mathfrak{l}_{\mathcal{C},x}$ for $m \in M$, $a \in A$, $n \in N$ and $h \in H$. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we set

$$\Sigma(\lambda) := \{\alpha \in \Sigma | \lambda(\alpha^\vee) \in \mathbb{Z}\}\,.$$

Lemma 6.1. *Let (V,η) be a spherical pair belonging to the twisted discrete series and assume that there is a quotient $\pi_{\lambda,\sigma} \twoheadrightarrow V$. Consider η as an H-fixed element of $\pi^{-\infty}_{\lambda,\sigma}$ and let $x \in G$ satisfy the support condition (5.2). (See Remark 5.2.) Then the following assertions hold.*

(i) $\lambda\big|_{\mathfrak{a}_x} \in (-\rho_P + \mathbb{Z}[\Pi])\big|_{\mathfrak{a}_x}$.

(ii) Let $\chi \in (\widehat{\mathfrak{h}/\mathfrak{h}})^*_{\mathbb{C}}$ be normalized unitary. If (V,η) belongs to the χ-twisted discrete series, then

$$\lambda\big|_{\mathfrak{a}_x^E} \in \frac{1}{2}\mathbb{Z}[\Pi]\big|_{\mathfrak{a}_x^E} + i\,\mathrm{Im}\,\chi_x\,.$$

Let $\mathcal{C}$ be a connected component of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$. Then the following hold.

(iii) $\Sigma(\mathfrak{a}, \mathfrak{l}_{\mathcal{C},x}) \subseteq \Sigma(\lambda)$.

(iv) $\mathrm{Re}\,\lambda(X) \leq 2\rho(\mathfrak{l}_{\mathcal{C},x} \cap \mathfrak{n})(X)$ *for all* $X \in -\overline{\mathcal{C}} \subseteq \mathfrak{a}^+$. *The inequality is strict for* $X \in -\overline{\mathcal{C}} \setminus \mathfrak{a}_x^E$.

Proof. Assertion (i) is immediate from Corollary 5.3(ii) for any choice of $\mathcal{C}$. We move on to (ii). By (3.2) we have

$$|\det \mathrm{Ad}(ha)\big|_{\mathfrak{g}/\mathfrak{h}}| = a^{2\rho_Q} \qquad (h \in H, a \in A_{Z,E})\,.$$

We thus see that $\mathrm{Re}\,\chi(Y') = -\frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}(Y')\big|_{\mathfrak{g}/\mathfrak{h}}$ for every $Y' \in \widehat{\mathfrak{h}}$. Let $Y \in \mathfrak{a}_x^E$. It follows from (4.1) in Lemma 4.1 that there exists an element Y' in $\mathrm{Ad}(x^{-1})(Y + \mathfrak{n}) \cap \widehat{\mathfrak{h}}$. Now

$$\mathrm{Re}\,\chi_x(Y) = \mathrm{Re}\,\chi(Y') = -\frac{1}{2}\,\mathrm{tr}\,\mathrm{ad}(Y')\big|_{\mathfrak{g}/\mathfrak{h}} \in \frac{1}{2}\mathbb{Z}\mathrm{spec}\big(\mathrm{ad}(Y')\big)\,.$$

The eigenvalues of $\mathrm{ad}(Y')$ are equal to the eigenvalues of $\mathrm{ad}(Y)$. Therefore,

$$\operatorname{Re}\chi_x \in \frac{1}{2}\mathbb{Z}[\Pi]\big|_{\mathfrak{a}_x^E}.$$

Since (V,η) is χ-twisted, assertion (ii) now follows from Corollary 5.3(iii) for any choice of $\mathcal{C}$.

Assertion (iii) is a consequence of (i) since $\mathfrak{l}_{\mathcal{C},x} \cap \mathfrak{a} = \mathfrak{a}_x$ and hence $\alpha^\vee \in \mathfrak{a}_x$ for all $\alpha \in \Sigma(\mathfrak{a}, \mathfrak{l}_{\mathcal{C},x})$.

Moving on to (iv) we first observe that if (V,η) is a spherical pair of the twisted discrete series and $\pi_{\lambda,\sigma} \twoheadrightarrow V$, then Corollary 5.3 (i) combined with the bound (4.9) and Proposition 4.4 results for $X \in -\overline{\mathcal{C}} \subseteq \mathfrak{a}^+$ in the inequality

$$\big(-\operatorname{Re}\lambda - \rho_P - 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})\big)(-X) + r_{-X} \leq -\rho(\mathfrak{h}_{\mathcal{C},x})(-X).$$

Hence

$$(-\operatorname{Re}\lambda)(X) \geq \big(\rho_P + 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x}) - \rho(\mathfrak{h}_{\mathcal{C},x})\big)(X) \tag{6.1}$$

for all $X \in -\overline{\mathcal{C}}$. If $X \in -\overline{\mathcal{C}} \setminus \mathfrak{a}_x^E$, then instead of (4.9) we may use (4.10) in conjunction with Proposition 4.5 and conclude that in that case the inequality is strict.

Let $\mathfrak{v}_{\mathcal{C},x}$ be an $\mathfrak{a}$-stable complement of $\mathfrak{l}_{\mathcal{C},x}$ in $\mathfrak{h}_{\mathcal{C},x}$. Note that

$$2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x}) - \rho(\mathfrak{h}_{\mathcal{C},x}) = -2\rho(\mathfrak{l}_{\mathcal{C},x} \cap \mathfrak{n}) + \rho(\mathfrak{v}_{\mathcal{C},x} \cap \overline{\mathfrak{n}}) - \rho(\mathfrak{v}_{\mathcal{C},x} \cap \mathfrak{n}).$$

Since $\mathfrak{v}_{\mathcal{C},x} \cap \theta(\mathfrak{v}_{\mathcal{C},x}) = 0$, it follows that

$$\rho_P + 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x}) - \rho(\mathfrak{h}_{\mathcal{C},x}) \in -2\rho(\mathfrak{l}_{\mathcal{C},x} \cap \mathfrak{n}) + \frac{1}{2}\mathbb{N}_0[\Sigma^+].$$

Now (iv) follows from (6.1). ☐

Corollary 6.2.

(i) Let $\chi \in \widehat{(\mathfrak{h}/\mathfrak{h})}^*_{\mathbb{C}}$ be normalized unitary. There exists a finite set S_χ of pairs $(\mathfrak{b},\nu)$, where $\mathfrak{b}$ is a subspace of $\mathfrak{a}$ and $\nu \in \mathfrak{b}^*$, with the following property. If (V,η) is a spherical pair belonging to the χ-twisted discrete series of representations, and there is a quotient $\pi_{\lambda,\sigma} \twoheadrightarrow V$, then there exists an $\omega \in \operatorname{span}_{\mathbb{R}}\big(\Sigma(\lambda)\big)$ and a pair $(\mathfrak{b},\nu) \in S_\chi$ such that

$$\begin{aligned}
&\lambda\big|_{\mathfrak{b}} \in \frac{1}{2}\mathbb{Z}[\Pi]\big|_{\mathfrak{b}} + i\nu,\\
&\operatorname{Re}\lambda(X) \leq \omega(X) \qquad (X \in \mathfrak{a}^+),\\
&\operatorname{Re}\lambda(X) < \omega(X) \qquad (X \in \mathfrak{a}^+ \setminus \mathfrak{b}).
\end{aligned}$$

(ii) If (V,η) is a spherical pair belonging to the discrete series of representations, and there is a quotient $\pi_{\lambda,\sigma} \twoheadrightarrow V$, then there exists an $\omega \in \operatorname{span}_{\mathbb{R}}\big(\Sigma(\lambda)\big)$ and a subspace $\mathfrak{b}$ of $\mathfrak{a}$ such that

$$\begin{aligned}
&\lambda\big|_{\mathfrak{b}} \in (-\rho_P + \mathbb{Z}[\Pi])\big|_{\mathfrak{b}} \subseteq \frac{1}{2}\mathbb{Z}[\Pi]\big|_{\mathfrak{b}},\\
&\operatorname{Re}\lambda(X) \leq \omega(X) \qquad (X \in \mathfrak{a}^+),\\
&\operatorname{Re}\lambda(X) < \omega(X) \qquad (X \in \mathfrak{a}^+ \setminus \mathfrak{b}).
\end{aligned}$$

Proof. Ad (i): Let S_χ be the set of pairs $(\mathfrak{a}_x^E, \chi_x)$ where x runs over a set of representatives in G of H-orbits in $P\backslash G$. Consider η as an H-fixed element of $\pi_{\lambda,\sigma}^{-\infty}$. Then there exists an H-orbit in $P\backslash G$ so that the support condition (5.2) is satisfied. See Remark 5.2. Let $x \in G$ be the representative of the orbit. The assertions now follow from (ii), (iii) and (iv) in Lemma 6.1 with $\omega = \sum_{\mathcal{C}} 2\rho(\mathfrak{l}_{\mathcal{C},x} \cap \mathfrak{n})$, $\mathfrak{b} = \mathfrak{a}_x^E$ and $\nu = \operatorname{Im} \chi_x$.
Ad (ii): If V belongs to a discrete series representation, then $\widehat{\mathfrak{h}} = \mathfrak{h}$ by Lemma 3.3, and therefore $\mathfrak{a}_x^E = \mathfrak{a}_x$. We set $\mathfrak{b} = \mathfrak{a}_x$ and use (i) in Lemma 6.1 instead of (ii). $\square$

7 Negativity versus integrality in root systems

In this section we develop some general theory which is independent of the results in previous sections.

7.1 Equivalence relations

Let Σ be a (possibly non-reduced) root system spanning the Euclidean space $\mathfrak{a}^*$. We denote by W the corresponding Weyl group. Let $\Pi \subseteq \Sigma$ be a basis, Σ^+ the corresponding positive system and $C \subseteq \mathfrak{a} = (\mathfrak{a}^*)^*$ be the closure of the corresponding positive Weyl chamber, i.e.

$$C = \{x \in \mathfrak{a} \mid (\forall \alpha \in \Pi)\ \alpha(x) \geq 0\}\,.$$

Further we use the notation $C^\times = C\backslash\{0\}$.

We define an equivalence relation on $\mathfrak{a}_{\mathbb{C}}^*$ by $\lambda \sim \mu$ provided that μ is obtained from λ via a sequence

$$\lambda = \mu_0, \mu_1, \dots, \mu_l = \mu\,,$$

where for all i:

(a) $\mu_{i+1} = s_i(\mu_i)$ with $s_i = s_{\alpha_i}$ the simple reflection associated to $\alpha_i \in \Pi$,

(b) $\mu_i(\alpha_i^\vee) \notin \mathbb{Z}$.

The equivalence class of λ is denoted by $[\lambda]$.

A *root subsystem* Σ^0 of the root system Σ is a subset of Σ that satisfies:

(a) Σ^0 is a root system in the subspace it spans,

(b) if α, β are in Σ^0, and $\gamma = \alpha + \beta \in \Sigma$, then $\gamma \in \Sigma^0$.

A root subsystem $\Sigma^0 \subseteq \Sigma$ has a unique system of positive roots $\Sigma^{0,+}$ contained in Σ^+.

Given now $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we define

$$\begin{aligned} \Sigma(\lambda)^\vee &:= \{\alpha^\vee \in \Sigma^\vee | \lambda(\alpha^\vee) \in \mathbb{Z}\} \\ \Sigma(\lambda) &:= \{\alpha \in \Sigma | \lambda(\alpha^\vee) \in \mathbb{Z}\}\,. \end{aligned}$$

Clearly $\Sigma(\lambda)^\vee$ is a root subsystem of $\Sigma^\vee$, but observe that $\Sigma(\lambda)$ might not be a root subsystem of Σ. We call an element $\mu \in \mathfrak{a}^*$ a weight of $\Sigma(\lambda)$ if $\mu(\alpha^\vee) \in \mathbb{Z}$ for every $\alpha \in \Sigma(\lambda)$. The set of weights of $\Sigma(\lambda)$ forms a lattice in $\mathfrak{a}^*$ which contains $\operatorname{Re}(\lambda)$.

Next we define an equivalence relation on W by $u \sim_\lambda v$ provided that uC and vC are connected by a gallery of chambers $(uC = C_0, C_1, ..., C_l = vC)$ such that for each i, C_i and C_{i+1} are separated by H_{β_i} with $\beta_i \in \Sigma\backslash\Sigma(\lambda)$ an indivisible root for each i.

Let $\Sigma(\lambda)^+ = \Sigma(\lambda) \cap \Sigma^+$. We denote the closure of the corresponding positive chamber by $C(\lambda) \subseteq \mathfrak{a}$.

Lemma 7.1. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then the following assertions hold:*

(i) $C(\lambda)$ equals the union of the sets $w(C)$ where w runs over $[e]_\lambda$, the equivalence class of $e \in W$.

(ii) Let $\mu \in \mathfrak{a}_{\mathbb{C}}^*$. Then $\lambda \sim \mu$ if and only if there exists a $w \in W$ with $w^{-1} \in [e]_\lambda$ such that $\mu = w\lambda$.

Proof. We start with the proof of (i). Let D be the union of the sets $w(C)$ where w runs over $[e]_\lambda$. By definition $C(\lambda)$ is the closure of a connected component of the complement of the union of the hyperplanes H_α with $\alpha \in \Sigma(\lambda)$, namely the connected component which contains $\mathrm{int}(C)$.

Clearly $C(\lambda)$ is the closure of the union of the open chambers it contains. These are of the form $w(\mathrm{int}(C))$, where w varies over a subset of $[e]_\lambda$; indeed, the latter follows since the hyperplanes intersecting $\mathrm{int}\big(C(\lambda)\big)$ are hyperplanes of roots which are not in $\Sigma(\lambda)$. Hence $C(\lambda) \subseteq D$, since D is closed. But clearly we can not extend any further beyond $C(\lambda)$ while staying in D, since all the walls of $C(\lambda)$ are hyperplanes of roots in $\Sigma(\lambda)$. Hence the equality is clear and (i) is established.

Moving on to (ii), let $\lambda = \mu_0, \mu_1, \dots, \mu_l = \mu$ be a sequence connecting λ and $\mu = w\lambda$ such that $\mu_{i+1} = s_i(\mu_i)$, with s_i a reflection in a simple root α_i, and $\mu_i(\alpha_i^\vee) \notin \mathbb{Z}$ for all i. Let $w_0 = e$ and $w_{i+1} = s_i w_i$. Furthermore, let $\beta_i = w_i^{-1}(\alpha_i)$, so that $w_{i+1} = w_i s_{\beta_i}$. Then β_i is an indivisible root and

$$\lambda(\beta_i^\vee) = w_i(\lambda)(\alpha_i^\vee) = \mu_i(\alpha_i^\vee) \notin \mathbb{Z}\,,$$

that is, $\beta_i \in \Sigma\backslash\Sigma(\lambda)$. We may assume that $w_l = w$. Therefore, the gallery

$$C, w_1^{-1}(C), w_2^{-1}(C), \dots, w^{-1}(C)$$

yields an equivalence $w^{-1} \sim_\lambda e$. The converse is also true. If the gallery

$$(C_0 = C, C_1, \dots, C_l = w^{-1}(C))$$

defines an equivalence $e \sim_\lambda w^{-1}$, then $C_{i+1} = s_{\beta_i}(C_i)$ with $\beta_i \in \Sigma\backslash\Sigma(\lambda)$ an indivisible root for all i. Let $w_i \in W$ so that $C_i = w_i^{-1}C$, and $\mu_i := w_i(\lambda)$. Since H_{β_i} is a common face of C_i and C_{i+1} (by definition of gallery), we have $s_{\beta_i} w_i^{-1} = w_i^{-1} s_{\alpha_i}$ for some simple root $\alpha_i = w_i\beta_i \in \Pi$. Note that

$$\mu_i(\alpha_i^\vee) = w_i\lambda(\alpha_i^\vee) = \lambda(\beta_i^\vee) \notin \mathbb{Z}\,.$$

This implies that λ and $w(\lambda)$ are equivalent and finishes the proof of (ii). $\square$

7.2 Integral-negative parameters

Let us call $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *weakly integral-negative* provided that there exists a $\omega_\lambda \in \operatorname{span}_{\mathbb{R}}(\Sigma(\lambda))$ and a subspace $\mathfrak{a}_\lambda \subseteq \mathfrak{a}$ such that

$$\begin{aligned}(\operatorname{Re}\lambda - \omega_\lambda)\big|_{C} &\leq 0\,,\\ (\operatorname{Re}\lambda - \omega_\lambda)\big|_{C\setminus\mathfrak{a}_\lambda} &< 0\,.\end{aligned}$$

Further, we call $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *integral-negative* provided that there exists a $\omega_\lambda \in \operatorname{span}_{\mathbb{R}}(\Sigma(\lambda))$ and a subspace $\mathfrak{a}_\lambda \subseteq \mathfrak{a}$ such that

$$\begin{aligned}\lambda\big|_{\mathfrak{a}_\lambda} &= \operatorname{Re}\lambda\big|_{\mathfrak{a}_\lambda}\,,\\ (\operatorname{Re}\lambda - \omega_\lambda)\big|_{C} &\leq 0\,,\\ (\operatorname{Re}\lambda - \omega_\lambda)\big|_{C\setminus\mathfrak{a}_\lambda} &< 0\,.\end{aligned}$$

Finally, we call $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *strictly integral-negative* if there exists a $\omega_\lambda \in \operatorname{span}_{\mathbb{R}}(\Sigma(\lambda))$ such that

$$(\operatorname{Re}\lambda - \omega_\lambda)\big|_{C\setminus\{0\}} < 0\,.$$

Remark 7.2. These definitions are motivated by our results from the previous section. Let $\mathfrak{a} \subseteq \mathfrak{g}$ and $\Sigma(\mathfrak{g}, \mathfrak{a})$ be as introduced in Section 2, and let Σ^+ be the positive system determined by the minimal parabolic subgroup P. Let (V, η) be a spherical pair and assume that there exists a quotient morphism $\pi_{\lambda,\sigma} \twoheadrightarrow V$ for some $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\sigma \in \widehat{M}$. Then from Corollary 6.2 we derive the following.

(i) 2λ is weakly integral-negative if V belongs to the twisted discrete series for Z. In fact we may take $\mathfrak{a}_\lambda$ and ω_λ to be equal to $\mathfrak{b}$ and ω as in Corollary 6.2(i).

(ii) λ is integral-negative if V belongs to the discrete series for Z.

Remark 7.3. Sometimes more is true for parameters of the discrete series and λ is actually strictly integral-negative. This for example happens in the group case $Z = G \times G/G \simeq G$.

Let us define the *edge* of λ by

$$\mathfrak{e} := \mathfrak{e}(\lambda) := \{X \in \mathfrak{a} \mid (\forall \alpha \in \Sigma(\lambda))\ \alpha(X) = 0\}\,,$$

i.e., $\mathfrak{e}$ is the intersection of all faces of $C(\lambda)$.

Notice the orthogonal decomposition

$$\mathfrak{a} = \mathfrak{e} \oplus \operatorname{span}_{\mathbb{R}} \Sigma(\lambda)^\vee\,. \tag{7.1}$$

Theorem 7.4. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then the following assertions hold:*

(i) Suppose that $[\lambda]$ consists of weakly integral-negative parameters. Then there exists a $w \in W$ with $w^{-1} \sim_\lambda e$ such that $\mathfrak{e} \subseteq w^{-1}\mathfrak{a}_{w\lambda}$. Moreover, $\operatorname{Re}\lambda\big|_{\mathfrak{e}} = 0$. Finally, there exists an $N \in \mathbb{N}$ only depending on Σ such that $\operatorname{Re}\lambda(\alpha^\vee) \in \frac{1}{N}\mathbb{Z}$ for all $\alpha \in \Sigma$.

(ii) If $[\lambda]$ *consists of integral-negative parameters, then* $\lambda\big|_{\mathfrak{e}} = 0$. *In particular,* $\lambda = \operatorname{Re}\lambda$.

(iii) If $[\lambda]$ *consists of strictly integral-negative parameters, then* $\mathfrak{e} = \{0\}$. *In particular,* $\Sigma(\lambda)^\vee$ *has full rank.*

Proof. We start with (i). Let $\mu \in [\lambda]$, that is $\mu = w\lambda$ for some $w \in W$ with $w^{-1} \sim_\lambda e$ by Lemma 7.1(ii). Since μ is weakly integral-negative there exists a subspace $\mathfrak{a}_\mu$ of $\mathfrak{a}$ and an $\omega_\mu \in \operatorname{span}_{\mathbb{R}} \Sigma(\mu)$ such that $(\operatorname{Re}\mu - \omega_\mu)\big|_{C\setminus\mathfrak{a}_\mu} < 0$ and $(\operatorname{Re}\mu - \omega_\mu)\big|_{C} \leq 0$. The latter conditions are equivalent to $(\operatorname{Re}\lambda - w^{-1}\omega_\mu)\big|_{w^{-1}C\setminus w^{-1}\mathfrak{a}_\mu} < 0$ and $(\operatorname{Re}\lambda - w^{-1}\omega_\mu)\big|_{w^{-1}C} \leq 0$. Now define a function $f : \mathfrak{a} \to \mathbb{R}$ by

$$f(X) := \max_{w^{-1}\sim_\lambda e} w^{-1}\omega_{w\lambda}(X) \qquad (X \in \mathfrak{a})\,.$$

By Lemma 7.1(i) we have $C(\lambda) = \bigcup_{w^{-1}\sim_\lambda e} w^{-1}C$, and thus

$$\big(\operatorname{Re}\lambda - f\big)\big|_{C(\lambda)\setminus\bigcup_{w^{-1}\sim_\lambda e} w^{-1}\mathfrak{a}_{w\lambda}} < 0\,, \tag{7.2}$$

$$\big(\operatorname{Re}\lambda - f\big)\big|_{C(\lambda)} \leq 0\,. \tag{7.3}$$

Recall that $\mathfrak{e}$ is the intersection of all faces of $C(\lambda)$. Since $w\Sigma(\lambda)^\vee = \Sigma(w\lambda)^\vee$ for every $w \in W$, we have $w^{-1}\omega_{w\lambda} \in \operatorname{span}_{\mathbb{R}}\big(\Sigma(\lambda)\big)$. It follows that $w^{-1}\omega_{w\lambda}\big|_{\mathfrak{e}} = 0$ and thus $f\big|_{\mathfrak{e}} = 0$. Since $\mathfrak{e}\setminus\bigcup_{w^{-1}\sim_\lambda e} w^{-1}\mathfrak{a}_{w\lambda}$ is invariant under multiplication by -1, it follows from (7.2) that $\mathfrak{e} \subseteq \bigcup_{w^{-1}\sim_\lambda e} w^{-1}\mathfrak{a}_{w\lambda}$. Hence $\mathfrak{e} \subseteq w^{-1}\mathfrak{a}_{w\lambda}$ for some $w \in W$ with $w^{-1} \sim_\lambda e$. It now follows from (7.3) that $\operatorname{Re}\lambda\big|_{\mathfrak{e}} = 0$.

We call a root subsystem Σ' of Σ parabolic if Σ' is the intersection of Σ with a subspace. Let $\Sigma_P(\lambda) \subseteq \Sigma$ be the parabolic closure of $\Sigma(\lambda) \subseteq \Sigma$, i.e., the smallest parabolic root subsystem of Σ containing $\Sigma(\lambda)$. Then $\Sigma_P(\lambda) = \mathfrak{e}^\perp \cap \Sigma$, and $\Sigma(\lambda)^\vee \subseteq \Sigma_P(\lambda)^\vee$ is a root subsystem of maximal rank of the corresponding dual parabolic subsystem $\Sigma_P(\lambda)^\vee$ of $\Sigma^\vee$. By the above, $\operatorname{Re}(\lambda) \in \mathfrak{e}^\perp$, and by definition of $\Sigma(\lambda)$, $\operatorname{Re}(\lambda)$ is a weight of $\Sigma(\lambda)$.

Let N be the index of the root lattice of $\Sigma_P(\lambda)$ in the weight lattice of $\Sigma(\lambda)$ (which is a lattice containing the weight lattice of $\Sigma_P(\lambda)$). Then $N\operatorname{Re}(\lambda)$ is in the root lattice of $\Sigma_P(\lambda)$ and thus, a fortiori, in the root lattice of Σ. In particular, $N\operatorname{Re}(\lambda)$ is integral for Σ (i.e., as a functional on $\Sigma^\vee$).

Since there are only finitely many root subsystems of maximal rank in any given root system, and only finitely many parabolic root subsystems, we see that we can choose the bound $N \in \mathbb{N}$ independent of λ (only depending on Σ). This completes the proof of (i).

We move on to (ii). From (i) it follows that there exists a $w \in W$ with $w^{-1} \sim_\lambda e$ such that $\mathfrak{e} \subseteq w^{-1}\mathfrak{a}_{w\lambda}$. Now $\lambda(\mathfrak{e}) \subseteq \lambda(w^{-1}\mathfrak{a}_{w\lambda}) = w\lambda(\mathfrak{a}_{w\lambda}) \subseteq \mathbb{R}$. It follows that $\lambda\big|_{\mathfrak{e}}$ is real and thus $\lambda\big|_{\mathfrak{e}} = 0$ by (i). It then follows from (7.1) that $\lambda = \operatorname{Re}\lambda$.

Finally for (iii) we observe that $[\lambda]$ being strictly integral-negative implies, as above, $\operatorname{Re}\lambda(X) < f(X)$ for all $X \in C(\lambda) \setminus \{0\}$ and therefore $\operatorname{Re}\lambda\big|_{\mathfrak{e}^\times} < 0$. The latter forces $\mathfrak{e}^\times = \emptyset$, i.e., $\mathfrak{e} = \{0\}$. □

7.3 Additional results

The assertions in this subsection are of independent interest, but not needed in the remainder of this article.

Given a full rank subsystem $(\Sigma^0)^\vee$ of $\Sigma^\vee$ we note that $\mathbb{Z}[(\Sigma^0)^\vee]$ has finite index in the full co-root lattice $\mathbb{Z}[\Sigma^\vee]$ and thus

$$\mathbb{Z}[\Sigma^\vee]/\mathbb{Z}[(\Sigma^0)^\vee] \simeq \bigoplus_{j=1}^{r} \mathbb{Z}/d_j\mathbb{Z}$$

for $d_j \in \mathbb{N}$. Set $N(\Sigma^0) := \operatorname{lcm}\{d_1, \ldots, d_r\}$ and note that $N(\Sigma^0)\alpha^\vee \in \mathbb{Z}[(\Sigma^0)^\vee]$ for all $\alpha \in \Sigma$.

The following corollary is particularly relevant for the group case $Z = G \times G/G$. See Remark 7.3.

Corollary 7.5. *Let $\lambda \in \mathfrak{a}_\mathbb{C}^*$ be such that $[\lambda]$ consist of strictly integral-negative parameters. Then*

$$\lambda(\alpha^\vee) \in \frac{1}{N(\Sigma(\lambda))}\mathbb{Z} \qquad (\alpha \in \Sigma)\,.$$

Note that

$$N_\Sigma := \operatorname{lcm}\{N(\Sigma^0) \mid (\Sigma^0)^\vee \text{ is full rank subsystem of } \Sigma^\vee\}\,.$$

is finite as there are only finitely many full rank subsystems of $\Sigma^\vee$. Therefore, N_Σ is an upper bound for the indices $N(\Sigma(\lambda))$ which only depends on Σ.

Remark 7.6. Full rank subsystems can be described by repeated applications of the "Borel-de Siebenthal" theorem. That is: The maximal such subsystems are obtained by removing a node from the affine extended root system (and we can repeat this procedure to obtain the non maximal cases).

In type A_n, there are no proper subsystems of this type, since the affine extension is a cycle, so removing a node will again yield A_n. Hence if Σ is of type A_n, then the condition that $[\lambda]$ consists of strictly integral-negative parameters implies that $[\lambda] = \{\lambda\}$, and λ is integral on all coroots.

8 Integrality properties of leading exponents of twisted discrete series

For every $\alpha \in \Pi$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we set $\lambda_\alpha := s_\alpha(\lambda)$ and $\sigma_\alpha := \sigma \circ s_\alpha$. Further we let $I_\alpha(\lambda) : \pi^\infty_{\lambda_\alpha,\sigma_\alpha} \to \pi^\infty_{\lambda,\sigma}$ be the rank one intertwining operator. If we identify the space of smooth vectors of $\pi_{\lambda,\sigma}$ with $C^\infty(K \times_M V_\sigma)$ then the assignment

$$\mathfrak{a}_\mathbb{C}^* \to \operatorname{End}(C^\infty(K \times_M V_\sigma)), \;\; \lambda \mapsto I_\alpha(\lambda)$$

is meromorphic. In the appendix we prove:

Lemma 8.1. *There exists a constant $N \in \mathbb{N}$ only depending on G with the following property: If $\lambda(\alpha^\vee) \notin \frac{1}{N}\mathbb{Z}$, then $I_\alpha(\lambda)$ is an isomorphism.*

Combining Lemma 8.1 with Remark 7.2 we obtain:

Corollary 8.2. *Let $N \in \mathbb{N}$ be as in Lemma 8.1. Let (V, η) be a representation of the twisted discrete series and $\pi_{\lambda,\sigma} \twoheadrightarrow V$ a quotient morphism. Then the equivalence class $[2N\lambda]$ consists of weakly integral-negative parameters. If moreover (V, η) belongs to the discrete series, then $[2N\lambda]$ consists of integral-negative parameters.*

Proof. If $\alpha \in \Pi$ and $\alpha \notin \Sigma(2N\lambda)$, then $I_\alpha(\lambda)$ is an isomorphism by Lemma 8.1. Therefore the composition of $I_\alpha(\lambda)$ with the quotient morphism $\pi_{\lambda,\sigma} \twoheadrightarrow V$ gives a quotient morphism $\pi_{\lambda_\alpha,\sigma_\alpha} \twoheadrightarrow V$. It then follows from Remark 7.2(i) that $2\lambda_\alpha$ and thus also $2N\lambda_\alpha$ is weakly integral-negative. By repeating this argument we obtain that the equivalence class $[2N\lambda]$ consists of weakly integral-negative elements. If (V, η) belongs to the discrete series, then we use (ii) in Remark 7.2 instead of (i). $\square$

Recall the set of spherical roots $S \subseteq \mathfrak{a}_Z^*$ and recall that $S \subseteq \mathbb{Z}[\Sigma]$. Let $\chi \in (\widehat{\mathfrak{h}/\mathfrak{h}})_\mathbb{C}^*$ be normalized unitary and let $\mu \in \mathfrak{a}_Z^*$ be a leading exponent of a χ-twisted discrete series representation (V, η). Then we know from (3.3), (3.4), and (3.5) that we may expand μ as

$$\mu = \rho_Q + \sum_{\alpha \in S} c_\alpha \alpha + i\nu \qquad (c_\alpha \in \mathbb{R}). \tag{8.1}$$

with

(a) $c_\alpha > 0$ for all $\alpha \in S$,

(b) $\nu \in \mathfrak{a}_Z^*$ with $\nu\big|_{\mathfrak{a}_{Z,E}} = \operatorname{Im}\chi\big|_{\mathfrak{a}_{Z,E}}$.

Theorem 8.3. *Let $Z = G/H$ be a unimodular real spherical space. There exists an $N \in \mathbb{N}$ and for every normalized unitary $\chi \in (\widehat{\mathfrak{h}/\mathfrak{h}})_\mathbb{C}^*$ a finite set $\mathfrak{Y}_\chi \subseteq \mathfrak{a}^*$ with the following property. Let (V, η) be a spherical pair corresponding to a χ-twisted discrete series representation and let μ be any leading exponent of (V, η), which we expand as $\mu = \rho_Q + \sum_{\alpha \in S} c_\alpha \alpha + i\nu$ as in (8.1). Then the following hold.*

(i) $c_\alpha \in \frac{1}{N}\mathbb{N}$ for all $\alpha \in S$ and $\nu \in \mathfrak{Y}_\chi$.

(ii) If in addition (V, η) belongs to the discrete series, then $\nu = 0$, i.e., $\mu \in \mathfrak{a}_Z^*$. In particular, the infinitesimal character of V is real.

Proof. We let $\lambda := w_0\overline{\mu} + \rho_P$ and recall from Lemma 3.4 that there exists a $\sigma \in \widehat{M}$ such that $\pi_{\lambda,\sigma} \twoheadrightarrow V$. By Corollary 8.2 there exists a constant $N(G) \in \mathbb{N}$, depending only on G, such that the equivalence class $[2N(G)\lambda]$ consists of weakly integral-negative elements. By Theorem 7.4 (i) there exists an $N' \in \mathbb{N}$, only depending on G, such that

$$\operatorname{Re}\lambda(\alpha^\vee) \in \frac{1}{N'}\mathbb{Z} \qquad (\alpha \in \Sigma).$$

This implies that $\operatorname{Re}\lambda \in \frac{1}{N''}\mathbb{Z}(\Pi)$ for some $N'' \in \mathbb{N}$ depending only on G. Since the spherical roots are integral linear combinations of simple roots, it follows that there exists a $N \in \mathbb{N}$ (only depending on Z) such that $c_\alpha \in \frac{1}{N}\mathbb{N}$. Moreover, it follows from Corollary 6.2(i) and Theorem 7.4(i) (cf. Remark 7.2) that the imaginary part of λ is contained in a finite subset of $\mathfrak{a}^*$ depending only on χ. This proves (i). For the second assertion we use (ii) in Theorem 7.4 instead of (i). The infinitesimal character of V is equal to the infinitesimal character of $\pi_{\lambda,\sigma}$, which is real since λ is real. □

Theorem 8.3 (ii) implies the following.

Corollary 8.4. *Fix a normalized unitary $\chi \in (\widehat{\mathfrak{h}/\mathfrak{h}})^*_{\mathbb{C}}$ and a K-type τ. There are only finitely many χ-twisted discrete series representations V for Z such that the τ-isotypical component $V[\tau]$ of V is non-zero.*

Proof. (cf. [12, Lemma 70, p. 84]) Let $\mathfrak{t} \subseteq \mathfrak{m}$ be a Cartan subalgebra of $\mathfrak{m}$. Set $\mathfrak{c} := \mathfrak{a} + i\mathfrak{t}$ and note that $\mathfrak{c}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We inflate $\Sigma^+ = \Sigma^+(\mathfrak{g},\mathfrak{a})$ to a positive system $\Sigma^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{c})$ and write ρ_B for the corresponding half sum. Observe that $\rho_B = \rho_P + \rho_M \in \mathfrak{c}^*$. We identify σ with its highest weight in $i\mathfrak{t}^*$ and write $\langle\cdot\rangle^2$ for the quadratic form on $\mathfrak{c}_{\mathbb{C}}$ obtained from the Cartan-Killing form. Let $C_{\mathfrak{g}}$ be the Casimir element of $\mathfrak{g}$. Note that $C_{\mathfrak{g}}$ acts on $\pi_{\lambda,\sigma}$ with $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ as the scalar

$$\langle \lambda + \sigma + \rho_M \rangle^2 - \langle \rho_B \rangle^2 .$$

Let $\mathfrak{t}_{\mathfrak{k}} \supseteq \mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ and $\rho_{\mathfrak{k}} \in i\mathfrak{t}^*_{\mathfrak{k}}$ be the Weyl half sum with respect to a fixed positive system of $\Sigma(\mathfrak{k}_{\mathbb{C}},\mathfrak{t}_{\mathfrak{k}}) \subseteq i\mathfrak{t}_{\mathfrak{k}^*}$. As before we identify $\tau \in \widehat{K}$ with its highest weight in $i\mathfrak{t}^*_{\mathfrak{k}}$. We write $\langle\cdot\rangle^2_{\mathfrak{k}}$ for the quadratic form on $\mathfrak{t}_{\mathfrak{k},\mathbb{C}}$ obtained from the Cartan-Killing form. Further we let $C_{\mathfrak{k}}$ denote the Casimir element of $\mathfrak{k}$. The element $\Delta := C_{\mathfrak{g}} + 2C_{\mathfrak{k}}$ is a Laplace element and thus $\langle \Delta v, v\rangle \leq 0$ for all K-finite vectors in a unitarizable Harish-Chandra module V.

Let now V be a χ-twisted discrete series representation and $\pi_{\lambda,\sigma} \twoheadrightarrow V$ a quotient morphism. For $0 \neq v \in V[\tau]$ we obtain

$$\begin{aligned} 0 \geq \langle \Delta v, v\rangle &= \langle (C_{\mathfrak{g}} + 2C_{\mathfrak{k}})v, v\rangle \\ &= \Big(\langle \lambda+\sigma+\rho_M\rangle^2 - \langle\rho_B\rangle^2 - 2\big(\langle\tau+\rho_{\mathfrak{k}}\rangle^2_{\mathfrak{k}} - \langle\rho_{\mathfrak{k}}\rangle^2_{\mathfrak{k}}\big)\Big)\langle v, v\rangle . \end{aligned}$$

This forces

$$\langle \lambda+\sigma+\rho_M\rangle^2 - \langle\rho_B\rangle^2 \leq 2\big(\langle\tau+\rho_{\mathfrak{k}}\rangle^2_{\mathfrak{k}} - \langle\rho_{\mathfrak{k}}\rangle^2_{\mathfrak{k}}\big)$$

and in particular

$$\langle \operatorname{Re}\lambda\rangle^2 - \langle \operatorname{Im}\lambda\rangle^2 - \langle\rho\rangle^2 \leq 2\big(\langle\tau+\rho_{\mathfrak{k}}\rangle^2_{\mathfrak{k}} - \langle\rho_{\mathfrak{k}}\rangle^2_{\mathfrak{k}}\big) .$$

By Theorem 8.3 (i) $\operatorname{Re}\lambda$ is discrete and $\operatorname{Im}\lambda$ is contained in a finite set that only depends on Z. The assertion now follows from the fact that the map $\mathfrak{X}$ from (1.1) has finite fibers. □

Appendices

Appendix A: Invariant Sobolev Lemma

The aim of this appendix is an invariant Sobolev lemma for functions on Z that transform under the right action of $A_{Z,E}$ by a unitary character.

Recall that a weight on Z is a locally bounded function $w : Z \to \mathbb{R}_{>0}$ with the property that for every compact subset $\Omega \subseteq G$ there exists a constant $C > 0$ such that

$$w(gz) \leq Cw(z) \qquad (z \in Z, g \in \Omega)\,.$$

Further recall that there is a natural identification between the space of smooth densities on $\widehat{Z}$ and the space of functions

$$C^\infty(G : \Delta_{\widehat{Z}}) := \{f \in C^\infty(G) : f(\,\cdot\,\widehat{h}) = \Delta_{\widehat{Z}}^{-1}(\widehat{h})f \text{ for } \widehat{h} \in \widehat{H}\}\,,$$

where $\Delta_{\widehat{Z}}$ is the modular character

$$\Delta_{\widehat{Z}} : \widehat{H} \to \mathbb{R}_{>0}; \qquad ha \mapsto a^{-2\rho_Q} \qquad (a \in A_{Z,E}, h \in H)\,.$$

See Sections 8.1 and 8.2 in [25]. Note that smooth functions $f : G \to \mathbb{C}$ satisfying

$$f(\,\cdot\,ha) = a^{\rho_Q + i\nu} f \qquad (h \in H, a \in A_{Z,E})$$

for some $\nu \in \mathfrak{a}_{Z,E}^*$, are in the same way identified with smooth half-densities on $\widehat{Z}$.

Let B be a ball in G, i.e., a compact symmetric neighborhood of e in G. Recall that the corresponding volume-weight $\mathbf{v}_B$ is defined by

$$\mathbf{v}_B(z) := \mathrm{vol}_Z(Bz) \qquad (z \in Z)\,.$$

Note that if B' is another ball in G, then there exists $c > 0$ such that

$$\frac{1}{c}\mathbf{v}_{B'} \leq \mathbf{v}_B \leq c\mathbf{v}_{B'}\,.$$

In the following we drop the index and write $\mathbf{v}$ instead of $\mathbf{v}_B$.

The following lemma is a generalization of the invariant Sobolev lemma of Bernstein. See the key lemma in [3] on p. 686 and [31, Lemma 4.2].

Lemma A.1. *For every $k > \dim G$ there exists a constant $C > 0$ with the following property. Let $\nu \in \mathfrak{a}_{Z,E}^*$ and let $f \in C^\infty(Z)$ be a smooth function which transforms as $f(z \cdot a) = f(z)a^{\rho_Q + i\nu}$ for all $a \in A_{Z,E}$, and let Ω_f be the attached half-density on $\widehat{Z} = G/\widehat{H}$. Then*

$$|f(z)| \leq C\mathbf{v}(z)^{-\frac{1}{2}} \|\Omega_f\|_{B\widehat{z},2;k} \qquad (z \in Z)\,.$$

Here $\widehat{z} \in \widehat{Z}$ is the image of $z \in Z$ and $\|\cdot\|_{B\widehat{z},2:k}$ is the k'th L^2-Sobolev norm on $B\widehat{z}$.

Let A_0 be a closed subgroup of A such that the multiplication map $A_0 \times A_E \to A$ is a diffeomorphism. Let A_0^- be the cone such that $A_0^- A_E/A_E = A_{\widehat{Z}}^-$. By taking inverse images of the projection $A_Z = A/(A \cap H) \to A_{\widehat{Z}} = A/A_E$ we get

$$A_0^- A_E/(A \cap H) = A_Z^- . \tag{A.1}$$

We recall from [25, Section 3.4] that there exists a finite sets $F, \mathcal{W} \subseteq G$ such that

$$\mathcal{W} A_{Z,E} \subseteq A_{Z,E} \mathcal{W} H \tag{A.2}$$

and

$$\widehat{Z} = FKA_{\widehat{Z}}^- \mathcal{W} \cdot \widehat{z}_0 .$$

For the proof of the invariant Sobolev lemma we need the following lemma.

Lemma A.2. *There exists an $a_1 \in A$ and a constant $c > 0$, depending only on the normalization of the Haar measures on K and $A_{\widehat{Z}}$, such that for all compactly supported measurable non-negative densities f on $\widehat{Z}$ we have*

$$\int_{\widehat{Z}} f \geq c \sum_{w \in \mathcal{W}} \int_K \int_{A_{\widehat{Z}}^-} f(ka_1 aw) a^{-2\rho_Q} \, da \, dk .$$

Proof. Let f be a compactly supported measurable non-negative density on $\widehat{Z}$ and let $\varphi : Z \to \mathbb{R}_{\geq 0}$ be a compactly supported continuous function such that

$$\int_{A_E/A\cap H} \varphi(za_E) a_E^{-2\rho_Q} \, da_E = f(\widehat{z}) \qquad (z \in Z) .$$

Here $\widehat{z} \in \widehat{Z}$ denotes the image of $z \in Z$. Then by the Fubini theorem for densities (see [1, Theorem A.8])

$$\int_{\widehat{Z}} f = \int_Z \varphi(z) \, dz.$$

We will use Lemma 3.3 (1) in [28] to obtain a lower bound for this integral. The estimate in that lemma involves the integration over the conjugate of the maximal compact subgroup by some element in A, which we shall denote by a_1. We apply the lemma to the function $z \mapsto \varphi(a_1 \cdot z)$ on Z, and write the estimate in terms of the original maximal compact subgroup K. By this we obtain a constant $c > 0$ such that

$$\int_Z \varphi(a_1 \cdot z) \, dz \geq c \sum_{w \in \mathcal{W}} \int_K \int_{A_Z^-} \varphi(ka_1 aw) a^{-2\rho_Q} \, da \, dk .$$

Using that the measure on Z is G-invariant and (A.1), we obtain

$$\int_Z \varphi(z) \, dz \geq c \sum_{w \in \mathcal{W}} \int_K \int_{A_0^-} \int_{A_E/A\cap H} \varphi(ka_1 a a_E w) a_E^{-2\rho_Q} a^{-2\rho_Q} \, da_E \, da \, dk .$$

In view of (A.2) we now have

$$\int_Z \varphi(z) \, dz \geq c \sum_{w \in \mathcal{W}} \int_K \int_{A_0^-} f(ka_1 aw) a^{-2\rho_Q} \, da \, dk = c \sum_{w \in \mathcal{W}} \int_K \int_{A_{\widehat{Z}}^-} f(ka_1 aw) a^{-2\rho_Q} \, da \, dk .$$

□

Proof of Lemma A.1. We will prove that there exists a constant $C > 0$ such that for every non-negative smooth density ϕ on $\widehat{Z}$ and every $x \in G$

$$\int_B \phi(gx)\, dg \leq C \frac{1}{\mathbf{v}(x)} \int_{Bx\widehat{H}} \phi\,. \tag{A.3}$$

On the left-hand side ϕ is considered as a function on G that transforms under the right-action of $\widehat{H}$ with the modular character. Before giving the proof of A.3 we derive the lemma from it. By the local Sobolev lemma, applied to the function $f(\,\cdot\, z)$ on G, we obtain the following bound by the k-th Sobolev norm of $f(\,\cdot\, z)$ over the neighborhood B of $e \in G$:

$$|f(z)| \leq C\|f(\,\cdot\, z)\|_{B,2;k}.$$

The constant C is independent of f and z. Choose $x \in G$ such that $z = xH$. Using A.3 for the square of each derivative up to k of Ω_f, we also have

$$\|f(\,\cdot\, z)\|^2_{B,2;k} \leq C \frac{1}{\mathbf{v}(x)}\, \|\Omega_f\|^2_{Bx\widehat{H},2;k}.$$

The lemma follows from these inequalities.

For a measurable function $\chi : Z \to \mathbb{R}_{\geq 0}$, let $\psi_\chi : G \to \mathbb{R}_{\geq 0}$ be such that

$$\chi = \int_H \psi_\chi(\,\cdot\, h)\, dh\,.$$

Then for every $a \in A_{Z,E}$ we have

$$\int_G \psi_\chi(xa)\, dx = \int_Z \int_H \psi_\chi(gha)\, dh\, dgH = \left|\det \operatorname{Ad}(a)\big|_{\mathfrak{h}}\right| \int_Z \chi(z \cdot a)\, dz\,.$$

Since $\left|\det \operatorname{Ad}(a)\big|_{\mathfrak{h}}\right| = a^{-2\rho_Q}$, and by the invariance of the Haar measure the left-hand side is independent of a, it follows that

$$\int_Z \chi(z \cdot a)\, dz = a^{2\rho_Q} \int_Z \chi(z)\, dz\,.$$

We may apply this to $\chi = \mathbf{1}_{Bz}$ and obtain

$$\mathbf{v}(\,\cdot\, a) = a^{-2\rho_Q}\mathbf{v} \qquad (a \in A_{Z,E})\,.$$

We conclude that $\frac{1}{\mathbf{v}}$ may be considered as a density on Z.

Let $B \subseteq G$ be a ball and define $\mathbf{w}_B : \widehat{Z} \to \mathbb{R}_{>0}$ by

$$\mathbf{w}_B(\widehat{z}) := \int_{B\widehat{z}} \frac{1}{\mathbf{v}} \qquad (\widehat{z} \in \widehat{Z})\,.$$

If B' is another ball in G, then we may cover B' by a finite number of sets of the form gB. Since $\mathbf{v}$ is a weight, it follows that there exists a $c > 0$ such that

$$\frac{1}{c}\mathbf{w}_{B'} \leq \mathbf{w}_B \leq c\mathbf{w}_{B'}\,. \tag{A.4}$$

Let Ω be a compact subset of G. Let $B' = \{g^{-1}bg : g \in \Omega, b \in B\}$. Then

$$\mathbf{w}_B(g\widehat{z}) = \int_{Bg\widehat{z}} \frac{1}{\mathbf{v}} \leq \int_{gB'\widehat{z}} \frac{1}{\mathbf{v}} = \mathbf{w}_{B'}(\widehat{z}) \qquad (\widehat{z} \in \widehat{Z}, g \in \Omega)\,.$$

From (A.4) it follows that there exits a $c > 0$ such that

$$\mathbf{w}_B(g\widehat{z}) \leq c\mathbf{w}_B(\widehat{z}) \qquad (\widehat{z} \in \widehat{Z}, g \in \Omega)\,.$$

We thus see that $\mathbf{w}_B$ is a weight.

We claim that there exists a $c_1 > 0$ such that for every $z \in \widehat{Z}$

$$\mathbf{w}_B(\widehat{z}) > c_1\,. \tag{A.5}$$

Since $\mathbf{w}_B$ is a weight, it suffices to show that $\inf_{a_0 \in A^-_{\widehat{Z}}, w_0 \in \mathcal{W}} \mathbf{w}_B(a_0 w_0 \cdot \widehat{z}_0) > 0$ to prove this claim.

Let $a_0 \in A^-_{\widehat{Z}}$ and $w_0 \in \mathcal{W}$. It follows from the inequality (3.6) in [28] and Lemma A.2 that there exists a an element $a_1 \in A$ and a constant $C > 0$ such that,

$$\begin{aligned}
\mathbf{w}_B(a_0 w_0 \cdot \widehat{z}_0) &\geq C \sum_{w \in \mathcal{W}} \int_K \int_{A^-_{\widehat{Z}}} \mathbf{1}_{Ba_0 w_0 \cdot z_0}(ka_1 aw \cdot \widehat{z}_0)\, da\, dk \\
&\geq C \int_K \int_{A^-_{\widehat{Z}}} \mathbf{1}_{Ba_0 w_0 \cdot z_0}(ka_1 aw_0 \cdot \widehat{z}_0)\, da\, dk \\
&\geq C \int_K \int_{a_1 A^-_{\widehat{Z}}} \mathbf{1}_{Ba_0 w_0 \cdot z_0}(kaw_0 \cdot \widehat{z}_0)\, da\, dk\,.
\end{aligned}$$

For the last equality we used the invariance of the measure on $A_{\widehat{Z}}$. Let A_c be a compact subset of A with non-empty interior and $A_c A_{Z,E}/A_{Z,E} \subseteq a_1 A^-_{\widehat{Z}}$. By enlarging B, we may assume that B is invariant under left translations by elements from K on the left and $A_c \subseteq B$. Since $\int_K dk = 1$, we have

$$\int_K \int_{a_1 A^-_{\widehat{Z}}} \mathbf{1}_{Ba_0 w_0 \cdot z_0}(kaw_0 \cdot \widehat{z}_0)\, da\, dk = \int_{a_1 A^-_{\widehat{Z}}} \mathbf{1}_{Ba_0 w_0 \cdot z_0}(aw_0 \cdot \widehat{z}_0)\, da\,.$$

If $a \in a_0 A_c A_{Z,E}/A_{Z,E}$, then $aw_0 \cdot \widehat{z}_0 \in A_c a_0 w_0 \cdot \widehat{z}_0 \subseteq Ba_0 w_0 \cdot \widehat{z}_0$. Therefore,

$$\int_{a_1 A^-_{\widehat{Z}}} \mathbf{1}_{Ba_0 w_0 \cdot z_0}(aw_0 \cdot \widehat{z}_0)\, da \geq \int_{a_1 A^-_{\widehat{Z}}} \mathbf{1}_{a_0 A_c A_{Z,E}/A_{Z,E}}(a)\, da\,.$$

Since $a_0 \in A^-_{\widehat{Z}}$ and $A^-_{\widehat{Z}} A^-_{\widehat{Z}} \subseteq A^-_{\widehat{Z}}$, the set $a_0 A_c A_{Z,E}/A_{Z,E}$ is contained in $a_1 A^-_{\widehat{Z}}$ and thus

$$\int_{a_1 A^-_{\widehat{Z}}} \mathbf{1}_{a_0 A_c A_{Z,E}/A_{Z,E}}(a)\, da = \int_{A_{\widehat{Z}}} \mathbf{1}_{a_0 A_c A_{Z,E}/A_{Z,E}}(a)\, da = \int_{A_{\widehat{Z}}} \mathbf{1}_{A_c A_{Z,E}/A_{Z,E}}(a)\, da\,,$$

and hence

$$\mathbf{w}_B(a_0 w_0 \cdot \widehat{z}_0) \geq C \int_{A_{\widehat{Z}}} \mathbf{1}_{A_c A_{Z,E}/A_{Z,E}}(a)\, da\,.$$

The claim (A.5) now follows as the right-hand side is independent of a_0 and strictly positive.

Let ϕ be a non-negative smooth density on $\widehat{Z}$ and let $x \in G$. To prove (A.3) we may assume that $\operatorname{supp} \phi \subseteq Bx\widehat{H}$ and that $B^{-1} = B$. Since $\mathbf{v}$ is a weight, there exists a constant $c_2 > 0$ such that $\mathbf{v}(x) \leq c_2 \mathbf{v}(y)$ for every $y \in Bx$. If $y = bx$ with $b \in B$, then

$$\mathbf{v}(x) \int_B \phi(gx)\, dg \leq c_2 \mathbf{v}(y) \int_B \phi(gb^{-1}y)\, dg \leq c_2 \mathbf{v}(y) \int_{B^2} \phi(gy)\, dg\,.$$

Note that $\mathbf{v}\phi$ is right $\widehat{H}$-invariant, hence

$$\mathbf{v}(x) \int_B \phi(gx)\, dg \leq c_2 \mathbf{v}(y) \int_{B^2} \phi(gy)\, dg \qquad (y \in Bx\widehat{H})\,.$$

Therefore,

$$\int_B \phi(gx)\, dg \int_{y \in Bx\widehat{H}} \frac{1}{\mathbf{v}(y)} \leq \frac{c_2}{\mathbf{v}(x)} \int_{y \in Bx\widehat{H}} \left[\int_{B^2} \phi(gy)\, dg \right].$$

Let $c_1 > 0$ be as in (A.5). Then

$$\int_B \phi(gx)\, dg \leq \frac{1}{c_1} \int_B \phi(gx)\, dg \int_{Bx\widehat{H}} \frac{1}{\mathbf{v}} \leq \frac{c_2}{c_1 \mathbf{v}(x)} \int_{y \in Bx\widehat{H}} \Big[\int_{B^2} \phi(gy)\, dg \Big].$$

Now we use Fubini's theorem to change the order of integration. We thus get

$$\begin{aligned} \int_B \phi(gx)\, dg &\leq \frac{c_2}{c_1 \mathbf{v}(x)} \int_{B^2} \Big[\int_{y \in Bx\widehat{H}} \phi(gy) \Big]\, dg \\ &\leq \frac{c_2}{c_1 \mathbf{v}(x)} \int_{B^2} \Big[\int_{y \in \widehat{Z}} \phi(gy) \Big]\, dg \\ &= \frac{c_2 \operatorname{vol}(B^2)}{c_1 \mathbf{v}(x)} \int_{\widehat{Z}} \phi\,. \end{aligned}$$

This implies (A.3) as by assumption $\operatorname{supp} \phi \subseteq Bx\widehat{H}$. ☐

Appendix B: Intertwining operators

The main result of this appendix is the following proposition.

Proposition B.1. *There exists a $N \in \mathbb{N}$ such that for every $\alpha \in \Pi$, $\sigma \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\lambda(\alpha^\vee) \notin \frac{1}{N}\mathbb{Z}$, the standard intertwining operator $I_\alpha(\lambda, \sigma) : \pi_{s_\alpha\lambda, s_\alpha\sigma} \to \pi_{\lambda,\sigma}$ is defined and an isomorphism.*

Before we prove the proposition, we first prove a lemma.

Lemma B.2. *Assume that the split rank of G is equal to 1 and let α be the simple root of $(\mathfrak{g}, \mathfrak{a})$. There exists a $N \in \mathbb{N}$ such that for every $\sigma \in \widehat{M}$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ with $\nu(\alpha^\vee) \notin \frac{1}{N}\mathbb{Z}$, the representation $\pi_{\nu,\sigma}$ is irreducible.*

Proof. Let $\mathfrak{t}$ be a maximal torus in $\mathfrak{m}$. Let $\mathfrak{h} = \mathfrak{a} \oplus i\mathfrak{t}$. Then $\mathfrak{h}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. We define $\Sigma(\mathfrak{h}) \subseteq \mathfrak{h}^*$ to be the set of roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, choose a positive system $\Sigma^+(\mathfrak{h})$ and define

$$\rho_M := \frac{1}{2} \sum_{\substack{\beta \in \Sigma^+(\mathfrak{h}) \\ \beta|_\mathfrak{a} = 0}} \dim(\mathfrak{g}^\beta)\beta \,.$$

Let $\xi \in \mathfrak{t}_\mathbb{C}^*$ be the Harish-Chandra parameter of some constituent σ_0 of the restriction of σ to the connected component of M. Then $\xi - \rho_M$ is the highest weight of σ_0.

We view $\mathfrak{t}_\mathbb{C}^*$ and $\mathfrak{a}_\mathbb{C}^*$ as subspaces of $\mathfrak{h}_\mathbb{C}^*$ by extending the functionals trivially with respect to the decomposition $\mathfrak{h}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C}$. We write $p_\mathfrak{a}$ and $p_\mathfrak{t}$ for the restrictions $\mathfrak{a}_\mathbb{C}$ and $\mathfrak{t}_\mathbb{C}$ respectively. Let θ be the involutive automorphism on $\mathfrak{h}_\mathbb{C}$ that is 1 on $\mathfrak{t}_\mathbb{C}$ and -1 on $\mathfrak{a}_\mathbb{C}$. We denote the adjoint of θ by θ as well.

Now assume that $\pi_{\nu,\sigma}$ is not irreducible. We write $\gamma = (\xi, \nu) \in \mathfrak{t}_\mathbb{C}^* \oplus \mathfrak{a}_\mathbb{C}^*$. By [40, Theorem 1.1] there exists a $\beta \in \Sigma(\mathfrak{h})$ such that $\gamma(\beta^\vee) \in \mathbb{Z}$ and either

(a) $\gamma(\beta^\vee) > 0$, $\gamma(\theta\beta^\vee) < 0$ and $\theta\beta \neq -\beta$, or

(b) $\theta\beta = -\beta$.

Note that in both cases (a) and (b) $p_\mathfrak{a}\beta$ is non-zero and is in fact a root of $(\mathfrak{g}, \mathfrak{a})$. Therefore, $p_\mathfrak{a}\beta \in \{\pm\alpha, \pm 2\alpha\}$. Let $k \in \{\pm 1, \pm 2\}$ be such that $p_\mathfrak{a}\beta = k\alpha$. Then

$$\nu(\alpha^\vee) = \frac{k\|\beta\|^2}{\|p_\mathfrak{a}\beta\|^2} \frac{2\langle \nu, p_\mathfrak{a}\beta\rangle}{\|\beta\|^2} = \frac{k\|\beta\|^2}{\|p_\mathfrak{a}\beta\|^2}\Big(\gamma(\beta^\vee) - \frac{2\langle \xi, p_\mathfrak{t}\beta\rangle}{\|\beta\|^2}\Big) \in \frac{k\|\beta\|^2}{\|p_\mathfrak{a}\beta\|^2}\Big(\mathbb{Z} - \frac{2\langle \xi, p_\mathfrak{t}\beta\rangle}{\|\beta\|^2}\Big) \,.$$

Let d be the determinant of the Cartan matrix of the root system $\Sigma_\mathfrak{m}(\mathfrak{t}_\mathbb{C})$ of $\mathfrak{m}_\mathbb{C}$ in $\mathfrak{t}_\mathbb{C}$. The lattice $\Lambda_\mathfrak{m}(\mathfrak{t}_\mathbb{C})$ of integral weights of $\mathfrak{m}_\mathbb{C}$ in $\mathfrak{t}_\mathbb{C}$ is contained in $\frac{1}{d}\mathbb{Z}[\Sigma_\mathfrak{m}(\mathfrak{t}_\mathbb{C})]$. Note that $p_\mathfrak{t}\beta, \xi \in \Lambda_\mathfrak{m}(\mathfrak{t}_\mathbb{C})$. Let l be the square of the length of the shortest root in $\Sigma(\mathfrak{h})$. Then $\|\Sigma(\mathfrak{h})\|^2 \subseteq \{l, 2l, 3l\}$ and $\langle \Sigma(\mathfrak{h}), \Sigma(\mathfrak{h})\rangle \in \frac{l}{2}\mathbb{Z}$. Therefore,

$$\langle \Lambda_\mathfrak{m}(\mathfrak{t}_\mathbb{C}), \Lambda_\mathfrak{m}(\mathfrak{t}_\mathbb{C})\rangle \subseteq \frac{1}{d^2}\langle \Sigma(\mathfrak{t}_\mathbb{C}), \Sigma(\mathfrak{t}_\mathbb{C})\rangle \subseteq \frac{l}{2d^2}\,\mathbb{Z}\,,$$

and since $p_\mathfrak{t}\beta, \xi \in \Lambda_\mathfrak{m}(\mathfrak{t}_\mathbb{C})$,

$$\frac{2\langle \xi, p_\mathfrak{t}\beta\rangle}{\|\beta\|^2} \in \frac{1}{6d^2}\,\mathbb{Z}\,.$$

Since $\theta\beta \in \Sigma(\mathfrak{h})$ and by the Cauchy-Schwartz inequality

$$\frac{2\langle \beta, \theta\beta\rangle}{\|\beta\|^2} \in \{0, \pm 1, \pm 2\}\,.$$

Taking into account that $0 < \|p_\mathfrak{a}\beta\|^2 \leq \|\beta\|^2$ we obtain

$$\frac{\|\beta\|^2}{\|p_\mathfrak{a}\beta\|^2} = \frac{2\|\beta\|^2}{\|\beta\|^2 - \langle \beta, \theta\beta\rangle} \in \{1, \frac{4}{3}, 2, 4\}$$

and thus

$$\nu(\alpha^\vee) \in \frac{k\|\beta\|^2}{\|p_\mathfrak{a}\beta\|^2}\Big(\mathbb{Z} - \frac{2\langle \xi, p_\mathfrak{t}\beta\rangle}{\|\beta\|^2}\Big) \subseteq \frac{1}{18d^2}\mathbb{Z}\,.$$

□

Proof of Proposition B.1. Let $N \in \mathbb{N}$ be as in Lemma B.2. For $\alpha \in \Pi$ let G_α be the connected subgroup of G with Lie algebra generated by the subspace $\mathfrak{g}^{-2\alpha} \oplus \mathfrak{g}^{-\alpha} \oplus \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{2\alpha}$ of $\mathfrak{g}$. Note that the real rank of G_α is equal to 1. We define the subgroups

$$A_\alpha := A \cap G_\alpha\,, \quad M_\alpha := M \cap G_\alpha\,, \quad P_\alpha := P \cap G_\alpha\,.$$

Write σ_α and λ_α for $\sigma\big|_{M_\alpha}$ and $\lambda\big|_{\mathfrak{a}_\alpha}$ respectively. Let $I^0_\alpha(\lambda_\alpha, \sigma_\alpha)$ be the standard intertwining operator

$$I^0_\alpha(\lambda_\alpha, \sigma_\alpha) : \mathrm{Ind}_{P_\alpha}^{G_\alpha}(s_\alpha\lambda_\alpha \otimes s_\alpha\sigma_\alpha) \to \mathrm{Ind}_{P_\alpha}^{G_\alpha}(\lambda_\alpha \otimes \sigma_\alpha)\,.$$

By equation (17.8) in [18] we have

$$I_\alpha(\lambda, \sigma) f(e) = I^0_\alpha(\lambda_\alpha, \sigma_\alpha)\big(f\big|_{G_\alpha}\big)(e) \qquad \big(f \in \pi^\infty_{s_\alpha\lambda, s_\alpha\sigma}\big)\,. \tag{B.1}$$

The poles of the meromorphic family $I^0_\alpha(\lambda_\alpha, \sigma_\alpha)$ are located at the $\lambda_\alpha \in \mathfrak{a}^*_\alpha$ such that $\lambda_\alpha(\alpha^\vee) \in -\mathbb{N}_0$. See [18, Theorem 3]. It follows from (B.1) that $I_\alpha(\lambda, \sigma)$ is defined for $\lambda(\alpha^\vee) \notin -\mathbb{N}_0$.

Now assume that $\lambda(\alpha^\vee) \notin \mathbb{Z}$. Let $\phi_0 \in C^\infty_c(\overline{N}, V_\sigma)$ be such that $\int_{\overline{N} \cap s_\alpha N} \phi_0(\overline{n})\, d\overline{n} \neq 0$. Define $\phi \in \pi^\infty_{\sigma\lambda}, s_{\alpha\sigma}$ by setting $\phi\big|_{\overline{N}} = \phi_0$. Then the integral

$$\int_{\overline{N} \cap s_\alpha N} \phi(\overline{n})\, dn$$

is absolutely convergent and non-zero. Hence $I_\alpha(\lambda, \sigma)\phi(e)$ is non-zero. In particular this shows that both $I_\alpha(\lambda, \sigma)$ and $I^0_\alpha(\lambda_\alpha, \sigma_\alpha)$ are non-zero.

If $I_\alpha(\lambda, \sigma)$ is not injective, then there exists a $\phi \in \pi^\infty_{s_\alpha\lambda, s_\alpha\sigma}$ such that $I_\alpha(\lambda, \sigma)\phi = 0$ and $\phi(e) \neq 0$. It then follows from (B.1) that $I^0_\alpha(\lambda_\alpha, \sigma_\alpha)$ is not injective either. Since $I^0_\alpha(\lambda_\alpha, \sigma_\alpha)$ is non-zero, $\mathrm{Ind}_{P_\alpha}^{G_\alpha}(s_\alpha\lambda_\alpha \otimes s_\alpha\sigma_\alpha)$ is not irreducible. Similarly, if $I_\alpha(\lambda, \sigma)$ is not surjective, then its adjoint $I_\alpha(\lambda, \sigma)^* = I_\alpha(-s_\alpha\lambda, s_\alpha\sigma^\vee)$ is not injective, hence it follows that $\mathrm{Ind}_{P_\alpha}^{G_\alpha}(-\lambda_\alpha \otimes \sigma^\vee_\alpha)$ is not irreducible. It now follows from Lemma B.2 that if $I_\alpha(\lambda, \sigma)$ is not an isomorphism then $\lambda(\alpha^\vee) \in \frac{1}{N}\mathbb{Z}$. ☐

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Chapter II

Ellipticity and discrete series[†]

Joint with Bernhard Krötz, Eric Opdam and Henrik Schlichtkrull.

Abstract

We explain by elementary means why the existence of a discrete series representation of a real reductive group G implies the existence of a compact Cartan subgroup of G. The presented approach has the potential to generalize to real spherical spaces.

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1 Introduction

Let $\underline{G}$ be a connected reductive algebraic group defined over $\mathbb{R}$ and $G := \underline{G}(\mathbb{R})$ its group of real points. In this article we give an elementary proof that Harish-Chandra's compact Cartan subgroup condition is necessary for G to have discrete series. To explain the background, we first describe the problem in the more general context of real spherical spaces.

1.1 Real spherical spaces

Let $\underline{H} \subset \underline{G}$ be an algebraic subgroup defined over $\mathbb{R}$ and $H = \underline{H}(\mathbb{R})$. A suitable framework for harmonic analysis on $Z := G/H$ is obtained by the request that Z is real spherical, i.e., there exists an open orbit on Z for the natural action of a minimal parabolic subgroup P of G.

[†]This article was published in J. reine angew. Math. **782** (2022), 109–119.

Our interest is to obtain a geometric criterion for the existence of discrete series on a unimodular real spherical space Z. We recall that by definition the discrete series for Z consists of the irreducible subrepresentations of the regular representation of G on $L^2(Z)$. The following condition for its existence was conjectured in [8, (1.2)]:

Conjecture 1.1. *Let Z be a unimodular real spherical space. A necessary and sufficient condition for the existence of a discrete series representation for Z is that the interior of $(\mathfrak{h}^\perp)_{\mathrm{ell}}$ in $\mathfrak{h}^\perp$ is non-empty.*

Let us explain the notation. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G and H. Then $\mathfrak{h}^\perp \simeq (\mathfrak{g}/\mathfrak{h})^*$ is the cotangent space $T^*_{z_0}Z$ at the base point $z_0 = H \in Z$, and the index 'ell' stands for elliptic elements.

The sufficiency of the condition has been established in [3]. We recall the result:

Theorem 1.2. *Let Z be a unimodular real spherical space. If the interior of $(\mathfrak{h}^\perp)_{\mathrm{ell}}$ in $\mathfrak{h}^\perp$ is non-empty, then there exist infinitely many representations in the discrete series for Z.*

A central tool in the proof of this theorem is a property of the infinitesimal characters of discrete series representations for Z, derived in [8]. The same property is crucial for our approach to necessity. Some notation is needed in order to describe it.

Let $G = KAN$ be an Iwasawa decomposition for G and $P = MAN$ the associated minimal parabolic subgroup, with $M = Z_K(A)$ the centralizer of A in K. Let $\mathfrak{t} \subset \mathfrak{m}$ be a maximal torus. Then $\mathfrak{c} = \mathfrak{a} + \mathfrak{t}$ is a maximally split Cartan subalgebra of G, unique up to conjugation. With $\mathfrak{c}_{\mathbb{R}} = \mathfrak{a} + i\mathfrak{t}$ we obtain a real form of $\mathfrak{c}_{\mathbb{C}}$ which is characterized by the property that all roots $\gamma \in \Sigma_{\mathfrak{c}} = \Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}}) \subset \mathfrak{c}^*_{\mathbb{C}}$ are real valued on $\mathfrak{c}_{\mathbb{R}}$. Let V be the Harish-Chandra module of a discrete series representation for Z, and let its infinitesimal character be denoted $\chi_V \in \mathrm{Hom}_{\mathrm{alg}}(\mathcal{Z}(\mathfrak{g}), \mathbb{C})$. Using the Harish-Chandra isomorphism we identify χ_V with a $W_{\mathfrak{c}}$-orbit $[\Lambda_V] = W_{\mathfrak{c}} \cdot \Lambda_V \in \mathfrak{c}^*_{\mathbb{C}}/W_{\mathfrak{c}}$, where $W_{\mathfrak{c}}$ is the big Weyl group, i.e. the Weyl group of the root system $\Sigma_{\mathfrak{c}}$ with respect to the Cartan subalgebra $\mathfrak{c}$.

The mentioned result of [8] asserts that there exists an explicit $W_{\mathfrak{c}}$-invariant rational lattice $\mathcal{L}$, such that

$$[\Lambda_V] \subset \mathcal{L} \subset \mathfrak{c}^*_{\mathbb{R}} \tag{1.1}$$

for all discrete series representations V of Z. Let us emphasize in particular that the parameters Λ_V of the discrete series are *real*, as the lattice $\mathcal{L}$ lies in the real form $\mathfrak{c}^*_{\mathbb{R}}$.

The purpose of this article is to explore whether this property of the infinitesimal character can be used to establish the conjectured necessity of the condition. To be more precise, we show that this is the case for the group, regarded as a spherical space. We believe the approach has the potential to generalize to all real spherical spaces.

1.2 The group case

In the remainder of this article we consider the group case. The group G is a real spherical space when looked upon as a geometric object under its both-sided symmetries of $G \times G$. Specialized to this case the conjecture is Harish-Chandra's beautiful geometric criterion for the existence of discrete series representations for G, which results from his deep study of discrete series [4, 5].

Theorem 1.1. (Harish-Chandra, [5, Theorem 13]) *A necessary and sufficient condition for G to admit discrete series is that it has a compact Cartan subgroup.*

As mentioned, we provide an elementary proof of the necessity, based on the property (1.1) for G. In the case at hand the proof of this property is also elementary, as explained in the introduction to [8].

Let us describe the argument. Let σ be the conjugation on $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{g}$. We call an element $\Lambda \in \mathfrak{c}_\mathbb{C}^*$ *strongly regular* provided that the stabilizer of Λ in the extended Weyl group $W_{\mathfrak{c},\text{ext}} := \langle W_\mathfrak{c}, -\sigma\rangle \subset \operatorname{Aut}(\Sigma_\mathfrak{c})$ is trivial. We show that the existence of a unitary representation with a strongly regular real infinitesimal character implies the existence of a compact Cartan subgroup, see Corollary 3.6. Knowing that infinitesimal characters of discrete series are real, the existence of a discrete series representation with strongly regular infinitesimal character therefore requires the existence of a compact Cartan subgroup. Finally, we complete the proof by using the Zuckerman translation principle [9] to produce from any representation of the discrete series a discrete series representation with strongly regular infinitesimal character, see Corollary 5.8. The tools used for this belong to general representation theory of Harish-Chandra modules. Beyond the characterization of square integrability in terms of the leading exponents of asymptotic expansions, the only property of discrete series used at this stage is the existence of the lattice $\mathcal{L}$ satisfying (1.1).

Acknowledgement: We thank Joseph Bernstein and the referee for valuable comments.

2 Notation

Throughout this article we let G be the open connected subgroup of $\underline{G}(\mathbb{R})$ where $\underline{G}$ is a connected reductive group defined over $\mathbb{R}$. We write $G_\mathbb{C}$ for the connected group $\underline{G}(\mathbb{C})$. As usual we denote the Lie algebra of G by $\mathfrak{g}$ and keep this terminology for subgroups of G, i.e., if $H \subset G$ is a subgroup, then we denote by $\mathfrak{h}$ its Lie algebra. If $\mathfrak{h}$ is a Lie algebra, then we write $\mathfrak{h}_\mathbb{C}$ for the complexification of $\mathfrak{h}$.

Fix a Cartan involution θ of G and denote by $K = G^\theta$ the corresponding maximal compact subgroup. The Lie algebra automorphism of $\mathfrak{g}$ induced by θ, and its linear extension to $\mathfrak{g}_\mathbb{C}$, will be denoted by θ as well. We write $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ for the associated Cartan decomposition. We fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ and write $A = \exp(\mathfrak{a})$. Further we let $M = Z_K(A)$ and select with $\mathfrak{t} \subset \mathfrak{m}$ a maximal torus. We write T for the Cartan subgroup $Z_M(\mathfrak{t})$ of M.

We denote by $\sigma : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ the complex conjugation with respect to the real form $\mathfrak{g}$, and let $U := K\exp(i\mathfrak{s})$ be the θ-stable maximal compact subgroup of $G_\mathbb{C}$, which is obtained as the fixed point subgroup of the antilinear extension $\theta \circ \sigma$ of the Cartan involution θ to $G_\mathbb{C}$.

We extend $\mathfrak{a}$ by $\mathfrak{t}$ to a Cartan subalgebra $\mathfrak{c} := \mathfrak{a} + \mathfrak{t}$ of $\mathfrak{g}$, and use the symbol σ also for the restriction of σ to $\mathfrak{c}_\mathbb{C}$. We write $\Sigma_\mathfrak{c} = \Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{c}_\mathbb{C})$ for the corresponding root system and $\Sigma_\mathfrak{a} := \Sigma_\mathfrak{c}|_\mathfrak{a} \backslash \{0\}$ for the corresponding restricted root system. Further we set $\mathfrak{c}_\mathbb{R} := \mathfrak{a} + i\mathfrak{t}$. Note that $\Sigma_\mathfrak{c} \subset \mathfrak{c}_\mathbb{R}^*$, that σ preserves $\Sigma_\mathfrak{c}$ and $\mathfrak{c}_\mathbb{R}$ and that $\sigma\big|_{\mathfrak{c}_\mathbb{R}} = -\theta\big|_{\mathfrak{c}_\mathbb{R}}$. We write $C_\mathbb{C}$ for the maximal torus of $G_\mathbb{C}$ with Lie algebra $\mathfrak{c}_\mathbb{C}$. As $G_\mathbb{C}$ is a connected algebraic reductive group, the torus $C_\mathbb{C}$ is connected. We further define $C := G \cap C_\mathbb{C}$ and $C_U := C_\mathbb{C} \cap U$. Note that $C = TA$ and $C_U = T\exp(i\mathfrak{a})$.

Let us denote by $W_\mathfrak{c}$ the Weyl group of the root system $\Sigma_\mathfrak{c}$ and likewise we denote by $W_\mathfrak{a}$ the Weyl group of the restricted root system $\Sigma_\mathfrak{a}$. With respect to $\Sigma_\mathfrak{a}$ we have the restricted root space decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma_\mathfrak{a}} \mathfrak{g}^\alpha .$$

In the sequel we fix with $\Sigma_\mathfrak{a}^+ \subset \Sigma_\mathfrak{a}$ a positive system. We then let $\Sigma_\mathfrak{c}^+ \subset \Sigma_\mathfrak{c}$ be any positive system which is compatible with $\Sigma_\mathfrak{a}^+$, i.e., $\Sigma_\mathfrak{a}^+ = \Sigma_\mathfrak{c}^+|_\mathfrak{a}\backslash\{0\}$.

The positive system $\Sigma_\mathfrak{a}^+$ defines a maximal nilpotent subalgebra $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma_\mathfrak{a}^+} \mathfrak{g}^\alpha$. Put $N = \exp\mathfrak{n}$ and note that $P = MAN \subset G$ defines a minimal parabolic subgroup of G. We write $\overline{P}$ and $\overline{\mathfrak{n}}$ for θP and $\theta\mathfrak{n}$, respectively.

3 Reading of the existence of a maximal compact Cartan subgroup from the infinitesimal character

As usual we write $\mathcal{Z}(\mathfrak{g})$ for the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}_\mathbb{C}$. Recall that according to Harish-Chandra the characters χ of $\mathcal{Z}(\mathfrak{g})$ are parametrized by $\mathfrak{c}_\mathbb{C}^*/W_\mathfrak{c}$ as follows. For any positive system S of $\Sigma_\mathfrak{c}$ we set

$$\mathfrak{u}_S := \bigoplus_{\alpha\in S} \mathfrak{g}_{\mathbb{C},\alpha},$$

and write ρ_S for half the trace of $\mathrm{ad}(\mathfrak{c})$ on $\mathfrak{u}_S$. Using the Poincaré-Birkhoff-Witt theorem we may decompose an element $Z \in \mathcal{Z}(\mathfrak{g})$ as

$$Z \in C_S + \mathfrak{u}_{-S}\mathcal{U}(\mathfrak{g})\mathfrak{u}_S \tag{3.1}$$

with $C_S \in \mathcal{U}(\mathfrak{c})$, see the proof of [7, Lemma 8.17]. The element $[\Lambda] \in \mathfrak{c}_\mathbb{C}^*/W_\mathfrak{c}$ parametrizing χ is then given by

$$\chi(Z) = (\Lambda - \rho_S)(C_S) \tag{3.2}$$

and does not depend on the choice of S.

Every irreducible Harish-Chandra module V admits an infinitesimal character

$$\chi_V : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$$

which then corresponds to a $W_\mathfrak{c}$-orbit

$$[\Lambda_V] := W_\mathfrak{c} \cdot \Lambda_V$$

for some $\Lambda_V \in \mathfrak{c}_\mathbb{C}^*$. The following lemma is standard. For convenience we include its short proof.

Lemma 3.1. *Let V be an irreducible Harish-Chandra module. The following hold.*

1. $[\Lambda_{\tilde{V}}] = [-\Lambda_V]$*, where $\tilde{V}$ is the contragredient of V.*
2. *If V is unitarizable, then $[\Lambda_V] = [-\sigma\Lambda_V]$.*

Proof. Let $Z \mapsto Z^\vee$ denote the principal anti-automorphism of $\mathcal{U}(\mathfrak{g})$. Then $\chi_{\tilde{V}}(Z) = \chi_V(Z^\vee)$ for $Z \in \mathcal{Z}(\mathfrak{g})$. Let S be any positive system of $\Sigma_{\mathfrak{c}}$. Let $Z \in \mathcal{Z}(\mathfrak{g})$ and let $C_S \in \mathcal{U}(\mathfrak{c})$ be as in (3.1). By (3.2)

$$\chi_{\tilde{V}}(Z) = (\Lambda_{\tilde{V}} - \rho_S)(C_S).$$

As

$$Z^\vee \in C_S^\vee + \mathfrak{u}_S\mathcal{U}(\mathfrak{g})\mathfrak{u}_{-S},$$

we have

$$\chi_V(Z^\vee) = (\Lambda_V - \rho_{-S})(C_S^\vee) = (-\Lambda_V - \rho_S)(C_S).$$

This proves *1.*

The conjugate representation $\overline{V}$ of V has infinitesimal character $[\Lambda_{\overline{V}}] = [\sigma(\Lambda_V)]$. If V is unitarizable, then the representation is isomorphic to its conjugate dual, hence assertion *2.* □

We recall that an element $\lambda \in \mathfrak{c}_{\mathbb{C}}^*$ is *regular* provided that the stabilizer of λ in $W_{\mathfrak{c}}$ is trivial. Notice that the complex conjugation σ and $-\operatorname{id}$ induce automorphisms of $\Sigma_{\mathfrak{c}}$, i.e., they determine elements of $\operatorname{Aut}(\Sigma_{\mathfrak{c}})$. In particular $-\sigma \in \operatorname{Aut}(\Sigma_{\mathfrak{c}})$. We define the extended Weyl group of $W_{\mathfrak{c}}$ as the following subgroup of $\operatorname{Aut}(\Sigma_{\mathfrak{c}})$:

$$W_{\mathfrak{c},\mathrm{ext}} := \langle W_{\mathfrak{c}}, -\sigma\rangle_{\mathrm{group}} \subset \operatorname{Aut}(\Sigma_{\mathfrak{c}})\,.$$

Furthermore $\lambda \in \mathfrak{c}_{\mathbb{C}}^*$ is called *strongly regular* in case the stabilizer in $W_{\mathfrak{c},\mathrm{ext}}$ is trivial.

According to Harish-Chandra (see [5, Theorem 16]) the infinitesimal characters of representations of the discrete series V of G are real, i.e., $\Lambda_V \in \mathfrak{c}_{\mathbb{R}}^*/W_{\mathfrak{c}}$. A simplified proof of this fact was recently given in the more general context of real spherical spaces, see [8, Theorem 1.1].

Proposition 3.2. *Assume that there exists a representation V of the discrete series for G with infinitesimal character $[\Lambda] \in \mathfrak{c}_{\mathbb{C}}^*/W_{\mathfrak{c}}$. Then the following assertions hold:*

1. $\Lambda \in \mathfrak{c}_{\mathbb{R}}^*$ *and there exists an element $w \in W_{\mathfrak{c}}$ such that $w \cdot \Lambda = -\sigma(\Lambda)$.*
2. *If in addition Λ is strongly regular, then there exists an element $w \in W_{\mathfrak{c}}$ such that $w = -\sigma$ on $\mathfrak{c}_{\mathbb{R}}^*$. In particular, $-\sigma|_{\mathfrak{c}_{\mathbb{R}}^*} \in W_{\mathfrak{c}} \subset \operatorname{Aut}(\mathfrak{c}_{\mathbb{R}}^*)$.*

Proof. As mentioned above, $\Lambda \in \mathfrak{c}_{\mathbb{R}}^*$. Since representations of the discrete series are also unitarizable, Lemma 3.1 gives $[-\sigma\Lambda] = [\Lambda]$. This shows the first assertion and the second is a consequence thereof. □

We recall that $W_{\mathfrak{c}} = N_{G_{\mathbb{C}}}(\mathfrak{c}_{\mathbb{C}})/C$ and $W_{\mathfrak{a}} = N_K(\mathfrak{a})/M$. We denote by $W_{\mathfrak{c}}^{\theta}$ the subgroup of $W_{\mathfrak{c}}$ consisting of the elements which commute with θ, and recall the exact sequence

$$1 \to W_{\mathfrak{m}} \to W_{\mathfrak{c}}^{\theta} \to W_{\mathfrak{a}} \to 1 \tag{3.3}$$

where $W_{\mathfrak{m}}$ is the Weyl group of the root system $\Sigma_{\mathfrak{m}} := \Sigma(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, which can be realized as $N_M(\mathfrak{t})/T$.

Lemma 3.3. *Let τ be an automorphism of $\mathfrak{g}_{\mathbb{C}}$ and $J_{\mathbb{C}}$ a Cartan subgroup of $G_{\mathbb{C}}$. If τ acts trivially on $\mathfrak{j}_{\mathbb{C}}$, then there exists a $t \in J_{\mathbb{C}}$ so that $\tau = \mathrm{Ad}(t)$.*

Proof. Since τ acts trivially on $\mathfrak{j}_{\mathbb{C}}$, it preserves all root spaces $\mathfrak{g}_{\mathbb{C}}^{\gamma}$, $\gamma \in \Sigma_{\mathfrak{j}}$. Hence there exists for all $\gamma \in \Sigma_{\mathfrak{j}}$ numbers $c_\gamma \in \mathbb{C}$ such that $\tau|_{\mathfrak{g}_{\mathbb{C}}^{\gamma}} = c_\gamma \cdot \mathrm{id}_{\mathfrak{g}_{\mathbb{C}}^{\gamma}}$. Let now $t \in J_{\mathbb{C}}$ be such that $\mathrm{Ad}(t)$ coincides with τ on all simple root spaces $\mathfrak{g}_{\mathbb{C}}^{\gamma}$, $\gamma \in \Pi_{\mathfrak{j}}$. Now $\phi := \mathrm{Ad}(t)^{-1} \circ \tau$ is an automorphism of $\mathfrak{g}_{\mathbb{C}}$ which acts trivially on $\mathfrak{b}_{\mathbb{C}} = \mathfrak{j}_{\mathbb{C}} + \bigoplus_{\gamma \in \Sigma_{\mathfrak{j}}^{+}} \mathfrak{g}_{\mathbb{C}}^{\gamma}$ and leaves all other $\mathfrak{g}_{\mathbb{C}}^{-\gamma}$, $\gamma \in \Sigma_{\mathfrak{j}}^{+}$, invariant. In fact, ϕ acts trivially on all negative root spaces. To see this, let $\gamma \in \Sigma_{\mathfrak{j}}^{+}$ and $0 \neq E_\gamma \in \mathfrak{g}_{\mathbb{C}}^{\gamma}$ and $0 \neq F_\gamma \in \mathfrak{g}_{\mathbb{C}}^{-\gamma}$. Then $0 \neq [E_\gamma, F_\gamma] \in \mathfrak{j}_{\mathbb{C}}$. As ϕ acts trivially on $\mathfrak{j}_{\mathbb{C}}$ we have

$$[E_\gamma, F_\gamma] = \phi[E_\gamma, F_\gamma] = [E_\gamma, \phi F_\gamma],$$

and hence $\phi F_\gamma = F_\gamma$. It follows that $\tau = \mathrm{Ad}(t)$. □

Proposition 3.4. *The following assertions are equivalent:*

1. $-\sigma|_{\mathfrak{c}_{\mathbb{R}}} \in W_{\mathfrak{c}}$.
2. $\theta|_{\mathfrak{c}_{\mathbb{C}}} \in W_{\mathfrak{c}}$.
3. *θ is an inner automorphism of $\mathfrak{g}_{\mathbb{C}}$.*
4. *There exists a $g \in U$ such that $\theta = \mathrm{Ad}(g)$ as an automorphism of $\mathfrak{g}_{\mathbb{C}}$.*

Proof. Since $-\sigma$ and θ coincide on $\mathfrak{c}_{\mathbb{R}}$, the equivalence of (1) and (2) is clear.

Suppose now that (2) holds. Since $\theta\big|_{\mathfrak{a}} \in W_{\mathfrak{a}}$ there exists a $k \in N_K(\mathfrak{a})$ so that $\theta\big|_{\mathfrak{a}} = \mathrm{Ad}(k)\big|_{\mathfrak{a}}$. Since $N_K(\mathfrak{a}) \subseteq N_K(M)$, the restriction of $\mathrm{Ad}(k)^{-1}\theta$ to $\mathfrak{c}_{\mathbb{R}}$ defines an element of $W_{\mathfrak{c}}$ whose restriction to $\mathfrak{a}$ is trivial. In view of (3.3) $\mathrm{Ad}(k)^{-1}\theta$ defines an element of $W_{\mathfrak{m}}$, and thus there exists an $m \in M$ so that $\mathrm{Ad}(k)^{-1}\theta\big|_{i\mathfrak{t}} = \mathrm{Ad}(m)\big|_{i\mathfrak{t}}$. Now $\mathrm{Ad}(km)$ and θ coincide on $\mathfrak{c}_{\mathbb{R}}$. Let $w = km$.

Let $\tau = \theta \circ \mathrm{Ad}(w)$. Since τ is an automorphism of $\mathfrak{g}_{\mathbb{C}}$ with $\tau|_{\mathfrak{c}_{\mathbb{C}}} = \mathrm{id}_{\mathfrak{c}_{\mathbb{C}}}$, it follows from Lemma 3.3 that there exists a $t \in C_{\mathbb{C}}$ so that $\tau = \mathrm{Ad}(t)$. Since θ commutes with $\mathrm{Ad}(w)$ (as w in K) we have $\tau^2 \in \mathrm{Ad}(K)$. Hence $\langle\tau\rangle = \langle\tau^2\rangle \cup \tau\langle\tau^2\rangle$ is a relatively compact subgroup of $\mathrm{Ad}(C_{\mathbb{C}})$. Consequently we see that t can in fact be chosen in C_U. It follows $\theta = \mathrm{Ad}(tw^{-1})$ with $g := tw^{-1} \in U$. This proves (4).

The implication of (3) from (4) is trivial.

Finally, if (3) holds, then there exists a $g \in G_{\mathbb{C}}$ so that $\theta = \mathrm{Ad}(g)$. Since θ preserves the Cartan subalgebra $\mathfrak{c}_{\mathbb{R}}$, we have $g \in N_{G_{\mathbb{C}}}(\mathfrak{c}_{\mathbb{R}})$. Therefore, $\theta|_{\mathfrak{c}_{\mathbb{C}}} = \mathrm{Ad}(g)|_{\mathfrak{c}_{\mathbb{C}}} \in W_{\mathfrak{c}}$. This proves (2). □

The following statement can also be found in [1, Lemma 1.6].

Corollary 3.5. *The Cartan involution θ is an inner automorphism of $\mathfrak{g}_\mathbb{C}$ if and only if $\mathfrak{k} \subset \mathfrak{g}$ is a reductive subalgebra of maximal rank. In that case $\mathfrak{g}$ admits a compact Cartan subalgebra.*

Proof. Assume that θ is an inner automorphism of $\mathfrak{g}_\mathbb{C}$. By Proposition 3.4 there exists a $g \in U$ so that $\mathrm{Ad}(g) = \theta$. As g is semi-simple, the group $K_\mathbb{C} := G_\mathbb{C}^\theta$ is equal to $Z_{G_\mathbb{C}}(g)$. The centralizer of a semi-simple element contains a maximal torus of $G_\mathbb{C}$, and therefore, $\operatorname{rank} K_\mathbb{C} = \operatorname{rank} G_\mathbb{C}$.

If $\mathfrak{k}$ is reductive of maximal rank, then there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{k}$. The Cartan involution θ acts trivially on $\mathfrak{h}$. Now Lemma 3.3 is applicable to $\tau = \theta$ and $\mathfrak{j}_\mathbb{C} = \mathfrak{h}_\mathbb{C}$. It follows that θ is inner. □

Corollary 3.6. *Suppose that there exists a representation of the discrete series for G with strongly regular infinitesimal character. Then G admits a compact Cartan subgroup.*

Proof. The assertion follows from Propositions 3.2 and 3.4 and Corollary 3.5. □

4 Power series expansion

In this section we summarize a few basic facts regarding the power series expansions of the matrix coefficients of an irreducible Harish-Chandra module V. We denote the dual Harish-Chandra module of V by $\widetilde{V}$. Recall that $\widetilde{V}$ is given by the K-finite vectors in the algebraic dual V^* of V. As before we identify the infinitesimal character of V with an $W_\mathfrak{c}$-orbit $[\Lambda_V] = W_\mathfrak{c} \cdot \Lambda_V \subset \mathfrak{c}_\mathbb{C}^*$.

Let us denote by $\mathfrak{a}^{++}$ the positive Weyl chamber in $\mathfrak{a}$ with respect to $\Sigma_\mathfrak{a}^+$ and denote by $\mathfrak{a}^+$ the closure of $\mathfrak{a}^{++}$. Likewise we set $A^{++} = \exp(\mathfrak{a}^{++})$ and $A^+ = \exp(\mathfrak{a}^+)$. As usual we denote by $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma_\mathfrak{a}^+}(\dim \mathfrak{g}^\alpha)\alpha \in \mathfrak{a}^*$ the Weyl half sum.

Now given an irreducible Harish-Chandra module V each K-bi-finite matrix coefficient

$$G \ni g \mapsto m_{v,\tilde{v}}(g) := \langle \pi(g)v, \tilde{v}\rangle$$

for $v \in V$ and $\tilde{v} \in \widetilde{V}$ admits a power series expansion on A^{++}, see [7, Ch. VIII]. To be precise, we have

$$m_{v,\tilde{v}}(a) = \sum_{\xi\in[\Lambda_V]|_\mathfrak{a} - \mathbb{N}_0[\Sigma_\mathfrak{a}^+]} p_{v,\tilde{v}}^\xi(\log a)\, a^{\xi-\rho} \qquad (a \in A^{++}, v \in V, \tilde{v} \in \widetilde{V})$$

with unique polynomials $p_{v,\tilde{v}}^\xi$ on $\mathfrak{a}$ which are of bounded degree and depend bilinearly on the pair $v, \tilde{v}$. In case V belongs to the discrete series only those elements ξ contribute for which $\operatorname{Re}\xi|_{\mathfrak{a}^+}$ is negative, i.e., $\operatorname{Re}\xi(X) < 0$ for all $X \in \mathfrak{a}^+\backslash\{0\}$.

By definition, an element $\xi \in [\Lambda_V]|_\mathfrak{a} - \mathbb{N}_0[\Sigma_\mathfrak{a}^+]$ is called an *exponent* of V if $p_{v,\tilde{v}}^\xi \neq 0$ for some $v, \tilde{v}$. The maximal elements in the set of exponents with respect to the ordering given by $\xi_1 \succeq \xi_2$ if $\xi_1 - \xi_2 \in \mathbb{N}_0[\Sigma_\mathfrak{a}^+]$ are called the *leading exponents*. We denote by $\mathcal{E}_V \subset \mathfrak{a}_\mathbb{C}^*$ the set of leading exponents and note that by [7, Theorem 8.33] we have $\mathcal{E}_V \subseteq [\Lambda_V]|_\mathfrak{a}$. Then

$$m_{v,\tilde{v}}(a) = \sum_{\xi \in \mathcal{E}_V - \mathbb{N}_0[\Sigma_\mathfrak{a}^+]} p_{v,\tilde{v}}^\xi(\log a) a^{\xi-\rho} \qquad (a \in A^{++}).$$

The coefficients $p_{v,\tilde{v}}^\lambda$ for $\lambda \in \mathcal{E}_V$ determine the principal asymptotics of the matrix coefficient in the sense that

$$m_{v,\tilde{v}}(a) = \sum_{\lambda \in \mathcal{E}_V} p_{v,\tilde{v}}^\lambda(\log a) a^{\lambda-\rho} + \text{lower order terms} \qquad (a \in A^{++}).$$

The condition that V belongs to the discrete series can be read off by its set of leading exponents. Let

$$\mathcal{C} := (\mathfrak{a}^+)^\star := \{\lambda \in \mathfrak{a}^* \mid \lambda(X) \geq 0,\ X \in \mathfrak{a}^+\} = \sum_{\alpha \in \Sigma^+} \mathbb{R}_{\geq 0}\alpha$$

be the dual Weyl chamber. By [7, Theorem 8.48] V belongs to the discrete series if and only if it satisfies the condition

$$\operatorname{Re} \mathcal{E}_V \subset -\operatorname{int} \mathcal{C}. \tag{4.1}$$

Lemma 4.1. *Let $F = F_\mu$ be a finite dimensional representation of G with highest weight μ with respect to $\Sigma_\mathfrak{c}^+$ and let V be a Harish-Chandra module of the discrete series. The following are equivalent:*

1. $\operatorname{Re} \mu|_\mathfrak{a} + \operatorname{Re} \mathcal{E}_V \subset -\operatorname{int} \mathcal{C}$.
2. *All matrix coefficients of $V \otimes F_\mu$ are contained in $L^2(G)$.*

Proof. If $v \otimes f \in V \otimes F_\mu$ and $\tilde{v} \otimes \tilde{f} \in \widetilde{V} \otimes F_\mu^*$, then

$$m_{v\otimes f, \tilde{v}\otimes\tilde{f}} = m_{v,\tilde{v}}\, m_{f,\tilde{f}}. \tag{4.2}$$

The assertion (1) $\Rightarrow$ (2) now follows from (4.1) as $\operatorname{spec}_\mathfrak{a} F_\mu \subset \mu|_\mathfrak{a} - \mathbb{N}_0[\Sigma_\mathfrak{a}^+] \subset \mu|_\mathfrak{a} - \mathcal{C}$. The other implication follows immediately from (4.2) with suitable choices of f and $\tilde{f}$. □

5 Application of the translation principle

For a Harish-Chandra module V we denote by $H_0(\overline{\mathfrak{n}}, V)$ the finite dimensional $\overline{\mathfrak{n}}$-homology of degree 0, and recall that the covariant functor $H_0(\overline{\mathfrak{n}}, \cdot\,)$ is right exact. Notice that $H_0(\overline{\mathfrak{n}}, V)$ is a module for MA. By the Harish-Chandra homomorphism we have

$$\mathcal{Z}(\mathfrak{m}) \simeq \mathcal{U}(\mathfrak{t})^{W_\mathfrak{m}}.$$

Moreover we note $\mathcal{Z}(\mathfrak{a}+\mathfrak{m}) = \mathcal{U}(\mathfrak{a}) \otimes \mathcal{Z}(\mathfrak{m})$. Therefore we can consider the spectrum of a finite dimensional $\mathcal{Z}(\mathfrak{a}+\mathfrak{m})$-module as a $W_\mathfrak{m}$-invariant subset of $\mathfrak{c}_\mathbb{C}^*$. In addition we consider ρ as a $W_\mathfrak{m}$-invariant element of $\mathfrak{c}_\mathbb{C}^*$ by extending it trivially on $\mathfrak{t}$.

Lemma 5.1. *Let V be an irreducible Harish-Chandra module with infinitesimal character $[\Lambda]$. Then the following assertions hold:*

1. $\operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})} H_0(\overline{\mathfrak{n}}, V) \subset -\rho + [\Lambda]$.
2. $\operatorname{spec}_{\mathfrak{a}} H_0(\overline{\mathfrak{n}}, V) \subset -\rho + \mathcal{E}_V - \mathbb{N}_0[\Sigma_{\mathfrak{a}}^+]$.

Proof. For (1) see [6, Cor. 3.32]. For the inclusion in (2), let $\lambda \in \operatorname{spec}_{\mathfrak{a}} H_0(\overline{\mathfrak{n}}, V)$. Recall that it follows from Casselman's version of Frobenius reciprocity that elements $\lambda \in \operatorname{spec}_{\mathfrak{a}} H_0(\overline{\mathfrak{n}}, V)$ correspond to embeddings of V into a minimal principal series representation $\operatorname{Ind}_{\overline{P}}^{G}\big(\sigma \otimes (\lambda + \rho)\big)$ (see [2] or [6, Theorem 4.9]). Without loss of generality, we may assume that $V \subset \operatorname{Ind}_{\overline{P}}^{G}\big(\sigma \otimes (\lambda + \rho)\big)$. As in the derivation of [8, (1.4)] one sees that $\lambda + \rho$ occurs as an exponent of V, and hence is contained in $\mathcal{E}_V - \mathbb{N}_0[\Sigma_{\mathfrak{a}}^+]$. □

For the rest of this section we let V be a Harish-Chandra module of the discrete series with infinitesimal character $[\Lambda] = W_{\mathfrak{c}} \cdot \Lambda \in \mathfrak{c}_{\mathbb{C}}^*/W_{\mathfrak{c}}$. We set

$$[\Lambda]^+ := \{\nu \in [\Lambda] \mid \operatorname{Re}\nu|_{\mathfrak{a}} \in -\operatorname{int}\mathcal{C}\} = \{\nu \in [\Lambda] \mid \operatorname{Re}\nu|_{\mathfrak{a}^+\setminus\{0\}} < 0\}\,.$$

Lemma 5.2. *Let V be a Harish-Chandra module of the discrete series with infinitesimal character $[\Lambda]$. Then*

$$\operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})} H_0(\overline{\mathfrak{n}}, V) \subset -\rho + [\Lambda]^+\,. \tag{5.1}$$

Proof. Immediate from Lemma 5.1 and (4.1). □

We pick the representative $\Lambda \in [\Lambda]$ such that $\lambda := \Lambda|_{\mathfrak{a}} \in \mathcal{E}_V$. In view of [8], Theorem 1.1 and Remark 1.2(3), there exists an $N \in \mathbb{N}$, independent of the discrete series representation V, so that $N\Lambda$ is integral. We select such an N and set $\mu_0 := N\Lambda$. Let μ be the unique dominant integral element in $W_{\mathfrak{c}} \cdot \mu_0$ and let F_μ be the corresponding finite dimensional representation of G with highest weight $\mu \in \mathfrak{c}_{\mathbb{R}}^*$.

We are interested in the $\mathcal{Z}(\mathfrak{g})$-isotypical decomposition of $V \otimes F_\mu$. Let

$$\chi_{\Lambda+\mu_0} : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$$

be the character corresponding to $[\Lambda+\mu_0]$. According to Zuckerman [9, Theorem 1.2 (1)] the element $[\Lambda + \mu_0]$ appears in $\operatorname{spec}_{\mathcal{Z}(\mathfrak{g})}(V \otimes F_\mu)$ and thus the corresponding isotypical component

$$W := \{v \in V \otimes F_\mu \mid (\exists k \in \mathbb{N})(\forall z \in \mathcal{Z}(\mathfrak{g}))\ (z - \chi_{\Lambda+\mu_0}(z))^k \cdot v = 0\} \tag{5.2}$$

is non-zero. Let $J \subset W$ be a maximal submodule and set $U := W/J$. Then U is an irreducible Harish-Chandra module with infinitesimal character

$$[\Lambda_U] = [\Lambda + \mu_0] = [(N+1)\Lambda]\,.$$

Lemma 5.3. *For any finite dimensional representation F and any $p \geq 0$ we have*

$$\operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})} H_0(\overline{\mathfrak{n}}, V \otimes F) \subset -\rho + [\Lambda]^+ + \operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})} F\,.$$

Proof. Filter F as $\overline{P}$-module as

$$F_0 = \{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_n = F$$

such that F_k/F_{k-1} is an irreducible $\overline{P}$-module for each $1 \leq k \leq n$. In particular, each F_k/F_{k-1} is a trivial $\overline{\mathfrak{n}}$-module and thus $H_0(\overline{\mathfrak{n}}, V \otimes F_k/F_{k-1}) = H_0(\overline{\mathfrak{n}}, V) \otimes F_k/F_{k-1}$ as MA-modules.

We apply now H_0 to the exact sequence of MA-modules

$$0 \to V \otimes F_{k-1} \to V \otimes F_k \to V \otimes F_k/F_{k-1} \to 0.$$

and obtain the right exact sequence

$$H_0(\overline{\mathfrak{n}}, V \otimes F_{k-1}) \to H_0(\overline{\mathfrak{n}}, V \otimes F_k) \to H_0(\overline{\mathfrak{n}}, V) \otimes F_k/F_{k-1} \to 0\,.$$

This implies

$$\operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})} H_0(\overline{\mathfrak{n}}, V \otimes F_k) \subset \operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})} H_0(\overline{\mathfrak{n}}, V \otimes F_{k-1}) \cup \operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})}(H_0(\overline{\mathfrak{n}}, V) \otimes F)$$

and the assertion follows by induction on k and (5.1). □

Lemma 5.4. *Let $\mu \in \mathfrak{c}_{\mathbb{R}}^*$ be dominant and integral and let F_μ be the highest weight representation with highest weight μ. Let $\mu_0 \in W_{\mathfrak{c}} \cdot \mu$ and let $\Lambda \in \mathbb{R}_+\mu_0$. Further, let $\nu \in [\Lambda]$, $\sigma \in \operatorname{spec}_{\mathfrak{c}} F_\mu$ and $w \in W_{\mathfrak{c}}$. If $w(\Lambda + \mu_0) = \nu + \sigma$, then $w\Lambda = \nu$ and $w\mu_0 = \sigma$.*

Proof. Let $r > 0$ be so that $\mu_0 = r\Lambda$. We have $\sigma \in \operatorname{spec}_{\mathfrak{c}} F_\mu \subset \operatorname{conv}(W_{\mathfrak{c}} \cdot \mu_0)$. In particular, $\|\sigma\| \leq \|\mu_0\|$. Moreover, $\|\nu\| = \|\Lambda\|$. The Cauchy-Schwarz inequality applied to ν and σ then gives that $\sigma = r\nu$. It follows that $\sigma = w\mu_0$ and $\nu = w\Lambda$. □

For a Harish-Chandra module U and infinitesimal character $[\Lambda_U]$ we define a subset $[\Lambda_U]_{\mathcal{E}} \subset [\Lambda_U]$ by

$$[\Lambda_U]_{\mathcal{E}} := \{\Upsilon \in [\Lambda_U] \mid \Upsilon|_{\mathfrak{a}} \in \mathcal{E}_U\}\,.$$

Proposition 5.5. *For $U = W/J$ as defined after* (5.2) *one has $[\Lambda_U]_{\mathcal{E}} \subset [\Lambda + \mu_0]^+$. In particular, U is square integrable.*

Proof. First recall that $W \subset V \otimes F_\mu$ is a direct summand as it is a generalized $\mathcal{Z}(\mathfrak{g})$-eigenspace. Thus $H_0(\overline{\mathfrak{n}}, W) \subset H_0(\overline{\mathfrak{n}}, V \otimes F_\mu)$ as MA-module and therefore

$$\operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})} H_0(\overline{\mathfrak{n}}, W) \subset -\rho + [\Lambda]^+ + \operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})} F_\mu$$

by Lemma 5.3. Now $U = W/J$ is a quotient of W and thus the natural map

$$H_0(\overline{\mathfrak{n}}, W) \twoheadrightarrow H_0(\overline{\mathfrak{n}}, U)$$

is surjective. We conclude that

$$\operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})} H_0(\overline{\mathfrak{n}}, U) \subset -\rho + [\Lambda]^+ + \operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})} F_\mu. \tag{5.3}$$

On the other hand we have $\operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})} H_0(\overline{\mathfrak{n}}, U) \subset -\rho + [\Lambda_U]$ by Lemma 5.1(1). Comparing this with (5.3) and applying Lemma 5.4 yields

$$\operatorname{spec}_{\mathcal{Z}(\mathfrak{a}+\mathfrak{m})} H_0(\overline{\mathfrak{n}}, U) \subset -\rho + [\Lambda_U]^+\,.$$

Finally, from (4.1) we deduce that U is square integrable. □

Repeated application of Proposition 5.5 yields:

Corollary 5.6. *There exists a $N \in \mathbb{N}$ such that if V is a representation of the discrete series with infinitesimal character $[\Lambda]$, then for every $k \in \mathbb{N}$ there exists a representation U of the discrete series with infinitesimal character $[(kN+1)\Lambda]$ and*

$$\mathcal{E}_U \subset [(kN+1)\Lambda]^+\big|_{\mathfrak{a}}.$$

Corollary 5.7. *Suppose that there exists a representation of the discrete series. Then there exists a representation of the discrete series with strongly regular infinitesimal character.*

Proof. Let V be a representation of the discrete series with infinitesimal character $[\Lambda]$ such that $\lambda = \Lambda\big|_{\mathfrak{a}} \in \mathcal{E}_V$. By Corollary 5.6 there exists a discrete series representation V_k for every $k \in \mathbb{N}$ with infinitesimal character $[(kN+1)\Lambda]$ and $\mathcal{E}_k := \mathcal{E}_{V_k} \subset [(kN+1)\Lambda]^+|_{\mathfrak{a}}$. Since $[\Lambda]^+|_{\mathfrak{a}} \subset -\operatorname{int}\mathcal{C}$, we have

$$\lim_{k\to\infty} \operatorname{dist}(\mathcal{E}_k, -\partial\mathcal{C}) \geq \lim_{k\to\infty} (kN+1) \operatorname{dist}([\Lambda]^+|_{\mathfrak{a}}, -\partial\mathcal{C}) = \infty\,.$$

It follows that for any $\mu \in \mathfrak{c}_{\mathbb{R}}^*$ there exists a k such that

$$\mathcal{E}_k + \operatorname{conv}\left(W_{\mathfrak{c}} \cdot \mu\big|_{\mathfrak{a}}\right) \subset -\operatorname{int}\mathcal{C}.$$

In view of Lemma 4.1 this implies that for every $m \in \mathbb{N}$ and any choice of fundamental representations $F_{\mu_1}, \ldots, F_{\mu_m}$ there exists a $n \in \mathbb{N}$ so that for every $k \in \mathbb{N}$ with $k \geq n$ all matrix coefficients of the representation

$$V_k \otimes F_{\mu_1} \otimes \ldots \otimes F_{\mu_m} \tag{5.4}$$

are contained in $L^2(G)$. Let $\tilde{\Lambda} \in [\Lambda]$ be the dominant element with respect to $\Sigma_{\mathfrak{c}}^+$. In view of [9, Theorem 1.2(1)] the representation (5.4) contains a subrepresentation with infinitesimal character $[(kN+1)\tilde{\Lambda} + \mu_1 + \ldots + \mu_m]$.

The proof will be finished by showing that $(kN+1)\tilde{\Lambda} + \mu_1 + \ldots + \mu_m$ is strongly regular for a suitable choice of $\mu_1, \ldots, \mu_m$ and for all k sufficiently large. The strongly regular elements comprise the complement of a finite union of proper subspaces of $\mathfrak{c}_{\mathbb{C}}^*$. We first choose m and $\mu_1, \ldots, \mu_m$ such that $\mu := \mu_1 + \cdots + \mu_m$ is outside of those subspaces which contain $\tilde{\Lambda}$. Then so is $(kN+1)\tilde{\Lambda} + \mu$ for any k. Clearly each remaining subspace can contain $(kN+1)\tilde{\Lambda} + \mu$ for at most one value of k. □

Corollary 5.8 (Harish-Chandra)**.** *If a real reductive group G admits a representation of the discrete series, then there exists a compact Cartan subalgebra.*

Proof. Combine Corollary 5.7 with Corollary 3.6. □

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Chapter III

On the little Weyl group of a real spherical space

Joint with Eitan Sayag.

Abstract

In the present paper we further the study of the compression cone of a real spherical homogeneous space $Z = G/H$. In particular we provide a geometric construction of the little Weyl group of Z introduced recently by Knop and Krötz. Our technique is based on a fine analysis of limits of conjugates of the subalgebra $\mathrm{Lie}(H)$ along one-parameter subgroups in the Grassmannian of subspaces of $\mathrm{Lie}(G)$. The little Weyl group is obtained as a finite reflection group generated by the reflections in the walls of the compression cone.

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†This article was published in Math. Ann (2020), https://doi.org/10.1007/s00208-022-02473-x

1 Introduction

In this article we present an elementary construction of the little Weyl group of a real homogeneous spherical space $Z = G/H$, which was first defined in [15]. Here G is the group of real points of an algebraic reductive group defined over $\mathbb{R}$ and H the set of real points of an algebraic subgroup. We assume that H is real spherical, i.e., a minimal parabolic subgroup P of G admits an open orbit in Z. Our construction does not rely on algebraic geometry. Instead we further develop the limit construction of spherical subalgebras from [19]. More specifically we use a fine analysis of limits of conjugates of the subalgebra $\mathrm{Lie}(H)$ along one-parameter subgroups in the Grassmannian of subspaces of $\mathrm{Lie}(G)$.

Our main interest is in G-invariant harmonic analysis on a real spherical homogeneous space Z. If Z admits a positive G-invariant Radon measure, then the space $L^2(Z)$ of square integrable functions on Z is a unitary representation for G. Recently large progress has been made towards a precise description of the Plancherel decomposition for real spherical spaces, see [18], [8], [19], [7], [5] and [20]. From the last two mentioned articles it is seen that the little Weyl group plays an important role in the multiplicities with which representations occur in $L^2(Z)$. Such a relationship was earlier observed in the work of Sakellaridis and Venkatesh on p-adic spherical spaces in [23] and the description of the Plancherel decomposition for real reductive symmetric spaces by Delorme, [6] and Van den Ban and Schlichtkrull [1], [2]. The theory we develop to construct the little Weyl group is central to our article [20], in which we determine the most continuous part of the Plancherel decomposition of a real spherical space.

For harmonic analysis it is important to understand the asymptotics of the generalized matrix-coefficients of H-invariant functionals on induced representations. An example of this is Theorem 5.1 in [19], where the asymptotics of an H-fixed linear functional is described in terms of a limit of translates of this functional. Such a limit-functional is no longer invariant under the action of $\mathrm{Lie}(H)$ or a conjugate of it, but rather by a corresponding limit of conjugates of $\mathrm{Lie}(H)$ in the Grassmannian of subspaces in $\mathrm{Lie}(G)$. In our approach the elements of the little Weyl group are obtained by examining such limit subalgebras.

We will now describe our construction and results. For convenience we assume that Z is quasi-affine, i.e., a Zariski open subvariety of an affine variety. For a point $z \in Z$ we write $\mathfrak{h}_z$ for its stabilizer subalgebra. We fix a minimal parabolic subgroup P of G and a Langlands decomposition $P = MAN$ of P. Given a direction $X \in \mathfrak{a} := \mathrm{Lie}(A)$ we consider the limit subalgebra

$$\mathfrak{h}_{z,X} = \lim_{t\to\infty} \mathrm{Ad}\left(\exp(tX)\right)\mathfrak{h}_z,$$

where the limit is taken in the Grassmannian. If X is contained in the negative Weyl chamber with respect to P, then the limit $\mathfrak{h}_{z,X}$ is up to M-conjugacy the same for all $z \in Z$ with the property that $P \cdot z$ is open. Such a limit is called a horospherical degeneration of $\mathfrak{h}_z$. We fix a horospherical degeneration $\mathfrak{h}_\emptyset$, i.e., $\mathfrak{h}_\emptyset = \mathfrak{h}_{z,X}$ for some choice of X in the negative Weyl chamber and $z \in Z$ for which $P \cdot z$ is open. The M-conjugacy class of a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ we denote by $[\mathfrak{s}]$. We define

$$\mathcal{N}_\emptyset := \{v \in N_G(\mathfrak{a}) : \mathrm{Ad}(v)[\mathfrak{h}_\emptyset] = [\mathfrak{h}_\emptyset]\},$$

which is a subgroup of G. For $z \in Z$ we further define

$$\mathcal{V}_z := \{v \in N_G(\mathfrak{a}) : [\mathfrak{h}_{z,X}] = \mathrm{Ad}(v)[\mathfrak{h}_\emptyset] \text{ for some } X \in \mathfrak{a}\}. \tag{1.1}$$

and the set of cosets

$$\mathcal{W}_z := \mathcal{V}_z/\mathcal{N}_\emptyset \subseteq G/\mathcal{N}_\emptyset.$$

The main result of the paper is that for a suitable choice of $z \in Z$ the above set admits the structure of a finite Coxeter group and

The group $\mathcal{W}_z$ is a finite crystallographic group, which can be identified with the little Weyl group as defined in [15].

Our strategy is to obtain the little Weyl group as a subquotient of the Weyl group $W(\mathfrak{g}, \mathfrak{a})$ by first determining a cone that can serve as a fundamental domain. The perspective of limit subalgebras suggests that for a given point $z \in Z$ we should consider all directions $X \in \mathfrak{a}$ for which the limit $\mathfrak{h}_{z,X}$ is M-conjugate to $\mathfrak{h}_\emptyset$, i.e., we should consider the cone

$$\mathcal{C}_z := \{X \in \mathfrak{a} : [\mathfrak{h}_{z,X}] = [\mathfrak{h}_\emptyset]\}.$$

If $P \cdot z$ is open, then $\mathcal{C}_z$ contains the negative Weyl-chamber and therefore has non-empty interior. However, in general the cone $\mathcal{C}_z$ strongly depends on the choice of z. It turns out that when $\mathcal{C}_z$ is maximal then it is a fundamental domain for a reflection group. Thus our first step is to identify points z for which the cone $\mathcal{C}_z$ is maximal. For this we introduce the concept of an *adapted point*.

The definition is motivated by the local structure theorem from [17]. The local structure theorem provides a canonical parabolic subgroup Q so that $P \subseteq Q$. Let $\mathfrak{l}_Q$ be the Levi-subalgebra of $\mathfrak{q} := \mathrm{Lie}(Q)$ that contains $\mathfrak{a}$. We denote by $^\perp$ the orthocomplement with respect to a G-invariant non-degenerate bilinear form on $\mathfrak{g}$. We say that a point $z \in Z$ is adapted to the Langlands decomposition $P = MAN$ if

(i) $P \cdot z$ is open

(ii) there exists an $X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp$ so that $Z_\mathfrak{g}(X) = \mathfrak{l}_Q$.

It follows from the local structure theorem that every open P-orbit in Z contains adapted points. Adapted points are special in the sense that their stabilizer subgroups H_z intersect with P in a clean way:

$$P \cap H_z = (M \cap H_z)(A \cap H_z)(N \cap H_z).$$

In fact $A \cap H_z$ and $N \cap H_z$ are the same for all adapted points z in Z. In the present article adapted points play a fundamental role because their cones $\mathcal{C}_z$ are of maximal size and identical. Therefore, $\mathcal{C} := \mathcal{C}_z$, where z is adapted, is an invariant of Z. It is called the *compression cone of Z*. The closure $\overline{\mathcal{C}}$ of the compression cone is a finitely generated convex cone. In general it is not a proper cone in the sense that it may contain a non-trivial subspace; in fact $\mathfrak{a}_\mathfrak{h} := \mathrm{Lie}(A \cap H_z)$ is contained in the edge of $\overline{\mathcal{C}}$. We denote the projection $\mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_\mathfrak{h}$ by $p_\mathfrak{h}$. The cone $p_\mathfrak{h}(\overline{\mathcal{C}})$ is finitely generated convex cone in $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$. It is this cone that will be a fundamental domain for the little Weyl group.

The passage from the cone $p_\mathfrak{h}(\overline{\mathcal{C}})$ to the reflection group requires some multiplication law among certain cosets of $W(\mathfrak{g}, \mathfrak{a})$. For this we consider $N_G(\mathfrak{a})$-conjugates of $\mathfrak{h}_\emptyset$ that appear as a limit $\mathfrak{h}_{z,X}$, i.e., we consider the set $\mathcal{V}_z$ defined in (1.1). If $z \in Z$ is adapted and $v \in \mathcal{V}_z$, then $v^{-1} \cdot z$ is again adapted. If moreover, $v' \in \mathcal{V}_{v^{-1}\cdot z}$ then there exists an $X \in \mathfrak{a}$ so that

$$[\mathfrak{h}_{z,\mathrm{Ad}(v)X}] = \mathrm{Ad}(v)[\mathfrak{h}_{v^{-1}\cdot z,X}] = \mathrm{Ad}(vv')[\mathfrak{h}_\emptyset],$$

and hence $vv' \in \mathcal{V}_z$. If $\mathcal{V}_{v^{-1}\cdot z}$ would be the same for all $v \in \mathcal{V}_z$, then this would define a product map on $\mathcal{V}_z$. However, a priori this is not the case for all adapted points z. It turns out that this can be achieved by restricting further to *admissible points*, i.e., adapted points z for which the limits $\mathfrak{h}_{z,X}$ are conjugate to $\mathfrak{h}_\emptyset$ for all $X \in \mathfrak{a}$ outside of a finite set of hyperplanes. One of the main results in this article is that admissible points exist; in fact every open P-orbit in Z contains admissible points. Moreover, the sets $\mathcal{V} := \mathcal{V}_z$ are the same for all admissible points.

The set $\mathcal{V}$ is contained in

$$\mathcal{N} := N_G(\mathfrak{a}) \cap N_G(\mathfrak{l}_{Q,\mathrm{nc}} + \mathfrak{a}_\mathfrak{h}),$$

where $\mathfrak{l}_{Q,\mathrm{nc}}$ is the sum of all non-compact simple ideals in $\mathfrak{l}_Q$. The group $\mathcal{N}_\emptyset$ is a normal subgroup of $\mathcal{N}$ and $\mathcal{N}/\mathcal{N}_\emptyset$ is finite. We define

$$\mathcal{W} := \mathcal{V}/\mathcal{N}_\emptyset \subseteq \mathcal{N}/\mathcal{N}_\emptyset. \tag{1.2}$$

Now $\mathcal{W}$ is finite and closed under multiplication in the group $\mathcal{N}/\mathcal{N}_\emptyset$. It therefore is a group. We now come to our main theorem, see Theorem 12.1.

Theorem 1.1. *The following assertions hold true.*

(i) The group $\mathcal{W}$ is a subgroup of $\mathcal{N}/\mathcal{N}_\emptyset$, and as such it is a subquotient of the Weyl group $W(\mathfrak{g}, \mathfrak{a})$ of the root system of $\mathfrak{g}$ in $\mathfrak{a}$.

(ii) The group $\mathcal{W}$ acts faithfully on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$ as a finite crystallographic group, i.e. it is a finite group generated by reflections $s_1, \ldots, s_l$ and for each i, j the order $m_{i,j}$ of $s_i s_j$ is contained in the set $\{1, 2, 3, 4, 6\}$.

(iii) The cone $p_\mathfrak{h}(\overline{\mathcal{C}})$ is a fundamental domain for the action of $\mathcal{W}$ on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$. Moreover, $\mathcal{W}$ is generated by the simple reflections in the walls $p_\mathfrak{h}(\overline{\mathcal{C}})$.

In fact, $\mathcal{W}$ is equal to the little Weyl group of Z as defined in [15, Section 9].

For the proof of the theorem we use two results from the literature. The first is the local structure theorem from [17], which we use to establish the existence of adapted points. The second is the polar decomposition from [16], which we use to describe the closure of $\mathrm{Ad}(G)\mathfrak{h}_z$ in the Grassmannian. Besides these two theorems the proof is essentially self-contained. It is based on an analysis of the limits $\mathfrak{h}_{z,X}$, where $z \in Z$ is adapted and $X \in \mathfrak{a}$.

The heart of the proof is to show the existence of admissible points. For this we first classify the adapted points and study the correspondence between adapted points in Z and in its boundary degenerations. For the horospherical boundary degeneration $G/H_\emptyset$ of Z the existence of such points is clear, but it cannot be used to deduce anything for Z. We thus consider the second most degenerate boundary degenerations, for which the existence of admissible points can be proven by a non-trivial direct computation. The existence of admissible points in Z is then proven by a reduction to these boundary degenerations. The realization of $\mathcal{W}$ as a reflection group is then obtained from the natural relation between the little Weyl group of a space and its degenerations.

For convenience of the reader we give a short description of each section in this paper. In §2 we recall the definition of a real spherical space and introduce our notations and basic assumptions. We then properly start in §3 by defining the notion of adapted points. We further prove several properties of adapted points, in particular that they satisfy the main conclusion from the local structure theorem, and we provide a kind of parametrization. In the short section §4 we provide a description of the stabilizer subalgebra $\mathfrak{h}_z$ of an adapted point z in terms of a linear map T_z. This description is a direct generalization of Brion's description in the complex case ([4, Proposition 2.5]) and was also used in [16]. In the following section, Section §5, we discuss limits in the Grassmannian of k-dimensional subspaces of the Lie algebra $\mathfrak{g}$ and we collect all properties of such limits that will be needed in the following sections. We introduce the compression cone in §6. The main result in the section is that the compression cone $\mathcal{C}_z$ is of maximal size if z is adapted and does not depend on the choice of the adapted point. It therefore is an invariant of Z.

In §7 we describe the relation between limits subalgebras, open P-orbits in Z and the compression cone. This description gives the first indication that the little Weyl group may be constructed from such limits. The sections §8 and §10 serve as a preparation for the proof of the existence of admissible points. In §8 we describe the $\mathrm{Ad}(G)$-orbits in the closure of $\mathrm{Ad}(G)\mathfrak{h}_z$ in the Grassmannian. Each of the subalgebras in this closure gives rise to a boundary degeneration of Z, i.e., a real spherical homogeneous space which is determined by a subalgebra contained in the closure of $\mathrm{Ad}(G)\mathfrak{h}_z$. In §10 we show that there is a correspondence between adapted points in Z and adapted points in a boundary degeneration. After these preparations we can prove the existence of admissible points in §11. This is done through a reduction to the same problem for the second-most degenerate boundary degenerations of Z.

In §12 we finally define the set $\mathcal{W}$ by (1.1) and (1.2) using an admissible point z. We then prove that $\mathcal{W}$ has the properties listed in Theorem 1.1. It is relatively easy to see that $\mathcal{W}$ is a group acting on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$. For the proof that it is generated by reflections an explicit computation on the walls of the compression cone is needed. This computation is performed in Section 9.

In Section 13 we prove that the group $\mathcal{W}$ is a crystallographic group and show how to attach to it a reduced root system, the spherical root system.

The technique developed in the body of the paper works under the assumption that

Z is quasi-affine. In Section 14 we extend many of the concepts that were studied in the previous sections to any real spherical space, in particular we construct the little Weyl group $\mathcal{W}$ and hence the reduced root system Σ_Z in this generality. This is done by a standard trick that is based on a theorem of Chevalley.

We end this introduction with a short account of related works on the little Weyl group. Recall that an algebraic $\mathbf{G}$-variety $\mathbf{Z}$ defined over $k = \mathbb{C}$ is called spherical if a Borel subgroup of $\mathbf{G}$ defined over k admits an open orbit in $\mathbf{Z}$. Here $\mathbf{G}$ is an algebraic connected reductive group defined over k. In [4] Brion first introduced the compression cone for complex spherical varieties and showed that the asymptotic behavior of such varieties is determined by a root system: the spherical root system. The little Weyl group is the Weyl group for this root system.

By now, there are several constructions of the little Weyl group for complex varieties. Next to the construction of Brion, Knop gave a vast generalization. In fact, in [11] he constructed the little Weyl group for an arbitrary irreducible $\mathbf{G}$-variety and connected it to the ring of $\mathbf{G}$-invariant differential operators on $\mathbf{Z}$, see [13]. We also mention here a second construction by Knop in [12] and the explicit calculation of these groups by Losev [21].

In the case where k is an algebraically closed field of characteristic different from 2, Knop gave in [10] a construction of the little Weyl group and the spherical root system. The technique is close in spirit to Brion's approach for $k = \mathbb{C}$. Moving to fields that are not necessarily algebraically closed, a natural concept is that of a k-spherical variety, i.e., a $\mathbf{G}$-variety $\mathbf{Z}$ defined over k for which a minimal parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ defined over k admits an open orbit. In [15], the authors assume that k is of characteristic 0 and use algebraic geometry to define the little Weyl group of such a space $Z = \mathbf{Z}(k)$. The construction is based on algebra geometric invariants attached to the variety $\mathbf{Z}$, especially the cone of $\mathbf{G}$-invariant central valuations on $\mathbf{Z}$, as is the case for Knop's construction for $k = \mathbb{C}$ in [11]. This valuation cone serves as a fundamental domain for the action of $W_{\mathbf{Z}}$.

The compression cone plays an important role in this work. It was first considered for real spherical spaces in [16] by employing the local structure theorem of [17]. In [4] Brion showed that in the complex case the closure of the compression cone may be identified with the valuation cone. This argument generalizes to real spherical spaces.

The compression cone can be viewed as a dual object to the weight-monoid used by algebraic geometers to study spherical spaces. In the present work the compression cone is defined purely in terms of limits of subalgebras in the Grassmannian and is from our point of view better suited for application in harmonic analysis, like in [19]. We mention here our article [20], in which we determine the Plancherel decomposition of the most continuous part of $L^2(Z)$. A major step towards this is the construction of H-fixed functionals on principal series representations. For the analysis of P-orbits that is needed for this, we use the theory of limits of subalgebras.

Our approach to the little Weyl group is closest to that taken by Brion in his article [4] on complex spherical spaces. However, there are notable differences. Brion studies the relation between the closure of $\mathrm{Ad}(G)\mathfrak{h}_z$ in the Grassmannian and the wonderful compactification. In our approach compactifications do not enter directly. Further, Brion uses explicit computations related to the structure of $\mathfrak{h}_z$ for a well chosen point z. Some of these computations are adapted in Section 9 to the case of real spherical spaces. It appears that Brion's computations do not generalize easily to real spherical spaces as they rely on the fact that root spaces are 1-dimensional. We therefore put more attention to the compression cone and the limits $\mathfrak{h}_{z,X}$ for generic elements $X \in \mathfrak{a}$ and adapted points $z \in Z$. We do not fix a specific point z, but rather study the dependence of compression cones and limit subalgebras $\mathfrak{h}_{z,X}$ on adapted points z. In particular we obtain the group law for the little Weyl group from these considerations as explained above, rather than from explicit computations.

We thank Bernhard Krötz, Friedrich Knop and Vladimir Zhgoon for various discussions on the subject matter of this paper.

2 Notation and assumptions

Let $\underline{G}$ be a reductive algebraic group defined over $\mathbb{R}$ and let G be an open subgroup of $\underline{G}(\mathbb{R})$. Let H be a closed subgroup of G. We assume that there exists a subgroup $\underline{H}$ of $\underline{G}$ defined over $\mathbb{R}$ so that $H = G \cap \underline{H}(\mathbb{C})$. We define

$$Z := G/H$$

We fix a minimal parabolic subgroup P of G and a Langlands decomposition $P = MAN$. We assume that Z is real spherical, i.e., there exists an open P-orbit in Z.

Until Section 14 we assume that Z is quasi-affine. The assumption is used in one place only, namely for Proposition 3.1. In Section 3 we will define a notion of adapted points in Z. Proposition 3.1, and therefore the assumption that Z is quasi-affine, is needed to show that adapted points exist. In Section 14 we will consider real spherical spaces Z that are not necessarily quasi-affine and describe a reduction to the quasi-affine case.

Groups are indicated by capital roman letters. Their Lie algebras are indicated by the corresponding lower-case fraktur letter. If $z \in Z$, then the stabilizer subgroup of Z is indicated by H_z and its Lie algebra by $\mathfrak{h}_z$.

The root system of $\mathfrak{g}$ in $\mathfrak{a}$ we denote by Σ. If Q is a parabolic subgroup containing A we write $\Sigma(Q)$ for the subset of Σ of roots that occur in the nilpotent radical of $\mathfrak{q}$. We write Σ^+ for $\Sigma(P)$. We further write $\mathfrak{a}^-$ for the open negative Weyl chamber, i.e.,

$$\mathfrak{a}^- := \{X \in \mathfrak{a} : \alpha(X) < 0 \text{ for all } \alpha \in \Sigma^+\}.$$

We fix a Cartan involution θ of G that stabilizes A. If Q is a parabolic subgroup containing A, then we write $\overline{Q}$ for the opposite parabolic subgroup containing A, i.e., $\overline{Q} = \theta(Q)$. The unipotent radical of Q we denote by N_Q. We further agree to write $\overline{N}_Q$ for $N_{\overline{Q}}$.

We fix an $\mathrm{Ad}(G)$-invariant bilinear form B on $\mathfrak{g}$ so that $-B(\,\cdot\,,\theta\,\cdot\,)$ is positive definite. For $E \subseteq \mathfrak{g}$, we define

$$E^\perp = \big\{X \in \mathfrak{g} : B(X,E) = \{0\}\big\}.$$

If E is a finite dimensional real vector space, then we write $E_\mathbb{C}$ for its complexification $E \otimes_\mathbb{R} \mathbb{C}$. If S is an algebraic subgroup of G, then we write $S_\mathbb{C}$ for the complexification of S.

3 Adapted points

In this section we introduce the notion of an adapted point in Z. We further parameterize the set of adapted points and end the section with some applications which will be of use in the following sections.

We recall that we have fixed a minimal parabolic subgroup P and a Langlands decomposition $P = MAN_P$ of P. For $z \in Z$, let H_z be the stabilizer of z in G and let $\mathfrak{h}_z$ be the Lie algebra of H_z.

The following proposition is a reformulation of the so-called local structure theorem [17, Theorem 2.3].

Proposition 3.1. *There exists a parabolic subgroup Q with $P \subseteq Q$, and a Levi decomposition $Q = L_Q N_Q$ with $A \subseteq L_Q$, so that for every open P-orbit $\mathcal{O}$ in Z*

$$Q \cdot \mathcal{O} = \mathcal{O},$$

and there exists a $z \in \mathcal{O}$, so that the following hold,

(i) $Q \cap H_z = L_Q \cap H_z$,

(ii) the map

$$N_Q \times L_Q/L_Q \cap H_z \to Z, \qquad \big(n, l(L_Q \cap H_z)\big) \mapsto nl \cdot z$$

is a diffeomorphism onto $\mathcal{O}$,

(iii) the sum $\mathfrak{l}_{Q,\mathrm{nc}}$ of all non-compact simple ideals in $\mathfrak{l}_Q$ is contained in $\mathfrak{h}_z$,

(iv) there exists an $X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp$ so that $L_Q = Z_G(X)$ and $\alpha(X) > 0$ for all $\alpha \in \Sigma(Q)$.

Remark 3.2.

(i) The point $z \in \mathcal{O}$ with the properties asserted in the above proposition is in general not unique. On the other hand the parabolic subgroup Q and its Levi-decomposition are uniquely determined by $\mathcal{O}$. (Of course, the parabolic subgroup Q and the Levi decomposition of Q do depend on the choice of the minimal parabolic P and its Langlands decomposition $P = MAN_P$, but these choices we have assumed to be fixed.)

(ii) Property (iv) in Proposition 3.1 is not explicitly stated in [17, Theorem 2.3], but does follow from the proof of the theorem if Z is quasi-affine. For completeness, we give here an account of how this follows.

Let $z_0 \in Z$ be so that $P \cdot z_0$ is open. In the proof of [17, Theorem 2.3] an iterative process is used to produce a sequence of parabolic subgroups

$$G = Q_0 \supseteq Q_1 \supseteq Q_2 \supseteq \dots,$$

each containing P. Further, for each $i \in \mathbb{N}$ a hyperbolic element $X_i \in (\mathfrak{l}_{i-1} \cap \mathfrak{h}_{z_0})^\perp$ is constructed, with the property that $L_i := Z_G(X_i)$ is a Levi subgroup of Q_i and the restriction of $\mathrm{ad}(X_i)$ to $\mathfrak{n}_{Q_i}$ has only strictly negative eigenvalues. Since the sequence of parabolic subgroups descends it stabilizes, and hence there exists a parabolic subgroup Q with $Q_i = Q$ for sufficiently large $i \in \mathbb{N}$. This is the parabolic subgroup Q in Proposition 3.1.

Since $A \subseteq Q \subseteq Q_i$ and $Z_G(X_i)$ is a Levi subgroup of Q_i, there exists an $n \in N_{Q_i}$ so that $A \subseteq L_i := nZ_G(X_i)n^{-1} = Z_G\big(\mathrm{Ad}(n)X_i\big)$. Moreover, as $\mathrm{Ad}(n)X_i$ is an hyperbolic element in $\mathfrak{l}_i$, there exists an $l \in L_i$ so that $X_i' := \mathrm{Ad}(ln)X_i \in \mathfrak{a}$. We set $z_i := ln \cdot z_0 \in P \cdot z_0$. Now $L_i = Z_G(X_i')$ and $X_i' \in (\mathfrak{a} \cap \mathfrak{h}_{z_i})^\perp$. We set $z := z_i$, $L_Q := L_i$ and $X' := X_i$ for some $i \in \mathbb{N}$ with $Q_i = Q$. Then Q, L_Q and z satisfy (i), (ii) and (iii) in the proposition.

Each iteration uses a finite dimensional representation as input. To be more precise, for the i-th iteration a finite dimensional representation of L_{i-1} is used as input with the property that it contains a cyclic vector whose stabilizer is $L_{i-1} \cap H_{z_{i-1}}$. As Z is assumed to be quasi-affine, the theorem of Chevalley guarantees the existence of such representations. The representation that is used can freely be chosen from the set of representations with the mentioned property. If for the first iteration a representation is chosen with the additional requirement that it contains a lowest weight that does not vanish on any of the $\alpha^\vee$ with $\alpha \in \Sigma(Q)$, then the process yields $Q_1 = Q$, and hence only one iteration is needed. Moreover, in this case $X := -X_1'$ has the property listed in (iv). It thus remains to show that there exists a finite dimensional representation of G with a lowest weight that does not vanish on $\alpha^\vee$ for every $\alpha \in \Sigma(Q)$ and that contains a cyclic vector whose stabilizer is equal to H_z.

It follows from [17, Lemma 3.4 & Remark 3.5] that the lowest weights of irreducible finite dimensional H_z-spherical representations span $\big(\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{h}_z)\big)^*$. Therefore, the lattice of lowest weights of H_z-spherical representations contains a weight λ so that $\lambda(\alpha^\vee) \neq 0$ for all $\alpha \in \Sigma$ with $\alpha^\vee \notin \mathfrak{a} \cap \mathfrak{h}_z$. Let $X' \in (\mathfrak{a} \cap \mathfrak{h}_z)^\perp$ be as above. As the centralizer of X' is equal to L_Q, it follows that for every $\alpha \in \Sigma$ we have $\alpha^\vee \in \mathfrak{a} \cap \mathfrak{h}_z$ only if $\mathfrak{g}_\alpha \subseteq \mathfrak{l}_Q$. Therefore, there exists an irreducible finite dimensional H_z-spherical representation V with lowest weight λ so that $\lambda(\alpha^\vee) \neq 0$ for all $\alpha \in \Sigma(Q)$. Let W be any finite dimensional representation that contains a cyclic vector whose stabilizer is equal to H_z. Then for sufficiently large $n \in \mathbb{N}$ the representation $W \otimes V^{\otimes n}$ contains a cyclic vector whose stabilizer is equal to H_z and admits a lowest weight does not vanish on any of the $\alpha^\vee$ with $\alpha \in \Sigma(Q)$, as requested.

The assumption that Z is quasi-affine is crucial here. Up until Section 14 this is the only place where the assumption is explicitly used.

Definition 3.3. We say that a point $z \in Z$ is adapted (to the Langlands decomposition $P = MAN_P$) if the following three conditions are satisfied.

(i) $P \cdot z$ is open in Z, i.e., $\mathfrak{p} + \mathfrak{h}_z = \mathfrak{g}$,

(ii) $\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathfrak{h}_z$,

(iii) There exists an $X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp$ so that $Z_\mathfrak{g}(X) = \mathfrak{l}_Q$.

Remark 3.4.

(a) It follows from Proposition 3.1, that every open P-orbit $\mathcal{O}$ in Z contains an adapted point.

(b) If a point $z \in Z$ satisfies (i) and (iii), then (ii) is automatically satisfied. We will give a proof of this fact later in this section, see Proposition 3.19.

(c) (iii) can be stated alternatively as

(iii') There exists an $X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp$ so that $\alpha(X) \neq 0$ for all $\alpha \in \Sigma(Q)$.

(d) The set of adapted points in Z is L_Q-stable. To see this, let $z \in Z$ be adapted. The Levi-subgroup L_Q decomposes as

$$L_Q = MAL_{Q,\mathrm{nc}}, \tag{3.1}$$

where $L_{Q,\mathrm{nc}}$ is the connected subgroup with Lie algebra $\mathfrak{l}_{Q,\mathrm{nc}}$. Note that

$$L_{Q,\mathrm{nc}} \subseteq H_z \tag{3.2}$$

since $\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathfrak{h}_z$. Let $m \in M$, $a \in A$ and $l \in L_{Q,\mathrm{nc}}$. Then $l \cdot z = z$, and therefore, $Pmal \cdot z = P \cdot z$ is open and

$$\mathfrak{a} \cap \mathfrak{h}_{mal\cdot z}^\perp = \mathfrak{a} \cap \mathrm{Ad}(ma)\mathfrak{h}_z^\perp = \mathrm{Ad}(ma)\big(\mathfrak{a} \cap \mathfrak{h}_z^\perp\big) = \mathfrak{a} \cap \mathfrak{h}_z^\perp.$$

Moreover, $\mathfrak{l}_{Q,\mathrm{nc}}$ is L_Q-stable and hence $\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathrm{Ad}(l)\mathfrak{h}_z = \mathfrak{h}_{l\cdot z}$ for all $l \in L_Q$. This proves the assertion.

Example 3.5. Let $Z = G/\overline{N}_P$ and let $z := e \cdot \overline{N}_P$. We claim that the set of adapted points is equal to $MA \cdot z$.

Let $W := N_G(A)/MA$ be the Weyl group of Σ. The Bruhat decomposition of G provides a description of $P\backslash Z$,

$$Z = \bigsqcup_{w \in W} Pw \cdot z.$$

There is only one open P-orbit in Z, namely $\mathcal{O} := P \cdot z$. Since for every $p \in P$

$$\mathfrak{p} \cap \mathfrak{h}_{p\cdot z} = \mathfrak{p} \cap \mathrm{Ad}(p)\overline{\mathfrak{n}}_P = \{0\},$$

we have $Q = P$. It is now easy to see that z satisfies (i) – (iv) in Proposition 3.1. Since the set of adapted points is MA-stable, it suffices to show that the only adapted point in $N_P \cdot z$ is z in order to prove the claim. Let $n \in N_P$ and assume that $n \cdot z$ is adapted. Now $\mathfrak{h}_{n\cdot z}^\perp = \mathrm{Ad}(n)\overline{\mathfrak{p}}$, and hence

$$\mathfrak{a} \cap \mathfrak{h}_{n\cdot z}^\perp = \mathfrak{a} \cap \mathrm{Ad}(n)\overline{\mathfrak{p}} \subseteq \mathfrak{p} \cap \mathrm{Ad}(n)\overline{\mathfrak{p}} = \mathrm{Ad}(n)\big(\mathfrak{p} \cap \overline{\mathfrak{p}}\big) = \mathrm{Ad}(n)(\mathfrak{m} \oplus \mathfrak{a}).$$

Since $n \cdot z$ is adapted, there exists a regular element $X \in \mathfrak{a} \cap \mathfrak{h}_{n\cdot z}^\perp$. It follows that $X \in \mathrm{Ad}(n)(\mathfrak{m} \oplus \mathfrak{a})$, and hence $\mathrm{Ad}(n^{-1})X \in \mathfrak{m} \oplus \mathfrak{a}$. This implies that n stabilizes X. Since X is regular, it follows that $n = e$.

Proposition 3.6. *Let $z \in Z$ be adapted. Then the following hold.*

(i) $Q \cap H_z = L_Q \cap H_z$,

(ii) The map

$$N_Q \times L_Q/L_Q \cap H_z \to Z, \qquad \big(n, l(L_Q \cap H_z)\big) \mapsto nl \cdot z$$

is a diffeomorphism onto $P \cdot z$.

Remark 3.7.

(a) The proposition shows that besides (iii) and a weaker version of (iv), which hold by definition, also (i) and (ii) in Proposition 3.1 hold for adapted points $z \in Z$.

(b) Let $z \in Z$ be adapted. We claim that

$$MA \cap H_z = (M \cap H_z)(A \cap H_z) = (M \cap H_z)\exp(\mathfrak{a} \cap \mathfrak{h}_z). \tag{3.3}$$

To prove the claim, we first note that $MA \cap H_z$, $M \cap H_z$ and $A \cap H_z$ are algebraic subgroups of G, and that $M \cap H_z$ is a normal subgroup of $MA \cap H_z$. Define A' and M' to be the images of the projections of $MA \cap H_z$ onto A and M, respectively. Then A' and M' are algebraic subgroups of A and M, respectively. Moreover, $A \cap H_z$ and $M \cap H_z$ are normal subgroups of A' and M', respectively. Let

$$\phi : A'/(A \cap H_z) \to M'/(M \cap H_z)$$

be the unique map so that

$$a\phi(a) \in (MA \cap H_z)/(M \cap H_z)(A \cap H_z) \qquad \big(a \in A'/(A \cap H_z)\big).$$

Then ϕ is an algebraic homomorphism. An algebraic homomorphism from a split torus to a compact group is necessarily trivial. It follows that $A' = A \cap H_z$, and hence $M' = M \cap H_z$. Moreover, the group $A \cap H_z$ is connected since $A \cap H_z$ is an algebraic subgroup of A and A is isomorphic to a vector space. This proves (3.3). From (3.1), (3.2) and (3.3) it follows that

$$M/(M \cap H_z) \times A/\exp(\mathfrak{a}_{\mathfrak{h}}) \to L_Q \cdot z; \qquad (m(M \cap H_z), a\exp(\mathfrak{a}_{\mathfrak{h}})) \mapsto ma \cdot z$$

is a diffeomorphism. Therefore, if $z \in Z$ is adapted, then (ii) in Proposition 3.6 can be replaced by

(ii') The map

$$\begin{aligned} N_Q \times M/(M \cap H_z) \times A/\exp(\mathfrak{a} \cap \mathfrak{h}_z) &\to Z; \\ \big(n, m(M \cap H_z), a\exp(\mathfrak{a} \cap \mathfrak{h}_z)\big) &\mapsto nma \cdot z \end{aligned}$$

is a diffeomorphism onto $P \cdot z$.

Before we prove the proposition, we first prove a lemma.

Lemma 3.8. *Let $z \in Z$ be adapted and let $q \in Q$. Then $q \in L_Q$ if and only if there exists an element*

$$X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp \cap \mathfrak{h}_{q\cdot z}^\perp$$

so that $\mathfrak{l}_Q = Z_\mathfrak{g}(X)$ (and thus $\alpha(X) \neq 0$ for all $\alpha \in \Sigma(Q)$). In that case

$$\mathfrak{a} \cap \mathfrak{h}_z^\perp = \mathfrak{a} \cap \mathfrak{h}_{q\cdot z}^\perp.$$

Proof. Assume that there exists an element $X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp \cap \mathfrak{h}_{q\cdot z}^\perp$ so that $\alpha(X) \neq 0$ for all $\alpha \in \Sigma(Q)$. Let $l \in L_Q$ and $n \in N_Q$ be so that $q = ln$. We will show that $n = e$. Since $\alpha^\vee \in \mathfrak{h}_z$ for every $\alpha \in \Sigma$ with $\mathfrak{g}_\alpha \subseteq \mathfrak{l}_Q$, we have $\alpha(X) = 0$ for these roots. This implies that $\mathfrak{l}_Q$ centralizes X. Now in view of (3.1) also the group L_Q centralizes X. It follows that

$$\mathrm{Ad}(n^{-1})X = \mathrm{Ad}(n^{-1}l^{-1})X = \mathrm{Ad}(q^{-1})X \in \mathrm{Ad}(q^{-1})\mathfrak{h}_{q\cdot z}^\perp = \mathfrak{h}_z^\perp,$$

and hence $\mathrm{Ad}(n^{-1})X - X$ is contained in both $\mathfrak{h}_z^\perp$ and $\mathfrak{n}_Q$. However, since $P \cdot z$ is open, we have

$$\mathfrak{h}_z^\perp \cap \mathfrak{n}_Q = (\mathfrak{h}_z + \mathfrak{q})^\perp = \mathfrak{g}^\perp = \{0\}.$$

Hence $\mathrm{Ad}(n^{-1})X = X$. Since $\alpha(X) \neq 0$ for every $\alpha \in \Sigma(Q)$, it follows that $n = e$.

Now assume that $q \in L_Q$. It follows from (3.1) that there exist $m \in M$, $a \in A$ and $l_{\mathrm{nc}} \in L_{Q,\mathrm{nc}}$ so that $q = mal_{\mathrm{nc}}$. Since l_{nc} is contained in H_z, it normalizes $\mathfrak{h}_z^\perp$ and hence

$$\mathfrak{a} \cap \mathfrak{h}_{q\cdot z}^\perp = \mathfrak{a} \cap \mathrm{Ad}(q)\mathfrak{h}_z^\perp = \mathfrak{a} \cap \mathrm{Ad}(ma)\mathfrak{h}_z^\perp = \mathrm{Ad}(ma)\big(\mathfrak{a} \cap \mathfrak{h}_z^\perp\big) = \mathfrak{a} \cap \mathfrak{h}_z^\perp.$$

The latter set contains an element X with $Z_\mathfrak{g}(X) = \mathfrak{l}_Q$ in view of Definition 3.3. ☐

Proof of Proposition 3.6. Let $q \in Q \cap H_z$. Then $\mathfrak{h}_{q\cdot z}^\perp = \mathrm{Ad}(q)\mathfrak{h}_z^\perp = \mathfrak{h}_z^\perp$. Since there exists an element $X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp$ so that $Z_\mathfrak{g}(X) = \mathfrak{l}_Q$, it follows from Lemma 3.8 that $q \in L_Q$. Therefore, $Q \cap H_z \subseteq L_Q \cap H_z$. The other inclusion is trivial. This proves (i).

The map $Q/(Q \cap H_z) \to Z$, $q \mapsto q \cdot z$ is a diffeomorphism onto $Q \cdot z$. Since also $N_Q \times L_Q \to Q$ is a diffeomorphism and $P \cdot z = Q \cdot z$ by Proposition 3.1, assertion (ii) follows from (i). ☐

We move on to give a description of the adapted points in Z. We begin with a lemma parameterizing the points that satisfy (ii) in Definition 3.3 and the infinitesimal version of (i) in Proposition 3.6.

Lemma 3.9. *Fix an adapted point $z \in Z$. Let $z' \in P \cdot z$. Then*

$$\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathfrak{q} \cap \mathfrak{h}_{z'} = \mathfrak{l}_Q \cap \mathfrak{h}_{z'} \tag{3.4}$$

if and only if there exist $m \in M$, $a \in A$ and $n \in Z_{N_Q}(\mathfrak{l}_Q \cap \mathfrak{h}_z)$ so that $z' = man \cdot z$. In that case

$$\mathfrak{l}_Q \cap \mathfrak{h}_{z'} = \mathrm{Ad}(m)\big(\mathfrak{l}_Q \cap \mathfrak{h}_z\big), \tag{3.5}$$

and hence in particular

$$\mathfrak{a} \cap \mathfrak{h}_{z'} = \mathfrak{a} \cap \mathfrak{h}_z. \tag{3.6}$$

Proof. Let $n \in Z_{N_Q}(\mathfrak{l}_Q \cap \mathfrak{h}_z)$. Since $\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathfrak{l}_Q \cap \mathfrak{h}_z$, the element n centralizes $\mathfrak{l}_{Q,\mathrm{nc}}$, and hence

$$\mathfrak{l}_{Q,\mathrm{nc}} = \mathrm{Ad}(n)\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathrm{Ad}(n)\big(\mathfrak{q} \cap \mathfrak{h}_z\big) = \mathfrak{q} \cap \mathrm{Ad}(n)\mathfrak{h}_z = \mathfrak{q} \cap \mathfrak{h}_{n\cdot z}.$$

Moreover, as

$$\mathfrak{l}_Q \cap \mathfrak{h}_z \subseteq \mathrm{Ad}(n)\mathfrak{h}_z = \mathfrak{h}_{n\cdot z}$$

and $\mathfrak{q} \cap \mathfrak{h}_z = \mathfrak{l}_Q \cap \mathfrak{h}_z$ by Proposition 3.6 (ii), we have

$$\mathfrak{l}_Q \cap \mathfrak{h}_z \subseteq \mathfrak{l}_Q \cap \mathfrak{h}_{n\cdot z} \subseteq \mathfrak{q} \cap \mathfrak{h}_{n\cdot z} = \mathrm{Ad}(n)(\mathfrak{q} \cap \mathfrak{h}_z) = \mathrm{Ad}(n)(\mathfrak{l}_Q \cap \mathfrak{h}_z) = \mathfrak{l}_Q \cap \mathfrak{h}_z.$$

It follows that $\mathfrak{q} \cap \mathfrak{h}_{n\cdot z} = \mathfrak{l}_Q \cap \mathfrak{h}_{n\cdot z}$. We have now proven (3.4) for $z' = n \cdot z$. The subalgebras $\mathfrak{l}_{Q,\mathrm{nc}}$, $\mathfrak{q}$ and $\mathfrak{l}_Q$ are MA-stable. Therefore,

$$\mathfrak{l}_{Q,\mathrm{nc}} = \mathrm{Ad}(ma)\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathrm{Ad}(ma)(\mathfrak{q} \cap \mathfrak{h}_{n\cdot z}) = \mathfrak{q} \cap \mathfrak{h}_{man\cdot z}$$

and

$$\mathrm{Ad}(ma)(\mathfrak{q} \cap \mathfrak{h}_{n\cdot z}) = \mathrm{Ad}(ma)(\mathfrak{l}_Q \cap \mathfrak{h}_{n\cdot z}) = \mathfrak{l}_Q \cap \mathfrak{h}_{man\cdot z}.$$

This proves that (3.4) holds as well for $z' = man \cdot z$.

For the converse implication, let $z' \in Z$ and assume that (3.4) holds. By Remark 3.7 (b) there exist $m \in M$, $a \in A$ and $n \in N_Q$ so that $z' = man \cdot z$. Since $\mathfrak{q} \cap \mathfrak{h}_z = \mathfrak{l}_Q \cap \mathfrak{h}_z$ by Proposition 3.6 (ii), we have

$$\mathrm{Ad}(n)\big(\mathfrak{l}_Q \cap \mathfrak{h}_z\big) = \mathrm{Ad}(n)\big(\mathfrak{q} \cap \mathfrak{h}_z\big) = \mathrm{Ad}(ma)^{-1}\big(\mathfrak{q} \cap \mathfrak{h}_{man\cdot z}\big) = \mathrm{Ad}(ma)^{-1}\big(\mathfrak{l}_Q \cap \mathfrak{h}_{man\cdot z}\big).$$

The space on the right-hand side is contained in $\mathfrak{l}_Q$. It follows that $\mathrm{Ad}(n)\big(\mathfrak{l}_Q \cap \mathfrak{h}_z\big) \subseteq \mathfrak{l}_Q$. Now for every $Y \in \mathfrak{l}_Q \cap \mathfrak{h}_z$

$$\mathrm{Ad}(n)Y \in (Y + \mathfrak{n}_Q) \cap \mathfrak{l}_Q = Y + (\mathfrak{n}_Q \cap \mathfrak{l}_Q) = Y + \{0\}.$$

We thus conclude that n centralizes $\mathfrak{l}_Q \cap \mathfrak{h}_z$.

We continue to prove the identities (3.5) and (3.6). Let $m \in M$, $a \in A$ and $n \in Z_{N_Q}(\mathfrak{l}_Q \cap \mathfrak{h}_z)$. Then

$$\mathfrak{l}_Q \cap \mathfrak{h}_{man\cdot z} = \mathfrak{q} \cap \mathfrak{h}_{man\cdot z} = \mathfrak{q} \cap \mathrm{Ad}(man)\mathfrak{h}_z = \mathrm{Ad}(man)\big(\mathfrak{q} \cap \mathfrak{h}_z\big) = \mathrm{Ad}(man)\big(\mathfrak{l}_Q \cap \mathfrak{h}_z\big).$$

Now a normalizes and n centralizes $\mathfrak{l}_Q \cap \mathfrak{h}_z$. This proves (3.5). Equation (3.6) follows from (3.5) by intersecting both sides with $\mathfrak{a}$. □

For an adapted point $z \in Z$ we define

$$\mathfrak{a}_z^\circ := \mathfrak{a} \cap (\mathfrak{a} \cap \mathfrak{h}_z)^\perp$$

and

$$\mathfrak{a}_{z,\mathrm{reg}}^\circ := \{X \in \mathfrak{a}_z^\circ : Z_\mathfrak{g}(X) = \mathfrak{l}_Q\} = \{X \in \mathfrak{a}_z^\circ : \alpha(X) \neq 0 \text{ for all } \alpha \in \Sigma(Q)\}.$$

If $z, z' \in Z$ are both adapted and $P \cdot z = P \cdot z'$, then in view of Proposition 3.6 we may apply Lemma 3.9 to z and z' and conclude that $\mathfrak{a} \cap \mathfrak{h}_z = \mathfrak{a} \cap \mathfrak{h}_{z'}$. It follows that $\mathfrak{a} \cap \mathfrak{h}_z$, $\mathfrak{a}_z^\circ$ and $\mathfrak{a}_{z,\mathrm{reg}}^\circ$ only depend on the open P-orbit $\mathcal{O} = P \cdot z$, not on the adapted point in $\mathcal{O}$. Later we will prove that $\mathfrak{a} \cap \mathfrak{h}_z$, $\mathfrak{a}_z^\circ$ and $\mathfrak{a}_{z,\mathrm{reg}}^\circ$ are in fact the same for all adapted points $z \in Z$. See Corollary 3.17.

For the next lemma we adapt the analysis in [7, Section 12.2].

Lemma 3.10. *Let $z \in Z$ be adapted. There exists a unique linear map*

$$T_z^{\perp} : \mathfrak{a}_z^{\circ} \to Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q \cap \mathfrak{h}_z)$$

with the property that for every $X \in \mathfrak{a}_z^{\circ}$

$$X + T_z^{\perp}(X) \in \mathfrak{h}_z^{\perp}.$$

Proof. Since $\mathfrak{g} = \mathfrak{h}_z + \mathfrak{p}$, we have

$$\mathfrak{h}_z^{\perp} \cap \mathfrak{n}_P = \mathfrak{h}_z^{\perp} \cap \mathfrak{p}^{\perp} = (\mathfrak{h}_z + \mathfrak{p})^{\perp} = \mathfrak{g}^{\perp} = \{0\}.$$

Therefore, $\mathfrak{h}_z^{\perp} + \mathfrak{n}_Q = \mathfrak{h}_z^{\perp} \oplus \mathfrak{n}_Q$. As $\mathfrak{l}_Q^{\perp} = \overline{\mathfrak{n}}_Q \oplus \mathfrak{n}_Q$, we further have

$$\mathfrak{h}_z^{\perp} \oplus \mathfrak{n}_Q \subseteq \mathfrak{h}_z^{\perp} + \mathfrak{l}_Q^{\perp} = (\mathfrak{l}_Q \cap \mathfrak{h}_z)^{\perp}.$$

Moreover,

$$\dim\big((\mathfrak{l}_Q \cap \mathfrak{h}_z)^{\perp}\big) = 2\dim(\mathfrak{n}_Q) + \dim(\mathfrak{l}_Q) - \dim(\mathfrak{l}_Q \cap \mathfrak{h}_z)$$

and, in view of Proposition 3.6 (ii) and the fact that $\mathfrak{g} = \mathfrak{h}_z + \mathfrak{q}$,

$$\dim(\mathfrak{h}_z^{\perp}) = \dim(\mathfrak{q}) - \dim(\mathfrak{q} \cap \mathfrak{h}_z) = \dim(\mathfrak{n}_Q) + \dim(\mathfrak{l}_Q) - \dim(\mathfrak{l}_Q \cap \mathfrak{h}_z).$$

It follows that $\dim\big((\mathfrak{l}_Q \cap \mathfrak{h}_z)^{\perp}\big) = \dim\big(\mathfrak{h}_z^{\perp} \oplus \mathfrak{n}_Q\big)$, and hence

$$(\mathfrak{l}_Q \cap \mathfrak{h}_z)^{\perp} = \mathfrak{h}_z^{\perp} \oplus \mathfrak{n}_Q. \tag{3.7}$$

In particular, for every $X \in \mathfrak{a}_z^{\circ}$ there exists a unique element $Y \in \mathfrak{n}_Q$ so that $X+Y \in \mathfrak{h}_z^{\perp}$, and thus there exists a unique linear map $T : \mathfrak{a}_z^{\circ} \to \mathfrak{n}_Q$ whose graph is contained in $\mathfrak{h}_z^{\perp}$. It remains to be shown that T actually maps to $Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q \cap \mathfrak{h}_z)$.

Let $X \in \mathfrak{a}_z^{\circ}$ and $Y \in \mathfrak{n}_Q$ satisfy $X + Y \in \mathfrak{h}_z^{\perp}$. We will show that $Y \in Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q \cap \mathfrak{h}_z)$. Note that $\mathfrak{a}_z^{\circ} = \mathfrak{a} \cap (\mathfrak{l}_Q \cap \mathfrak{h}_z)^{\perp}$. As $[\mathfrak{l}_Q \cap \mathfrak{h}_z, (\mathfrak{l}_Q \cap \mathfrak{h}_z)^{\perp}] \subseteq (\mathfrak{l}_Q \cap \mathfrak{h}_z)^{\perp}$ and $[\mathfrak{l}_Q \cap \mathfrak{h}_z, \mathfrak{a}] \subseteq \mathfrak{l}_Q \cap \mathfrak{h}_z$ we have $[\mathfrak{l}_Q \cap \mathfrak{h}_z, \mathfrak{a}_z^{\circ}] = \{0\}$. From $[\mathfrak{l}_Q \cap \mathfrak{h}_z, Y] \subseteq [\mathfrak{l}_Q \cap \mathfrak{h}_z, \mathfrak{n}_Q] \subseteq \mathfrak{n}_Q$ and

$$[\mathfrak{l}_Q \cap \mathfrak{h}_z, Y] = [\mathfrak{l}_Q \cap \mathfrak{h}_z, X + Y] \subseteq [\mathfrak{l}_Q \cap \mathfrak{h}_z, \mathfrak{h}_z^{\perp}] \subseteq \mathfrak{h}_z^{\perp},$$

it follows that

$$[\mathfrak{l}_Q \cap \mathfrak{h}_z, Y] \subseteq \mathfrak{h}_z^{\perp} \cap \mathfrak{n}_Q = \{0\},$$

and thus $Y \in Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q \cap \mathfrak{h}_z)$. □

Given an adapted point z, the following lemma gives a characterization of the adapted points in the open P-orbit $P \cdot z$.

Lemma 3.11. *Let $z \in Z$ be adapted and $n \in N_Q$. Then $n \cdot z$ is adapted if and only if there exists an $X \in \mathfrak{a}_{z,\mathrm{reg}}^{\circ}$ so that*

$$\mathrm{Ad}(n^{-1})X = X + T_z^{\perp}(X).$$

Proof. Assume that $n\cdot z$ is adapted. By definition there exists an element $X\in\mathfrak{a}^\circ_{n\cdot z,\mathrm{reg}}=\mathfrak{a}^\circ_{z,\mathrm{reg}}$. Now

$$\mathrm{Ad}(n^{-1})X\in\mathrm{Ad}(n^{-1})\mathfrak{h}_{n\cdot z}^\perp=\mathfrak{h}_z^\perp.$$

Moreover, in view of Proposition 3.6 and Lemma 3.9 we have $n\in Z_{N_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)$. The Lie algebra $\mathfrak{l}_Q\cap\mathfrak{h}_z$ is normalized by $\mathfrak{a}$ and the roots of $\mathfrak{l}_Q\cap\mathfrak{h}_z$ in $\mathfrak{a}$ vanish on $\mathfrak{a}_z^\circ$. Therefore, X centralizes $\mathfrak{l}_Q\cap\mathfrak{h}_z$. It follows that

$$\begin{aligned}\mathrm{Ad}(n^{-1})X\in\mathrm{Ad}(N_Q)X\cap\mathrm{Ad}\big(Z_G(\mathfrak{l}_Q\cap\mathfrak{h}_z)\big)X&\subseteq(X+\mathfrak{n}_Q)\cap Z_{\mathfrak{g}}(\mathfrak{l}_Q\cap\mathfrak{h}_z)\\&=X+Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z).\end{aligned}$$

Since also $X+T_z^\perp(X)\in\mathfrak{h}_z^\perp\cap\big(X+Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)\big)$, it follows from (3.7) that

$$\mathrm{Ad}(n^{-1})X=X+T_z^\perp(X).$$

We move on to prove the other implication. Assume that there exists an $X\in\mathfrak{a}^\circ_{z,\mathrm{reg}}$ so that

$$\mathrm{Ad}(n^{-1})X=X+T_z^\perp(X).$$

First note that

$$Pn\cdot z=P\cdot z$$

is an open P-orbit in Z. Further, $\mathrm{Ad}(n^{-1})X\in\mathfrak{h}_z^\perp$ and thus

$$X=\mathrm{Ad}(n)\,\mathrm{Ad}(n^{-1})X\in\mathrm{Ad}(n)\mathfrak{h}_z^\perp=\mathfrak{h}_{n\cdot z}^\perp.$$

Finally, we claim that $n\in Z_{N_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)$. From the claim and Lemma 3.9 it follows that

$$\mathfrak{l}_{Q,\mathrm{nc}}\subseteq\mathfrak{h}_{n\cdot z},$$

and hence that $n\cdot z$ is adapted.

It thus remains to prove the claim. Since $\alpha(X)\neq0$ for all roots $\alpha\in\Sigma(Q)$, the map

$$\Xi:N_Q\to\mathfrak{n}_Q,\quad u\mapsto\mathrm{Ad}(u)X-X$$

is a diffeomorphism. The image of $Z_{N_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)$ under Ξ is a submanifold of $Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)$ that contains 0. Moreover, its dimension coincides with the dimension of $Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)$, and hence it is an open neighborhood of 0 in $Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)$. As $\mathfrak{l}_Q\cap\mathfrak{h}_z$ is normalized by A, also $Z_{N_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)$ is normalized by A. Therefore, $\Xi\big(Z_{N_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)\big)$ is A-stable. The only A-stable open neighborhood of 0 in $Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)$ is $Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)$ itself. We thus conclude that

$$\Xi\big(Z_{N_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)\big)=Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z).$$

The claim now follows as

$$n^{-1}=\Xi^{-1}\big(\mathrm{Ad}(n^{-1})X-X\big)=\Xi^{-1}\big(T_z^\perp(X)\big)\in\Xi^{-1}\big(Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z)\big)=Z_{N_Q}(\mathfrak{l}_Q\cap\mathfrak{h}_z).$$

□

We can now describe the adapted points in a given open P-orbit in Z.

Proposition 3.12. *Let $z \in Z$ be adapted. There exists a unique smooth rational map*

$$\Phi_z : \mathfrak{a}^\circ_{z,\mathrm{reg}} \to \mathfrak{n}_Q$$

so that the following hold.

(i) A point $z' \in P \cdot z$ is adapted if and only if there exist $m \in M$, $a \in A$ and $X \in \mathfrak{a}^\circ_{z,\mathrm{reg}}$ so that

$$z' = ma \exp\big(\Phi_z(X)\big) \cdot z \tag{3.8}$$

(ii) For every $X \in \mathfrak{a}^\circ_{z,\mathrm{reg}}$ we have $\mathbb{R}X \subseteq \mathfrak{h}^\perp_{\exp(\Phi_z(X))\cdot z}$.

The map Φ_z is determined by the identity

$$\mathrm{Ad}\big(\exp(-\Phi_z(X))\big)X = X + T^\perp_z(X) \in \mathfrak{h}^\perp_z \qquad (X \in \mathfrak{a}^\circ_{z,\mathrm{reg}}). \tag{3.9}$$

Finally, if $z' \in P \cdot z$ is adapted and $X \in \mathfrak{a}^\circ_{z,\mathrm{reg}} \cap \mathfrak{h}^\perp_{z'}$, then

$$z' \in MA\exp\big(\Phi_z(X)\big) \cdot z$$

Proof. Define the map

$$\Psi : \mathfrak{a}^\circ_{z,\mathrm{reg}} \times \mathfrak{n}_Q \to \mathfrak{a}^\circ_{z,\mathrm{reg}} \times \mathfrak{n}_Q; \quad (X, Y) \mapsto \big(X, \mathrm{Ad}\big(\exp(-Y)\big)X - X\big).$$

Ψ is a diffeomorphism and its inverse defines a smooth rational map from $\mathfrak{a}^\circ_{z,\mathrm{reg}} \times \mathfrak{n}_Q$ to itself. Define $\Phi_z : \mathfrak{a}^\circ_{z,\mathrm{reg}} \to \mathfrak{n}_Q$ to be the map determined by

$$\big(X, \Phi_z(X)\big) = \Psi^{-1}\big(X, T^\perp_z(X)\big) \qquad (X \in \mathfrak{a}^\circ_{z,\mathrm{reg}}). \tag{3.10}$$

By construction (3.9) holds. It follows from Lemma 3.11 that a point $z' \in P \cdot z$ is adapted if and only if (3.8) holds. Moreover, (3.9) implies that for every $X \in \mathfrak{a}^\circ_{z,\mathrm{reg}}$

$$X \in \mathrm{Ad}\big(\exp(\Phi_z(X))\big)\mathfrak{h}^\perp_z = \mathfrak{h}^\perp_{\Phi_z(X)\cdot z}.$$

This shows that Φ_z has all the desired properties.

We move on to show uniqueness. Let $\Phi' : \mathfrak{a}^\circ_{z,\mathrm{reg}} \to \mathfrak{n}_Q$ be a second map satisfying the properties (i) and (ii). If $X \in \mathfrak{a}^\circ_{z,\mathrm{reg}}$, then

$$X \in \mathfrak{a} \cap \mathfrak{h}^\perp_{\exp(\Phi_z(X))\cdot z} \cap \mathfrak{h}^\perp_{\exp(\Phi'_z(X))\cdot z}$$

In view of Lemma 3.8

$$\exp\big(\Phi_z(X)\big)\exp\big(-\Phi'_z(X)\big) \in N_Q \cap L_Q = \{e\},$$

and hence $\Phi_z(X) = \Phi'_z(X)$. This shows that Φ_z is unique.

Finally, let $z' \in P \cdot z$ be adapted and $X \in \mathfrak{a}^\circ_{z,\mathrm{reg}} \cap \mathfrak{h}^\perp_{z'}$. Then

$$X \in \mathfrak{a} \cap \mathfrak{h}^\perp_{z'} \cap \mathfrak{h}^\perp_{\exp(\Phi_z(X))\cdot z}$$

By Lemma 3.8

$$z' \in L_Q \exp\big(\Phi_z(X)\big) \cdot z$$

Since $\exp\big(\Phi_z(X)\big) \cdot z$ is adapted we have $L_{Q,\mathrm{nc}} \subseteq H_{\exp(\Phi_z(X))\cdot z}$, and thus

$$L_Q \exp\big(\Phi_z(X)\big) \cdot z = MA\exp\big(\Phi_z(X)\big) \cdot z.$$

This proves the final assertion. ☐

We complete the description of the adapted points in Z with a proposition, which provides a set of adapted points that intersects with all open P-orbits in Z.

Proposition 3.13. *Let $z \in Z$ be adapted. Every open P-orbit in Z contains a point $f \cdot z$ with $f \in G \cap \exp(i\mathfrak{a})H_{z,\mathbb{C}}$. For every $f \in G \cap \exp(i\mathfrak{a})H_{z,\mathbb{C}}$ the point $f \cdot z$ is adapted. Moreover,*

$$\begin{aligned} \mathfrak{l}_Q \cap \mathfrak{h}_{f\cdot z} &= \mathfrak{l}_Q \cap \mathfrak{h}_z, \\ \mathfrak{a} \cap \mathfrak{h}_{f\cdot z} &= \mathfrak{a} \cap \mathfrak{h}_z \\ \mathfrak{a} \cap \mathfrak{h}_{f\cdot z}^{\perp} &= \mathfrak{a} \cap \mathfrak{h}_z^{\perp}. \end{aligned}$$

To prove the proposition we make use of the following complex version of Lemma 3.8. The proof for this lemma is the same as the proof for Lemma 3.8.

Lemma 3.14. *Let $z \in Z$ be adapted and let $q \in Q_{\mathbb{C}}$. Then $q \in L_{Q,\mathbb{C}}$ if and only if there exists an element*

$$X \in \mathfrak{a}_{\mathbb{C}} \cap \mathfrak{h}_{z,\mathbb{C}}^{\perp} \cap \mathrm{Ad}(q)\mathfrak{h}_{z,\mathbb{C}}^{\perp}$$

so that $\mathfrak{l}_{Q,\mathbb{C}} = Z_{\mathfrak{g}_{\mathbb{C}}}(X)$ (and thus $\alpha(X) \neq 0$ for all $\alpha \in \Sigma(Q)$). In that case

$$\mathfrak{a}_{\mathbb{C}} \cap \mathfrak{h}_{z,\mathbb{C}}^{\perp} = \mathfrak{a}_{\mathbb{C}} \cap \mathrm{Ad}(q)\mathfrak{h}_{z,\mathbb{C}}^{\perp}.$$

Proof of Proposition 3.13. We adapt the arguments from [16, Sections 2.4 & 2.5].

Let $\mathcal{O}$ be an open P-orbit in Z. By [16, Lemma 2.1] the set

$$G \cap P_{\mathbb{C}}H_{z,\mathbb{C}}$$

is the union of all open $P \times H_z$-double cosets in G. Therefore, there exist $p \in P_{\mathbb{C}}$ and $h \in H_{z,\mathbb{C}}$ so that $ph \in G$ and $\mathcal{O} = Pph \cdot z$. Let $X \in \mathfrak{a}_{z,\mathrm{reg}}^{\circ} \cap \mathfrak{h}_z^{\perp}$. It follows from Proposition 3.12 that we may choose p so that $X \in \mathfrak{h}_{ph\cdot z}$. In view of Lemma 3.14 we have

$$p \in P_{\mathbb{C}} \cap L_{Q,\mathbb{C}} = M_{\mathbb{C}}A_{\mathbb{C}}(N_{P,\mathbb{C}} \cap L_{Q,\mathbb{C}}).$$

As $N_{P,\mathbb{C}} \cap L_{Q,\mathbb{C}} \subseteq H_{z,\mathbb{C}}$, $M_{\mathbb{C}} = M\exp(i\mathfrak{m})$ and $A_{\mathbb{C}} = A\exp(i\mathfrak{a})$, we may further choose p so that $p \in \exp(i\mathfrak{m})\exp(i\mathfrak{a})$. We claim that now $ph \in \exp(i\mathfrak{a})H_{z,\mathbb{C}}$.

To prove the claim, define $g \mapsto \overline{g}$ to be the conjugation on $G_{\mathbb{C}}$ with respect to G. Note that $M_{\mathbb{C}}\exp(i\mathfrak{a})$ is a group that is stable under this conjugation. Since $ph \in G$, we have $ph = \overline{p}\overline{h}$. Moreover, since $p \in \exp(i\mathfrak{m})\exp(i\mathfrak{a})$ we have $\overline{p} = p^{-1}$, and hence

$$p^2 = \overline{p}^{-1}p = \overline{h}h^{-1}.$$

The group $M_{\mathbb{C}}\exp(i\mathfrak{a}) \cap H_{z,\mathbb{C}}$ is algebraic and hence it has finitely many connected components. Therefore, $\exp(i\mathfrak{m}) \cap H_{z,\mathbb{C}}$ is connected, and thus equal to $\exp(i\mathfrak{m} \cap \mathfrak{h}_z)$. It follows that $p \in \exp(i\mathfrak{a})H_{z,\mathbb{C}}$. This proves the claim. We have now proven that the set

$$\big(G \cap \exp(i\mathfrak{a})H_{z,\mathbb{C}}\big) \cdot z$$

intersects with every open P-orbit in Z. We move on to show that all points in this set are adapted.

Let $a \in \exp(i\mathfrak{a})$ and $h \in H_{z,\mathbb{C}}$ be so that $ah \in G$. Then

$$\mathfrak{p}_{\mathbb{C}} + \mathfrak{h}_{ah\cdot z,\mathbb{C}} = \mathfrak{p}_{\mathbb{C}} + \mathrm{Ad}(ah)\mathfrak{h}_{z,\mathbb{C}} = \mathrm{Ad}(a)\big(\mathfrak{p}_{\mathbb{C}} + \mathfrak{h}_{z,\mathbb{C}}\big).$$

Since $P \cdot z$ is open, the right-hand side is equal to $\mathfrak{g}_{\mathbb{C}}$. Intersection both sides with $\mathfrak{g}$ now yields

$$\mathfrak{p} + \mathfrak{h}_{ah\cdot z} = \mathfrak{g}$$

and therefore $Pah \cdot z$ is open in Z. Furthermore, since $\mathfrak{l}_{Q,\mathrm{nc},\mathbb{C}}$ is stable under the action of $A_{\mathbb{C}}$ and $\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathfrak{h}_z$, we have

$$\mathfrak{l}_{Q,\mathrm{nc},\mathbb{C}} = \mathrm{Ad}(a)\mathfrak{l}_{Q,\mathrm{nc},\mathbb{C}} \subseteq \mathrm{Ad}(a)\mathfrak{h}_{z,\mathbb{C}} = \mathrm{Ad}(ah)\mathfrak{h}_{z,\mathbb{C}} = \mathfrak{h}_{ah\cdot z,\mathbb{C}}.$$

Intersecting both sides with $\mathfrak{g}$ yields

$$\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathfrak{h}_{ah\cdot z}.$$

Finally,

$$\mathfrak{a}_{\mathbb{C}} \cap \mathfrak{h}^{\perp}_{ah\cdot z,\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} \cap \mathrm{Ad}(ah)\mathfrak{h}^{\perp}_{z,\mathbb{C}} = \mathrm{Ad}(a)\big(\mathfrak{a}_{\mathbb{C}} \cap \mathfrak{h}^{\perp}_{z,\mathbb{C}}\big) = \mathfrak{a}_{\mathbb{C}} \cap \mathfrak{h}^{\perp}_{z,\mathbb{C}},$$

and hence intersecting with $\mathfrak{g}$ yields

$$\mathfrak{a} \cap \mathfrak{h}^{\perp}_{ah\cdot z,} = \mathfrak{a} \cap \mathfrak{h}^{\perp}_{z}.$$

Since z is adapted, $\mathfrak{a} \cap \mathfrak{h}_z^{\perp}$ contains an element X so that $Z_{\mathfrak{g}}(X) = \mathfrak{l}_Q$. This concludes the proof that $ah \cdot z$ is adapted.

It remains to prove that $\mathfrak{l}_Q \cap \mathfrak{h}_{ah\cdot z} = \mathfrak{l}_Q \cap \mathfrak{h}_z$ and $\mathfrak{a} \cap \mathfrak{h}_{ah\cdot z} = \mathfrak{a} \cap \mathfrak{h}_z$. These identities follow by intersecting with $\mathfrak{g}$ and $\mathfrak{a}$, respectively, from

$$\mathfrak{l}_{Q,\mathbb{C}} \cap \mathfrak{h}_{ah\cdot z,\mathbb{C}} = \mathfrak{l}_{Q,\mathbb{C}} \cap \mathrm{Ad}(ah)\mathfrak{h}_{z,\mathbb{C}} = \mathrm{Ad}(a)\big(\mathfrak{l}_{Q,\mathbb{C}} \cap \mathfrak{h}_{z,\mathbb{C}}\big) = \mathfrak{l}_{Q,\mathbb{C}} \cap \mathfrak{h}_{z,\mathbb{C}}.$$

□

We end this section with three corollaries of the previous results in this section. We begin with a description of the normalizer of $\mathfrak{h}_z$.

Corollary 3.15. *Let $z \in Z$ be adapted. Then*

$$N_G(\mathfrak{h}_z) \subseteq MA\Big(G \cap \exp(i\mathfrak{a})H_{z,\mathbb{C}}\Big).$$

In particular,

$$N_{\mathfrak{g}}(\mathfrak{h}_z) = \mathfrak{h}_z + N_{\mathfrak{a}}(\mathfrak{h}_z) + N_{\mathfrak{m}}(\mathfrak{h}_z).$$

Remark 3.16. The second assertion in the corollary was proven in [16, (5.10)] and a slightly weaker version in [17, Lemma 4.2]. In these articles the requirement (iii) in Definition 3.3 is not mentioned, but in general it cannot be omitted. If for example $H = \overline{N}_P$ and $z = man\cdot\overline{N}_P$, then $N_{\mathfrak{g}}(\mathfrak{h}_z) = \mathrm{Ad}(man)\overline{\mathfrak{p}}$. This is only contained in $\mathfrak{m}\oplus\mathfrak{a}\oplus\overline{\mathfrak{n}}_P$ if $n = e$. In Example 3.5 we showed that the latter condition is equivalent to the existence of regular elements in $\mathfrak{a} \cap \mathfrak{h}_z^{\perp}$. The additional requirement (iii) in Definition 3.3 is therefore necessary in this case.

Proof of Corollary 3.15. Let $g \in N_G(\mathfrak{h}_z)$. Then $\mathfrak{h}_{g\cdot z} = \mathrm{Ad}(g)\mathfrak{h}_z = \mathfrak{h}_z$. It follows that the properties (i) – (iii) in Definition 3.3 hold for the point $g \cdot z$ and thus $g \cdot z$ is adapted. By Proposition 3.13 there exists an $f \in G \cap \exp(i\mathfrak{a})H_{z,\mathbb{C}}$ so that $Pg \cdot z = Pf \cdot z$. Moreover, $f \cdot z$ is adapted and

$$\mathfrak{a} \cap \mathfrak{h}_{f\cdot z}^{\perp} = \mathfrak{a} \cap \mathfrak{h}_z^{\perp} = \mathfrak{a} \cap \mathfrak{h}_{g\cdot z}^{\perp}.$$

Since z is adapted, these spaces contain elements X so that $\mathfrak{l}_Q = Z_{\mathfrak{g}}(X)$. It follows from Lemma 3.8 that $g \cdot z \in L_Q f \cdot z$. Since $L_{Q,\mathrm{nc}} \subseteq H_{f\cdot z}$ and $L_Q = MAL_{Q,\mathrm{nc}}$, there exist $m \in M$ and $a \in A$ so that $g \cdot z = maf \cdot z$. This proves the first assertion in the Corollary.

We move on to the second assertion. Consider the set Λ consisting of all $\lambda \in \mathfrak{a}^*$ for which there exists a regular function $\phi_\lambda \in \mathbb{R}[G]$ so that $\phi(e) = 1$ and

$$\phi_\lambda(manxh) = a^\lambda \phi(x) \qquad \big(m \in M, a \in A, n \in N_P, h \in H_z, x \in G\big).$$

It follows from [17, Lemma 3.4 & Remark 3.5] that Λ spans $(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$. For each $\lambda \in \Lambda$ the function ϕ_λ extends to regular function on $G_{\mathbb{C}}$ which satisfies

$$\phi_\lambda(ah) = a^\lambda \qquad \big(a \in \exp(i\mathfrak{a}), h \in (H_{z,\mathbb{C}})_e\big),$$

where $(H_{z,\mathbb{C}})_e$ is the connected open subgroup of $H_{z,\mathbb{C}}$. Note that a^λ with $a \in \exp(i\mathfrak{a})$ is real if and only if $a^\lambda = \pm 1$. From this it follows that $\big(G \cap \exp(i\mathfrak{a})H_{z,\mathbb{C}}\big)/H_z$ is discrete. Since it is algebraic, it is in fact a finite set, and hence H_z is a relatively open subset of $G \cap \exp(i\mathfrak{a})H_{z,\mathbb{C}}$. Therefore, there exists a subspace $\mathfrak{s}$ of $\mathfrak{m} \oplus \mathfrak{a}$ so that

$$N_{\mathfrak{g}}(\mathfrak{h}_z) = \mathfrak{h}_z + \mathfrak{s}.$$

We may assume that $N_{\mathfrak{m}}(\mathfrak{h}_z) \oplus N_{\mathfrak{a}}(\mathfrak{h}_z) \subseteq \mathfrak{s}$. To prove the second assertion, it now suffices to show that $\mathfrak{s} = (\mathfrak{s} \cap \mathfrak{m}) \oplus (\mathfrak{s} \cap \mathfrak{a})$. The latter follows from (3.3) with H_z replaced by the real spherical subgroup $N_G(\mathfrak{h}_z)$. $\square$

The spaces $\mathfrak{a} \cap \mathfrak{h}_z$ and $\mathfrak{a}_z^\circ$ play an important role in this article. By the following corollary these spaces do not depend on the adapted point z.

Corollary 3.17. *If $z, z' \in Z$ are adapted, then $\mathfrak{a} \cap \mathfrak{h}_z = \mathfrak{a} \cap \mathfrak{h}_{z'}$.*

Proof. By Proposition 3.13 there exits an $f \in G \cap \exp(i\mathfrak{a})H_{z,\mathbb{C}}$ and a $p \in P$ so that $z' = pf \cdot z$. Moreover, $f \cdot z$ is adapted and $\mathfrak{a} \cap \mathfrak{h}_{f\cdot z} = \mathfrak{a} \cap \mathfrak{h}_z$. It follows from Proposition 3.6 that we may apply Lemma 3.9 to the points $f \cdot z$ and $pf \cdot z$. It follows that $\mathfrak{a} \cap \mathfrak{h}_{pf\cdot z} = \mathfrak{a} \cap \mathfrak{h}_{f\cdot z} = \mathfrak{a} \cap \mathfrak{h}_z$. $\square$

In view of Corollary 3.17 we may make the following definition.

Definition 3.18. We define

$$\mathfrak{a}_{\mathfrak{h}} := \mathfrak{a} \cap \mathfrak{h}_z,$$

where $z \in Z$ is any adapted point. We further define

$$\mathfrak{a}^\circ := \mathfrak{a} \cap \mathfrak{a}_{\mathfrak{h}}^{\perp}$$

and

$$\mathfrak{a}^\circ_{\mathrm{reg}} := \{X \in \mathfrak{a}^\circ : \alpha(X) \neq 0 \text{ for all } \alpha \in \Sigma(Q)\} = \{X \in \mathfrak{a}^\circ : Z_{\mathfrak{g}}(X) = \mathfrak{l}_Q\}.$$

The subalgebras $\mathfrak{m} \cap \mathfrak{h}_z$ and $\mathfrak{m} \cap \mathfrak{h}_{z'}$ may not be equal for all adapted points z and z'. However, there exists an $m \in M$ so that

$$\mathfrak{m} \cap \mathfrak{h}_z = \mathrm{Ad}(m)\big(\mathfrak{m} \cap \mathfrak{h}_{z'}\big).$$

We note that $\mathfrak{a}_z^\circ = \mathfrak{a}^\circ$ and $\mathfrak{a}_{z,\mathrm{reg}}^\circ = \mathfrak{a}_{\mathrm{reg}}^\circ$ for all adapted points $z \in Z$.

Finally, we give the alternative characterization of adapted points which we announced in Remark 3.4 (b).

Proposition 3.19. *Let $z \in Z$. Then z is adapted if and only if the following hold.*

(i) $P \cdot z$ is open in Z, i.e., $\mathfrak{p} + \mathfrak{h}_z = \mathfrak{g}$,

(ii) There exists an $X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp$ so that $Z_{\mathfrak{g}}(X) = \mathfrak{l}_Q$, i.e., $\mathfrak{a}_{\mathrm{reg}}^\circ \cap \mathfrak{h}_z^\perp \neq \emptyset$.

Proof. By definition adapted points satisfy (i) and (ii). For the converse implication, assume that z satisfies (i) and (ii). Let $X \in \mathfrak{a}_{\mathrm{reg}}^\circ \cap \mathfrak{h}_z^\perp$. It follows from Proposition 3.12 that there exists an adapted point $z' \in P \cdot z$ so that $X \in \mathfrak{a}_{\mathrm{reg}}^\circ \cap \mathfrak{h}_{z'}^\perp$. By Lemma 3.8 there exists a $l \in L_Q$ so that $z = l \cdot z'$. Since the set of adapted points is L_Q-stable, it follows that z is adapted. $\square$

4 Description of $\mathfrak{h}_z$ in terms of a graph

As in [4, Proposition 2.5] we may describe of the stabilizer subalgebra $\mathfrak{h}_z$ of an adapted point z in terms of a graph. It follows from Proposition 3.6 that for every adapted point $z \in Z$ there exists a unique linear map

$$T_z : \overline{\mathfrak{n}}_Q \to \big(\mathfrak{m} \cap (\mathfrak{m} \cap \mathfrak{h}_z)^\perp\big) \oplus \mathfrak{a}^\circ \oplus \mathfrak{n}_Q,$$

so that

$$\mathfrak{h}_z = (\mathfrak{l}_Q \cap \mathfrak{h}_z) \oplus \mathcal{G}(T_z).$$

Here $\mathcal{G}(T_z)$ denotes the graph of T_z.

Lemma 4.1. *Let $z \in Z$ be adapted. The subspace $\big(\mathfrak{m} \cap (\mathfrak{m} \cap \mathfrak{h}_z)^\perp\big) \oplus \mathfrak{a}^\circ \oplus \mathfrak{n}_Q$ of $\mathfrak{g}$ is $(L_Q \cap H_z)$-stable. Moreover, the map T_z is $(L_Q \cap H_z)$-equivariant.*

Proof. Note that

$$\big(\mathfrak{m} \cap (\mathfrak{m} \cap \mathfrak{h}_z)^\perp\big) \oplus \mathfrak{a}^\circ \oplus \mathfrak{n}_Q = \mathfrak{q} \cap (\mathfrak{l}_Q \cap \mathfrak{h}_z)^\perp.$$

As $L_Q \cap H_z$ stabilizes both $\mathfrak{q}$ and $\mathfrak{l}_Q \cap \mathfrak{h}_z$, and the adjoint representation preserves Killing-orthocomplements, the first assertion follows.

It follows from the first assertion that the decomposition

$$\mathfrak{g} = \overline{\mathfrak{n}}_Q \oplus (\mathfrak{l}_Q \cap \mathfrak{h}_z) \oplus \Big(\big(\mathfrak{m} \cap (\mathfrak{m} \cap \mathfrak{h}_z)^\perp\big) \oplus \mathfrak{a}^\circ \oplus \mathfrak{n}_Q\Big)$$

is stable under the adjoint action of $L_Q \cap H_z$. The second assertion now follows from the uniqueness of T_z. $\square$

For $\alpha \in \Sigma(Q)$ let p_α be the projection $\mathfrak{g} \to \mathfrak{g}_\alpha$ with respect to the root space decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha.$$

Likewise, we write $p_\mathfrak{m}$ and $p_\mathfrak{a}$ for the projections $\mathfrak{g} \to \mathfrak{m}$ and $\mathfrak{g} \to \mathfrak{a}$ with respect to this decomposition. For an adapted point $z \in Z$ and $Y \in \overline{\mathfrak{n}}_Q$ we define the z-support of Y to be

$$\operatorname{supp}_z(Y) := \big\{\beta \in \Sigma(Q) \cup \{\mathfrak{m}, \mathfrak{a}\} : p_\beta\big(T_z(Y)\big) \neq 0\big\}.$$

For every $\alpha \in \Sigma(Q)$ we have

$$-\alpha(X)Y + \big(\operatorname{ad}(X) \circ T_z\big)(Y) = \big[X, Y + T_z(Y)\big] \subseteq \mathfrak{h}_z \qquad \big(X \in \mathfrak{a}_\mathfrak{h}, Y \in \mathfrak{g}_{-\alpha}\big)$$

The uniqueness of the map T implies that

$$\operatorname{ad}(X) \circ T_z\big|_{\mathfrak{g}_{-\alpha}} = -\alpha(X) T_z\big|_{\mathfrak{g}_{-\alpha}}$$

In particular,

$$\alpha\big|_{\mathfrak{a}_\mathfrak{h}} = \begin{cases} -\beta|_{\mathfrak{a}_\mathfrak{h}} & \text{if } \beta \in \Sigma(Q) \text{ with } \beta \in \operatorname{supp}_z(\mathfrak{g}_{-\alpha}), \\ 0 & \text{if } \mathfrak{m} \in \operatorname{supp}_z(\mathfrak{g}_{-\alpha}) \text{ or } \mathfrak{a} \in \operatorname{supp}_z(\mathfrak{g}_{-\alpha}). \end{cases} \tag{4.1}$$

The map T_z possesses several symmetries, some of which are described in the following lemma.

Lemma 4.2. *Let $z \in Z$ be adapted. If $X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp$, then*

$$B\big([X, Y_1], T_z(Y_2)\big) = B\big([X, Y_2], T_z(Y_1)\big) \qquad \big(Y_1, Y_2 \in \overline{\mathfrak{n}}_Q\big).$$

Remark 4.3. If $\alpha, \beta \in \Sigma(Q)$ and $Y_{-\alpha} \in \mathfrak{g}_{-\alpha}$ and $Y_{-\beta} \in \mathfrak{g}_{-\beta}$, then the identity in the lemma specializes to

$$B\big(Y_{-\alpha}, p_\alpha T_z(Y_{-\beta})\big)\alpha(X) = B\big(Y_{-\beta}, p_\beta T_z(Y_{-\alpha})\big)\beta(X). \tag{4.2}$$

This identity was proved by Brion in [4, Proposition 2.5] in case G and H are complex algebraic groups and for one specific choice of X.

Proof of Lemma 4.2. Since $[X, Y_1], Y_2 \in \overline{\mathfrak{n}}_Q$ we have

$$B([X, Y_1], Y_2) = 0,$$

and since $[X, T_z(Y_1)] \in \mathfrak{n}_Q$ and $T_z(Y_2) \in \mathfrak{q}$, we have

$$B\big([X, T_z(Y_1)], T_z(Y_2)\big) = 0.$$

Therefore,

$$\begin{aligned} B\big([X, Y_1], T_z(Y_2)\big) - B\big(T_z(Y_1), [X, Y_2]\big) &= B\big([X, Y_1], T_z(Y_2)\big) + B\big([X, T_z(Y_1)], Y_2\big) \\ &= B\big([X, Y_1 + T_z(Y_1)], Y_2 + T_z(Y_2)\big). \end{aligned}$$

The right-hand side equals 0 as $[X, Y_1 + T_z(Y_1)] \in \mathfrak{h}_z^\perp$ and $Y_2 + T_z(Y_2) \in \mathfrak{h}_z$. $\square$

5 Limits of subspaces

For $k \in \mathbb{N}$ let $\mathrm{Gr}(\mathfrak{g}, k)$ be the Grassmannian of k-dimensional subspaces of the Lie algebra $\mathfrak{g}$. For our approach to the little Weyl group we will need to consider certain limits of the stabilizer subalgebras $\mathfrak{h}_z$ in $\mathrm{Gr}\left(\mathfrak{g}, \dim(\mathfrak{h}_z)\right)$. In this section we introduce the relevant limits and discuss their properties.

Definition 5.1. We say that an element $X \in \mathfrak{a}$ is *order-regular* if

$$\alpha(X) \neq \beta(X)$$

for all $\alpha, \beta \in \Sigma$ with $\alpha \neq \beta$.

If $X \in \mathfrak{a}$ is order-regular, then in particular $\alpha(X) \neq -\alpha(X)$ and therefore $\alpha(X) \neq 0$ for every $\alpha \in \Sigma$. This implies that order-regular elements in $\mathfrak{a}$ are regular. Every order-regular element $X \in \mathfrak{a}$ determines a linear order $\geq$ on Σ by setting

$$\alpha \geq \beta \quad \text{if and only if} \quad \alpha(X) \geq \beta(X)$$

for $\alpha, \beta \in \Sigma$.

Proposition 5.2. *Let $E \in \mathrm{Gr}(\mathfrak{g}, k)$ and let $X \in \mathfrak{a}$. The limit*

$$E_X := \lim_{t\to\infty} \mathrm{Ad}\left(\exp(tX)\right)E,$$

exists in the Grassmannian $\mathrm{Gr}(\mathfrak{g}, k)$. If $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ are the eigenvalues and $p_1, \dots, p_n$ the corresponding projections onto the eigenspaces V_i of $\mathrm{ad}(X)$, then E_X is given by

$$E_X = \bigoplus_{i=1}^{n} p_i\big(E \cap \bigoplus_{j=1}^{i} V_j\big). \tag{5.1}$$

The following hold.

(i) If E is a Lie subalgebra of $\mathfrak{g}$, then E_X is a Lie subalgebra of $\mathfrak{g}$.

(ii) If $X \in \mathfrak{a}$ is order-regular, then E_X is $\mathfrak{a}$-stable.

(iii) Let $\mathcal{R} \subseteq \mathfrak{a}$ be a connected component of the set of order-regular elements in $\mathfrak{a}$. If $X \in \overline{\mathcal{R}}$ and $Y \in \mathcal{R}$, then $\left(E_X\right)_Y = E_Y$. In particular, if $X, Y \in \mathcal{R}$, then $E_X = E_Y$.

(iv) If $g, g' \in G$ and

$$\lim_{t\to\infty} \exp(tX) g \exp(-tX) = g',$$

then

$$\left(\mathrm{Ad}(g)E\right)_X = \mathrm{Ad}(g')E_X$$

(v) Let $E_{\mathbb{C},X}$ be the limit of $\mathrm{Ad}\left(\exp(tX)\right)E_{\mathbb{C}}$ for $t \to \infty$ in the Grassmannian of k-dimensional complex subspaces in the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Then

$$E_{\mathbb{C},X} = (E_X)_{\mathbb{C}}.$$

Proof. The proofs for all assertions with the exception of (iv) and (v) are given in [19, Lemma 4.1]. Although order-regular elements are in [19] assumed to have the additional property that they are contained in $\mathfrak{a}^-$, this is not used anywhere in the proof of Lemma 4.1 *loc. cit.*

We move on to prove (iv). If $A_t \to \mathbf{1}$ for $t \to \infty$ in $\mathrm{End}(E)$, then $A_t \operatorname{Ad}\big(\exp(tX)\big)E$ tends to E_X as $t \to \infty$. The identity now follow straightforwardly from the fact that $(g')^{-1}\exp(tX)g\exp(-tX)$ converges to e in G.

Finally we prove (v). The space $E_{\mathbb{C},X}$ is a complex subspace of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Since E is contained in $E \otimes_{\mathbb{R}} \mathbb{C}$ as a real subspace, the limit E_X is contained in $E_{\mathbb{C},X}$ as a real subspace. Therefore, $E_X \otimes_{\mathbb{R}} \mathbb{C} \subseteq E_{\mathbb{C},X}$. A dimension count shows that equality holds. □

Remark 5.3. If X is not order-regular, then E_X need not to be stable under the action of $\mathfrak{a}$, even if X is regular. An example of this can be constructed as follows. Let $\alpha, \beta \in \Sigma(\mathfrak{a})$ be such that $\alpha \neq \beta$ and $\alpha(X) = \beta(X)$. Let $Y_\alpha \in \mathfrak{g}_\alpha$ and $Y_\beta \in \mathfrak{g}_\beta$ and define $E = \mathbb{R}(Y_\alpha + Y_\beta)$. Now E consists of eigenvectors for $\mathrm{ad}(X)$, hence $E_X = E$. However E is not $\mathfrak{a}$-stable.

6 The compression cone

In this section we introduce the compression cone of a point $z \in Z$. It consists of all $X \in \mathfrak{a}$ for which the limit $\mathfrak{h}_{z,X}$ is equal to a given limit subalgebra. The main result in this section is that the compression cones are the same for all adapted points. (See Proposition 6.5.) The compression cone for an adapted point is therefore an invariant of the space Z, which we call the compression cone of Z. For a non-adapted point the compression cone may be strictly smaller than the compression cone of Z. The closure of the compression cone of Z will serve as a fundamental domain of the little Weyl group.

We fix an adapted point $z_0 \in Z$ and define the subalgebra

$$\mathfrak{h}_\emptyset := (\mathfrak{l}_Q \cap \mathfrak{h}_{z_0}) + \overline{\mathfrak{n}}_Q. \tag{6.1}$$

This subalgebra was defined in the introduction as a limit subalgebra. From Lemma 6.4 it will follow that the two definitions indeed agree.

Clearly $\mathfrak{h}_\emptyset$ depends on the choice of the adapted point $z_0 \in Z$. However in view of the following lemma, another choice of z_0 would yield an M-conjugate of $\mathfrak{h}_\emptyset$.

Lemma 6.1. *Let $z \in Z$ be adapted. There exists an $m \in M$ so that*

$$(\mathfrak{l}_Q \cap \mathfrak{h}_z) \oplus \overline{\mathfrak{n}}_Q = \mathrm{Ad}(m)\mathfrak{h}_\emptyset.$$

Proof. The assertion follows directly from Proposition 3.13 and Lemma 3.9. The latter lemma we may apply in view of Proposition 3.6. □

Definition 6.2. For $z \in Z$ and $X \in \mathfrak{a}$ we define

$$\mathfrak{h}_{z,X} := (\mathfrak{h}_z)_X = \lim_{t\to\infty} \operatorname{Ad}\big(\exp(tX)\big)\mathfrak{h}_z.$$

Here the limit is taken in the Grassmannian of $\dim(\mathfrak{h}_z)$-dimensional subspaces of $\mathfrak{g}$. We further define for $z \in Z$ the cone in $\mathfrak{a}$

$$\mathcal{C}_z := \{X \in \mathfrak{a} : \mathfrak{h}_{z,X} = \mathrm{Ad}(m)\mathfrak{h}_\emptyset \text{ for some } m \in M\}.$$

Lemma 6.3. *Let $z \in Z$ be adapted. We define the set*

$$\begin{aligned} S_z := \{\alpha + \beta : \alpha \in \Sigma(Q), \beta \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha}) \cap \Sigma(Q)\} \\ \cup \{\alpha \in \Sigma(Q) : \mathfrak{a} \in \mathrm{supp}_z(\mathfrak{g}_\alpha) \textit{ or } \mathfrak{m} \in \mathrm{supp}_z(\mathfrak{g}_\alpha)\}. \end{aligned} \tag{6.2}$$

The cone $\mathcal{C}_z$ is given by

$$\mathcal{C}_z = \{X \in \mathfrak{a} : \gamma(X) < 0 \textit{ for all } \gamma \in S_z\}. \tag{6.3}$$

In particular $\mathcal{C}_z$ is an open cone in $\mathfrak{a}$, $\mathfrak{a}^-$ is contained in $\mathcal{C}_z$, and

$$\mathcal{C}_z + \mathfrak{a}_\mathfrak{h} = \mathcal{C}_z. \tag{6.4}$$

The dual cone

$$\mathcal{C}_z^\vee := \{\lambda \in \mathfrak{a}^* : \lambda(X) \geq 0 \textit{ for all } X \in \mathcal{C}_z\}$$

is equal to the finitely generated cone

$$\mathcal{C}_z^\vee = \sum_{\gamma \in S_z} \mathbb{R}_{\leq 0}\gamma.$$

Finally, $\mathcal{C}_z$ is equal to the interior of the double dual cone $(\mathcal{C}_z^\vee)^\vee$ and thus it is equal to the smallest convex open cone containing the order-regular elements in $\mathcal{C}_z$.

Proof. Let $X \in \mathfrak{a}$. Since $\mathfrak{h}_z = (\mathfrak{l}_Q \cap \mathfrak{h}_z) \oplus \mathcal{G}(T_z)$ we have $\mathfrak{h}_{z,X} = \mathrm{Ad}(m)\mathfrak{h}_\emptyset$ for some $m \in M$ if and only if

$$\lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)\mathcal{G}(T_z) = \bar{\mathfrak{n}}_Q.$$

In view of (5.1) the latter is equivalent to the conditions

$$\begin{cases} -\alpha(X) > \beta(X) & \text{if } \alpha, \beta \in \Sigma(Q) \text{ and } \beta \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha}), \\ -\alpha(X) > 0 & \text{if } \alpha \in \Sigma(Q), \text{ and } \mathfrak{m} \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha}) \text{ or } \mathfrak{a} \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha}). \end{cases}$$

This proves (6.3). The identity (6.4) follows from (4.1). All other assertions are trivial consequences of (6.3). □

Lemma 6.4. *Let $z \in Z$. The following hold.*

(i) For every $m \in M$ and $a \in A$ we have $\mathcal{C}_{ma\cdot z} = \mathcal{C}_z$.

(ii) $\mathcal{C}_z \neq \emptyset$ if and only if $P \cdot z$ is open. In that case $\mathfrak{a}^- \subseteq \mathcal{C}_z$.

Proof. Let $X \in \mathfrak{a}$. It follows from Proposition 5.2 (iv) that $\mathfrak{h}_{ma\cdot z,X} = \mathrm{Ad}(ma)\mathfrak{h}_{z,X}$. Since $\mathfrak{h}_\emptyset$ is A-stable it follows that $X \in \mathcal{C}_{ma\cdot z}$ if and only if $X \in \mathcal{C}_z$. This proves (i).

We move on to prove (ii). Assume that $\mathcal{C}_z \neq \emptyset$ and let $X \in \mathcal{C}_z$. Then $\overline{\mathfrak{n}}_P \subseteq \mathfrak{h}_{z,X}$, and hence $\mathfrak{h}_{z,X} + \mathfrak{p} = \mathfrak{g}$. This implies that $\mathfrak{g} = \mathrm{Ad}\big(\exp(tX)\big)\mathfrak{h}_z + \mathfrak{p}$ for large $t > 0$. Since $\mathfrak{g}$ and $\mathfrak{p}$ are both stable under the adjoint action of A, it follows that $\mathfrak{g} = \mathfrak{h}_z + \mathfrak{p}$ and thus $P \cdot z$ is open.

Assume now that $\mathcal{O} := P \cdot z$ is open. We will show that $\mathfrak{a}^- \subseteq \mathcal{C}_z$. To do so, let $z' \in \mathcal{O}$ be adapted and let $m \in M$, $a \in A$ and $n \in N_P$ so that $z = man \cdot z'$. It follows from Lemma 6.3 that $\mathfrak{a}^-$ is contained in $\mathcal{C}_{z'}$. In view of Proposition 5.2 (iv) we have $\mathfrak{h}_{n\cdot z',X} = \mathfrak{h}_{z',X}$ for every $X \in \mathfrak{a}^-$. Therefore, $\mathfrak{a}^- \subseteq \mathcal{C}_{n\cdot z'}$. It follows from (i) that $\mathcal{C}_z = \mathcal{C}_{n\cdot z'}$, and hence we have $\mathfrak{a}^- \subseteq \mathcal{C}_z$. This proves (ii). □

Proposition 6.5. *Let $z \in Z$ be adapted. For every $z' \in Z$ such that $P \cdot z'$ is open, we have*

$$\mathfrak{a}^- \subseteq \mathcal{C}_{z'} \subseteq \mathcal{C}_z.$$

Moreover, if z' is adapted, then

$$\mathcal{C}_{z'} = \mathcal{C}_z.$$

If $P \cdot z'$ is open, but z' is not adapted, then the inclusion $\mathcal{C}_{z'} \subseteq \mathcal{C}_z$ may be strict. Before we prove the proposition, we first consider the example of $Z = G/\overline{N}_P$ where this phenomenon is readily seen.

Example 6.6. Let $Z = G/\overline{N}_P$ and let $z = e \cdot \overline{N}_P$. We recall from Example 3.5 that the only open P-orbit in Z is $P \cdot z$ and the set of adapted points is equal to $MA \cdot z$.

Since $\overline{\mathfrak{n}}_P$ is $\mathfrak{a}$-stable, we have

$$\mathcal{C}_z = \mathfrak{a}.$$

Let $Y \in \mathfrak{n}_P$ and write $Y = \sum_{\alpha\in\Sigma^+} Y_\alpha$ with $Y_\alpha \in \mathfrak{g}_\alpha$. We claim that

$$\mathcal{C}_{\exp(Y)\cdot z} = \{X \in \mathfrak{a} : \alpha(X) < 0 \text{ for all } \alpha \in \Sigma^+ \text{ with } Y_\alpha \neq 0\}.$$

In view of Proposition 5.2 (iv) the set on the right-hand side is contained in $\mathcal{C}_{\exp(Y)\cdot z}$. For the other inclusion it suffices to show that no order-regular element in the complement of the set on the right-hand side is contained in $\mathcal{C}_{\exp(Y)\cdot z}$. Let $X \in \mathfrak{a}$ be order-regular, and assume that there exists a root $\alpha \in \Sigma^+$ so that $Y_\alpha \neq 0$ and $\alpha(X) > 0$. Let $\alpha_0 \in \Sigma^+$ be so that $\alpha_0(X)$ is minimal among the numbers $\alpha(X)$ with $\alpha \in \Sigma^+$, $Y_\alpha \neq 0$ and $\alpha(X) > 0$. Now

$$\mathrm{Ad}(Y)\theta Y_{\alpha_0} \in \theta Y_{\alpha_0} + [Y_{\alpha_0}, \theta Y_{\alpha_0}] + \mathfrak{n}_P,$$

and hence

$$\mathrm{Ad}\big(\exp(tX)\big)\,\mathrm{Ad}(Y)\theta Y_{\alpha_0} \in e^{-t\alpha_0(X)}\theta Y_{\alpha_0} + [Y_{\alpha_0}, \theta Y_{\alpha_0}] + \mathfrak{n}_P$$

The limit of $\mathrm{Ad}\big(\exp(tX)\big)\mathbb{R}\Big(\mathrm{Ad}(Y)\theta Y_{\alpha_0}\Big)$ in $\mathbb{P}(\mathfrak{g})$ is a line contained in $\mathfrak{p}$ as $-\alpha_0(X) < 0$ and $[Y_{\alpha_0}, \theta Y_{\alpha_0}] \in \mathfrak{a} \setminus \{0\}$. It follows that $X \notin \mathcal{C}_{\exp(Y)\cdot z}$.

The proof of Proposition 6.5 relies on the following lemma, which will also be of use later on.

Lemma 6.7. *Let $z \in Z$ and $f \in G \cap \exp(i\mathfrak{a})H_{z,\mathbb{C}}$. For every order-regular element $X \in \mathfrak{a}$*

$$\mathfrak{h}_{z,X} = \mathfrak{h}_{f\cdot z,X}.$$

Proof. Let $a \in \exp(i\mathfrak{a})$ and $h \in H_{z,\mathbb{C}}$ be so that $f = ah$. In view of Proposition 5.2 (v) limits and complexifications can be interchanges. Therefore, for every order-regular element $X \in \mathfrak{a}$

$$\begin{aligned}(\mathfrak{h}_{f\cdot z,X})_{\mathbb{C}} &= \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)\mathfrak{h}_{ah\cdot z,\mathbb{C}}\\ &= \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)ah\big)\mathfrak{h}_{z,\mathbb{C}}\\ &= \mathrm{Ad}(a)(\mathfrak{h}_{z,X})_{\mathbb{C}}.\end{aligned}$$

By Proposition 5.2 (ii) the space $\mathfrak{h}_{z,X}$ is $\mathfrak{a}$-stable and therefore $(\mathfrak{h}_{z,X})_{\mathbb{C}}$ is normalized by a. It follows that $(\mathfrak{h}_{f\cdot z,X})_{\mathbb{C}} = (\mathfrak{h}_{z,X})_{\mathbb{C}}$. Intersecting both sides with $\mathfrak{g}$ now yields the desired identity. □

Proof of Proposition 6.5. By Proposition 3.13 there exists an $f \in G \cap \exp(i\mathfrak{a})H_{z,\mathbb{C}}$ so that $f\cdot z$ is adapted and $z' \in Pf\cdot z$. By Lemma 6.7 we have $\mathfrak{h}_{z,X} = \mathfrak{h}_{f\cdot z,X}$ for every order-regular element $X \in \mathfrak{a}$, and hence $\mathcal{C}_{f\cdot z} = \mathcal{C}_z$. By replacing z by $f\cdot z$ we may thus without loss of generality assume that $z' \in P\cdot z$.

It follows from Lemma 6.4 (ii) that $\mathfrak{a}^-$ is contained in $\mathcal{C}_{z'}$. We move on to show that $\mathcal{C}_{z'}$ is contained in $\mathcal{C}_z$. Let $m \in M$, $a \in A$ and $n \in N_Q$ be so that $z' = man\cdot z$. Such elements exist by Proposition 3.6; see Remark 3.7 (b). In view of Lemma 6.4 (i) we have $\mathcal{C}_{z'} = \mathcal{C}_{n\cdot z}$.

Let $X \in \mathcal{C}_{n\cdot z}$ be order-regular. We may write $n = n_-n_+$ with

$$\log(n_\pm) \in \bigoplus_{\substack{\alpha\in\Sigma(Q)\\ \pm\alpha(X)>0}} \mathfrak{g}_\alpha.$$

In view of Proposition 5.2 (iv) we have $\mathfrak{h}_{n\cdot z,X} = \mathfrak{h}_{n_+\cdot z,X}$. We claim that $n_+ = e$. Assuming the claim is true, we have $\mathfrak{h}_{n\cdot z,X} = \mathfrak{h}_{z,X}$ and thus $X \in \mathcal{C}_z$.

To prove the claim we assume that $n_+ \neq e$ and work towards a contradiction. Let $X_\perp \in \mathfrak{a}^\circ_{\mathrm{reg}} \cap \mathfrak{h}_z^\perp$. For $\beta \in \Sigma(Q)$ let $U_\beta \in \mathfrak{g}_\beta$ be so that

$$\mathrm{Ad}(n_+)X_\perp = X_\perp + \sum_{\beta\in\Sigma(Q)} U_\beta.$$

Note that there exists an $\beta \in \Sigma(Q)$ so that $U_\beta \neq 0$, and that $U_\beta \neq 0$ only if $\beta(X) > 0$. Let $\alpha \in \Sigma(Q)$ be the maximal root for the order defined by X for which $U_\alpha \neq 0$. Set $Y_{-\alpha} := \theta U_\alpha$. There exists an $m' \in M$ so that $\mathfrak{h}_{n_+\cdot z,X} = \mathfrak{h}_{n\cdot z,X} = \mathrm{Ad}(m^{-1})\mathfrak{h}_{z',X} = \mathrm{Ad}(m')\mathfrak{h}_\emptyset$. It follows that

$$\mathbb{R}Y_{-\alpha} \subseteq \overline{\mathfrak{n}}_Q = \mathrm{Ad}(m')\overline{\mathfrak{n}}_Q \subseteq \mathrm{Ad}(m')\mathfrak{h}_\emptyset = \mathfrak{h}_{n_+\cdot z,X}.$$

We will exploit this fact.

Let $Y \in \mathfrak{h}_{n_+ \cdot z}$ be so that $(\mathbb{R}Y)_X = \mathbb{R}Y_{-\alpha}$. The existence of such an element Y is guaranteed by equation (5.1) in Proposition 5.2. The projection of Y onto $\mathfrak{g}_{-\alpha}$ along the root space decomposition is up to scaling equal to $Y_{-\alpha}$. After rescaling Y, we may therefore assume that Y decomposes as

$$Y = Y_{-\alpha} + \sum_{\substack{\beta \in \Sigma \cup \{0\} \\ \beta \neq \alpha}} Y_{-\beta}$$

with $Y_{-\beta} \in \mathfrak{g}_{-\beta}$ if $\beta \in \Sigma$ and $Y_0 \in \mathfrak{m} \oplus \mathfrak{a}$. Since $(\mathbb{R}Y)_X = \mathbb{R}Y_{-\alpha}$, the element $Y_{-\beta}$ can only be non-zero if $\beta(X) \geq \alpha(X) > 0$ (and since X is order-regular, equality holds if and only if $\beta = \alpha$). Therefore, $B(X_\perp, Y_{-\beta}) = B(U_{\beta'}, Y_{-\beta}) = 0$ for all $\beta \in \Sigma \cup \{0\}$ for which $Y_{-\beta} \neq 0$ and all $\beta' \in \Sigma(Q)$ for which $U_{\beta'} \neq 0$. It follows that

$$B(U_\alpha, Y_{-\alpha}) = B\big(\operatorname{Ad}(n_+)X_\perp, Y\big) = B\big(X_\perp, \operatorname{Ad}(n_+^{-1})Y\big) = 0.$$

For the last equality we used that $\operatorname{Ad}(n_+^{-1})Y \in \mathfrak{h}_z$. Since $-B(\,\cdot\,, \theta\,\cdot\,)$ is positive definite on the semisimple part of $\mathfrak{g}$, we conclude that $U_\alpha = 0$. This is a contradiction.

We have now proven that $\mathcal{C}_{z'} \subseteq \mathcal{C}_z$. For the second assertion in the proposition we may reverse the role of z' and z and further obtain the other inclusion $\mathcal{C}_z \subseteq \mathcal{C}_{z'}$. $\square$

Proposition 6.5 allows us to make the following definition.

Definition 6.8. We define $\mathcal{C} \subseteq \mathfrak{a}$ to be the cone given by $\mathcal{C} := \mathcal{C}_z$, where z is any adapted point in Z. The cone $\mathcal{C}$ is called the compression cone of Z.

Let $\mathfrak{a}_E$ be the edge of $\overline{\mathcal{C}}$, i.e.,

$$\mathfrak{a}_E := \overline{\mathcal{C}} \cap (-\overline{\mathcal{C}}). \tag{6.5}$$

We note that $\mathfrak{a}_E$ is a subspace of $\mathfrak{a}$. We end this section with a description of the relation between $\mathfrak{a}_E$ and the normalizer of $\mathfrak{h}_z$.

Recall the set S_z from (6.2).

Proposition 6.9. *Let $z \in Z$ be adapted.*

(i) The space $\mathfrak{a}_E$ is equal to the joint kernel of S_z, i.e.,

$$\mathfrak{a}_E = \{X \in \mathfrak{a} : \sigma(X) = 0 \text{ for all } \sigma \in S_z\}.$$

(ii) $\mathfrak{a}_E = N_{\mathfrak{a}}(\mathfrak{h}_z)$.

Proof. Assertion (i) follows from Lemma 6.3. We move on to (ii). It follows from (i) that $\mathfrak{a}_E$ normalizes the graph $\mathcal{G}(T_z)$. Moreover, $\mathfrak{a}$ normalizes $\mathfrak{l}_Q \cap \mathfrak{h}_z$, and hence $\mathfrak{a}_E$ normalizes $\mathfrak{h}_z$. This shows that $\mathfrak{a}_E \subseteq N_{\mathfrak{a}}(\mathfrak{h}_z)$.

Let $X \in N_{\mathfrak{a}}(\mathfrak{h}_z)$. For every $Y \in \mathfrak{a}$ we have $\mathfrak{h}_{z,X+Y} = \mathfrak{h}_{z,Y}$. In particular $N_{\mathfrak{a}}(\mathfrak{h}_z) + \mathcal{C} = \mathcal{C}$. It follows that $N_{\mathfrak{a}}(\mathfrak{h}_z) \subseteq \overline{\mathcal{C}}$ and thus $N_{\mathfrak{a}}(\mathfrak{h}_z) \subseteq \overline{\mathcal{C}} \cap (-\overline{\mathcal{C}}) = \mathfrak{a}_E$. $\square$

7 Limit subalgebras and open P-orbits

In this section we describe a relation between limits of $\mathfrak{h}_z$ and open P-orbits in $PN_G(\mathfrak{a})\cdot z$ for an adapted point z.

We define the group

$$\mathcal{N} := N_G(\mathfrak{a}) \cap N_G(\mathfrak{l}_{Q,\mathrm{nc}} + \mathfrak{a}_{\mathfrak{h}}). \tag{7.1}$$

The group $\mathcal{N}$ is relevant because of the following lemma.

Lemma 7.1. *Let $v \in N_G(\mathfrak{a})$, $z \in Z$, and $X \in \mathfrak{a}$. If z is adapted and $\mathfrak{h}_{z,X} = \mathrm{Ad}(v)\mathfrak{h}_\emptyset$, then $v \in \mathcal{N}$.*

Proof. Since z is adapted, the $\mathfrak{a}$-stable subalgebra $\mathfrak{l}_{Q,\mathrm{nc}}+\mathfrak{a}_{\mathfrak{h}}$ is contained in $\mathfrak{h}_z$. Therefore, $\mathfrak{l}_{Q,\mathrm{nc}} + \mathfrak{a}_{\mathfrak{h}}$ is also contained in $\mathfrak{h}_{z,X} = \mathrm{Ad}(v)\mathfrak{h}_\emptyset$. From (6.1) it is easily seen that the maximal θ-stable subspace of $\mathrm{Ad}(v)\mathfrak{h}_\emptyset$ is $\mathrm{Ad}(v)(\mathfrak{l}_Q\cap\mathfrak{h})$. Since $\mathfrak{l}_{Q,\mathrm{nc}}$ is θ-stable, it follows that $\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathrm{Ad}(v)(\mathfrak{l}_Q \cap \mathfrak{h})$. From the fact that $\mathfrak{l}_{Q,\mathrm{nc}}$ is the sum of all non-compact simple ideals in $\mathfrak{l}_Q \cap \mathfrak{h}$, it follows that in fact $\mathfrak{l}_{Q,\mathrm{nc}} = \mathrm{Ad}(v)\mathfrak{l}_{Q,\mathrm{nc}}$. Thus we find that v normalizes $\mathfrak{l}_{Q,\mathrm{nc}}$. Moreover,

$$\mathfrak{a}_{\mathfrak{h}} \subseteq \mathfrak{a}\cap\mathfrak{h}_{z,X} = \mathfrak{a}\cap\mathrm{Ad}(v)\mathfrak{h}_\emptyset = \mathrm{Ad}(v)\big(\mathfrak{a}\cap\mathfrak{h}_\emptyset\big) = \mathrm{Ad}(v)\mathfrak{a}_{\mathfrak{h}}.$$

Therefore, $v \in \mathcal{N}$. ☐

The main result of this section is the following proposition.

Proposition 7.2. *Let $z \in Z$ be adapted and let $w \in \mathcal{N}$. The following are equivalent.*

(i) There exists a $X \in \mathfrak{a}$ so that $\mathfrak{h}_{z,X} = \mathrm{Ad}(wm)\mathfrak{h}_\emptyset$ for some $m \in M$,

(ii) $Pw^{-1}\cdot z$ is open in Z,

(iii) $X \in \mathrm{Ad}(w)\mathcal{C}$ if and only if $\mathfrak{h}_{z,X} = \mathrm{Ad}(wm)\mathfrak{h}_\emptyset$ for some $m \in M$.

Before we prove the proposition, we first prove a lemma.

Lemma 7.3. *Let $z \in Z$ and $v \in \mathcal{N}$. If z is adapted and $Pv^{-1}\cdot z$ is open, then $v^{-1}\cdot z$ is adapted.*

Proof. Assume that z is adapted and $Pv^{-1}\cdot z$ is open. As v normalizes $\mathfrak{a}$ and $\mathfrak{l}_{Q,\mathrm{nc}}+\mathfrak{a}_{\mathfrak{h}}$, it also normalizes $\mathfrak{m}$ and hence $\mathfrak{a}+\mathfrak{m}+\mathfrak{l}_{Q,\mathrm{nc}} = \mathfrak{l}_Q$. If $X \in \mathfrak{a}\cap\mathfrak{h}_z^\perp$ is so that $\mathfrak{l}_Q = Z_{\mathfrak{g}}(X)$, then

$$\mathfrak{l}_Q = \mathrm{Ad}(v^{-1})\mathfrak{l}_Q = Z_{\mathfrak{g}}(\mathrm{Ad}(v^{-1})X).$$

Moreover, $\mathrm{Ad}(v^{-1})X \in \mathrm{Ad}(v^{-1})\big(\mathfrak{a}\cap\mathfrak{h}_z^\perp\big) = \mathfrak{a}\cap\mathfrak{h}_{v^{-1}\cdot z}^\perp$. The assertion now follows from Proposition 3.19. ☐

Proof of Proposition 7.2. *(i)⇒(ii):* Let $X \in \mathfrak{a}$. If $\mathfrak{h}_{z,X} = \mathrm{Ad}(wm)\mathfrak{h}_\emptyset$ for some $m \in M$, then

$$\mathfrak{h}_{w^{-1}\cdot z, \mathrm{Ad}(w^{-1})X} = \mathrm{Ad}(w^{-1})\mathfrak{h}_{z,X} = \mathrm{Ad}(m)\mathfrak{h}_\emptyset,$$

and hence $\mathrm{Ad}(w^{-1})X \in \mathcal{C}_{w^{-1}\cdot z}$. Now $Pw^{-1}\cdot z$ is open in view of Lemma 6.4 (ii).
(ii)⇒(iii): By Lemma 7.3 the point $w^{-1}\cdot z$ is adapted. It follows from Proposition 6.5 that $\mathcal{C}_{w^{-1}\cdot z} = \mathcal{C}$. Therefore $X \in \mathrm{Ad}(w)\mathcal{C}$ if and only if

$$\mathfrak{h}_{w^{-1}\cdot z, \mathrm{Ad}(w^{-1})X} = \mathrm{Ad}(m)\mathfrak{h}_\emptyset$$

for some $m \in M$. The implication now follows from the identity

$$\mathrm{Ad}(w^{-1})\mathfrak{h}_{z,X} = \mathfrak{h}_{w^{-1}\cdot z, \mathrm{Ad}(w^{-1})X}.$$

(iii)⇒(i): This implication is trivial.

□

8 Limits of $\mathfrak{h}_z$

In this section we describe the closure of $\mathrm{Ad}(G)\mathfrak{h}_z$ in the Grassmannian. We will show that this closure is a finite union of G-orbits, each of the form $\mathrm{Ad}(G)\mathfrak{h}_{z,X}$, where z is an adapted point in Z and $X \in \overline{\mathcal{C}}$. The crucial tool for this is the polar decomposition ([16]) for Z.

Recall the set S_z defined in (6.2). For an adapted point $z \in Z$ let $\mathcal{M}_z$ be the monoid generated by S_z, i.e.,

$$\mathcal{M}_z := \mathbb{N}S_z. \tag{8.1}$$

We note that the negative dual cone

$$-\mathcal{C}^\vee := -\{\xi \in \mathfrak{a}^* : \xi(X) \geq 0 \text{ for all } X \in \mathcal{C}\}$$

of $\mathcal{C}$ is equal to the cone generated by $\mathcal{M}_z$. A priori $\mathcal{M}_z$ may depend on the adapted point $z \in Z$, but the cone spanned by $\mathcal{M}_z$ is independent of z. We write $\mathcal{S}_z$ for the set of indecomposable elements in $\mathcal{M}_z$. Note that $\mathcal{S}_z \subseteq S_z$.

The closure of the compression cone $\overline{\mathcal{C}}$ is finitely generated and hence polyhedral as $-\mathcal{C}^\vee$ is finitely generated. We call a subset $\mathcal{F} \subseteq \overline{\mathcal{C}}$ a face of $\overline{\mathcal{C}}$ if $\mathcal{F} = \overline{\mathcal{C}}$ or there exists a closed half-space $\mathcal{H}$ so that

$$\mathcal{F} = \overline{\mathcal{C}} \cap \mathcal{H} \quad \text{and} \quad \mathcal{C} \cap \partial\mathcal{H} = \emptyset.$$

There exist finitely many faces of $\overline{\mathcal{C}}$ and each face is polyhedral cone. A face $\mathcal{F}$ of $\overline{\mathcal{C}}$ is said to be a wall of $\mathcal{C}$ or $\overline{\mathcal{C}}$ if $\mathrm{span}(\mathcal{F})$ has codimension 1.

Let $z \in Z$ be adapted. Each subset S of $\mathcal{S}_z$ corresponds to a face $\mathcal{F}$ of $\overline{\mathcal{C}}$, namely

$$\mathcal{F} = \overline{\mathcal{C}} \cap \bigcap_{\alpha \in S} \ker(\alpha). \tag{8.2}$$

The map from the power set of $\mathcal{S}_z$ to the set of faces of $\overline{\mathcal{C}}$ is surjective. If $\mathcal{C}^\vee$ is generated by a set of linearly independent elements, then this map is also injective. If $\mathcal{F}$ is a wall of $\mathcal{C}$, then there exists an element $\alpha \in \mathcal{S}_z$ so that

$$\mathcal{F} = \overline{\mathcal{C}} \cap \ker(\alpha). \tag{8.3}$$

For an adapted point $z \in Z$ and a face $\mathcal{F}$ of $\overline{\mathcal{C}}$ we define

$$\mathcal{M}_{z,\mathcal{F}} := \{\sigma \in \mathcal{M}_z : \sigma\big|_{\mathcal{F}} = 0\}. \tag{8.4}$$

Note that $\mathcal{M}_{z,\mathcal{F}}$ is a submonoid of $\mathcal{M}_z$. We further note that the annihilator of $\mathcal{M}_{z,\mathcal{F}}$ is equal to

$$\mathfrak{a}_{\mathcal{F}} := \operatorname{span}(\mathcal{F}),$$

i.e.,

$$\bigcap_{\sigma\in\mathcal{M}_{z,\mathcal{F}}} \ker(\sigma) = \mathfrak{a}_{\mathcal{F}}. \tag{8.5}$$

Lemma 8.1. *Let $z \in Z$ be adapted and let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. Let $\mathcal{M}_{z,\mathcal{F}}$ be as in (8.4). For every X in the (relative) interior of $\mathcal{F}$*

$$\mathfrak{h}_{z,X} = (\mathfrak{l}_Q \cap \mathfrak{h}_z) \oplus \bigoplus_{\alpha\in\Sigma(Q)} \mathcal{G}\Big(\sum_{\sigma\in-\alpha+\mathcal{M}_{z,\mathcal{F}}} p_\sigma \circ T_z\big|_{\mathfrak{g}_{-\alpha}}\Big). \tag{8.6}$$

In particular, for all X and X' in the (relative) interior of $\mathcal{F}$

$$\mathfrak{h}_{z,X} = \mathfrak{h}_{z,X'}.$$

Proof. Let X be an element from the interior of $\mathcal{F}$. Then

$$\mathcal{M}_{z,\mathcal{F}} = \{\sigma \in \mathcal{M}_z : \sigma(X) = 0\}.$$

For $\sigma \in \Sigma \cup \{0\}$, let $p_\sigma : \mathfrak{g} \to \mathfrak{g}_\sigma$ be the projection onto $\mathfrak{g}_\sigma$ along the Bruhat decomposition, where $\mathfrak{g}_0$ denotes $\mathfrak{m} \oplus \mathfrak{a}$. If $\alpha \in \Sigma(Q)$ and $Y \in \mathfrak{g}_{-\alpha}$, then

$$\begin{aligned}&\operatorname{Ad}\big(\exp(tX)\big)\big(Y + T_z(Y)\big)\\ &\quad= e^{-t\alpha(X)}\Big(Y + \sum_{\sigma\in-\alpha+\mathcal{M}_{z,\mathcal{F}}} p_\sigma T_z(Y)\Big) + \sum_{\sigma\in(-\alpha+\mathcal{M}_z)\setminus(-\alpha+\mathcal{M}_{z,\mathcal{F}})} e^{t\sigma(X)} p_\sigma T_z(Y).\end{aligned}$$

If $\sigma \in -\alpha + \mathcal{M}_z$ but $\sigma \notin -\alpha + \mathcal{M}_{z,\mathcal{F}}$, then $\sigma(X) < -\alpha(X)$. Therefore,

$$\Big(\mathbb{R}\big(Y + T_z(Y)\big)\Big)_X = \mathbb{R}\Big(Y + \sum_{\sigma\in-\alpha+\mathcal{M}_{z,\mathcal{F}}} p_\sigma T_z(Y)\Big),$$

and hence

$$\bigoplus_{\alpha\in\Sigma(Q)} \mathcal{G}\Big(\sum_{\sigma\in-\alpha+\mathcal{M}_{z,\mathcal{F}}} p_\sigma \circ T_z\big|_{\mathfrak{g}_{-\alpha}}\Big) \subseteq \big(\mathcal{G}(T_z)\big)_X.$$

In fact, equality holds since the dimensions of both spaces are equal. As

$$\mathfrak{h}_{z,X} = (\mathfrak{l}_Q \cap \mathfrak{h}_z) \oplus \big(\mathcal{G}(T_z)\big)_X,$$

this proves (8.6). It follows from (8.6) that $\mathfrak{h}_{z,X}$ does not depend on the choice of X in the interior of $\mathcal{F}$. $\square$

Lemma 8.1 allows us to make the following definition.

Definition 8.2. For an adapted point $z \in Z$ and a face $\mathcal{F}$ of $\overline{\mathcal{C}}$, define

$$\mathfrak{h}_{z,\mathcal{F}} := \mathfrak{h}_{z,X}$$

with X contained in the interior of $\mathcal{F}$.

We note that for every adapted point $z \in Z$ there exists an $m \in M$ so that

$$\mathfrak{h}_{z,\overline{\mathcal{C}}} = \mathrm{Ad}(m)\mathfrak{h}_{\emptyset}.$$

Lemma 8.3. *Let $z \in Z$ be adapted and let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. The Lie algebra $\mathfrak{h}_{z,\mathcal{F}}$ is a real spherical subalgebra of $\mathfrak{g}$. Moreover,*

$$N_{\mathfrak{g}}(\mathfrak{h}_{z,\mathcal{F}}) = \mathfrak{h}_{z,\mathcal{F}} + \mathfrak{a}_{\mathcal{F}} + N_{\mathfrak{m}}(\mathfrak{h}_{z,\mathcal{F}}).$$

Finally,

$$\mathfrak{h}_{z,\mathcal{F}} \cap \mathfrak{a} = \mathfrak{a}_{\mathfrak{h}}.$$

Proof. By Proposition 5.2 (iii) there exists an $m \in M$ so that for all $X \in \mathcal{C}$

$$(\mathfrak{h}_{z,\mathcal{F}})_X = \mathfrak{h}_{z,X} = \mathrm{Ad}(m)\mathfrak{h}_{\emptyset}.$$

Since $\overline{\mathfrak{n}}_P \subseteq \mathfrak{h}_{\emptyset}$, it follows that $(\mathfrak{h}_{z,\mathcal{F}})_X + \mathfrak{p} = \mathfrak{g}$ and hence $\mathrm{Ad}\big(\exp(tX)\big)\mathfrak{h}_{z,\mathcal{F}} + \mathfrak{p} = \mathfrak{g}$ for sufficiently large $t > 0$. Since $\mathfrak{p}$ and $\mathfrak{g}$ are both stable under the action of A, we find

$$\mathfrak{h}_{z,\mathcal{F}} + \mathfrak{p} = \mathfrak{g}.$$

In particular $\mathfrak{h}_{z,\mathcal{F}}$ is a real spherical subalgebra of $\mathfrak{g}$.

By Corollary 3.15

$$N_{\mathfrak{g}}(\mathfrak{h}_{z,\mathcal{F}}) = \mathfrak{h}_{z,\mathcal{F}} + N_{\mathfrak{a}}(\mathfrak{h}_{z,\mathcal{F}}) + N_{\mathfrak{m}}(\mathfrak{h}_{z,\mathcal{F}}).$$

To prove the second assertion in the lemma, it suffices to show that $N_{\mathfrak{a}}(\mathfrak{h}_{z,\mathcal{F}}) = \mathfrak{a}_{\mathcal{F}}$. It follows from equation (8.6) that $\mathfrak{h}_{z,\mathcal{F}}$ is normalized by $\mathfrak{a}_{\mathcal{F}}$, and hence $\mathfrak{a}_{\mathcal{F}} \subseteq N_{\mathfrak{a}}(\mathfrak{h}_{z,\mathcal{F}})$. To prove the other inclusion, let $X \in N_{\mathfrak{a}}(\mathfrak{h}_{z,\mathcal{F}})$. It follows from (8.6) that $\sigma(X) = 0$ for all $\sigma \in \mathcal{M}_{z,\mathcal{F}}$ so that $-\alpha + \sigma \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha})$ for some $\alpha \in \Sigma(Q)$. The submonoid $\mathcal{M}_{z,\mathcal{F}}$ is generated by the indecomposable elements from $\mathcal{M}_z$ that vanish on $\mathcal{F}$. Therefore, there exists a set of generators σ of $\mathcal{M}_{z,\mathcal{F}}$ with $-\alpha + \sigma \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha})$ for some $\alpha \in \Sigma(Q)$. It follows that X is in the joint kernel of a set of generators of $\mathcal{M}_{z,\mathcal{F}}$, and hence $\sigma(X) = 0$ for all $\sigma \in \mathcal{M}_{z,\mathcal{F}}$. By (8.5) the annihilator of $\mathcal{M}_{z,\mathcal{F}}$ is equal to $\mathfrak{a}_{\mathcal{F}}$. Therefore, $X \in \mathfrak{a}_{\mathcal{F}}$. This proves the second assertion.

Finally, for every $X \in \mathcal{C}$

$$\mathfrak{a}_{\mathfrak{h}} \subseteq \mathfrak{a} \cap \mathfrak{h}_{z,\mathcal{F}} = (\mathfrak{a} \cap \mathfrak{h}_{z,\mathcal{F}})_X \subseteq \mathfrak{a} \cap (\mathfrak{h}_{z,\mathcal{F}})_X = \mathfrak{a} \cap \mathfrak{h}_{\emptyset} \subseteq \mathfrak{a}_{\mathfrak{h}}.$$

Here we used Proposition 5.2 (iii) for the second equality. It follows that $\mathfrak{a} \cap \mathfrak{h}_{z,\mathcal{F}} = \mathfrak{a}_{\mathfrak{h}}$. ∎

The following proposition describes the dependence of the Lie algebras $\mathfrak{h}_{z,\mathcal{F}}$ on the adapted point z.

Proposition 8.4. *Let $z, z' \in Z$ be adapted and let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. If $P \cdot z = P \cdot z'$, then*

$$\mathrm{Ad}(G)\mathfrak{h}_{z,\mathcal{F}} = \mathrm{Ad}(G)\mathfrak{h}_{z',\mathcal{F}}.$$

The proof for the proposition relies on the following two lemmas. Recall the map $T_z^\perp : \mathfrak{a}^\circ \to Z_{\mathfrak{n}_Q}(\mathfrak{l}_Q \cap \mathfrak{h}_z)$ from Lemma 3.10.

Lemma 8.5. *Let $z \in Z$ be adapted. Then*

$$\mathrm{Im}\,(T_z^\perp) \subseteq \bigoplus_{\substack{\alpha\in\Sigma(Q)\\ \mathfrak{a}\in\mathrm{supp}_z(\mathfrak{g}_{-\alpha})}} \mathfrak{g}_\alpha.$$

Proof. Let $p_\mathfrak{a}$ be the projection $\mathfrak{g} \to \mathfrak{a}$ along the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$. We claim that

$$\mathrm{Im}\,(T_z^\perp) \subseteq \big(\ker(p_\mathfrak{a} \circ T_z)\big)^\perp \cap \mathfrak{n}_Q \tag{8.7}$$

To prove the claim, let $X \in \mathfrak{a}^\circ$ and $Y \in \ker(p_\mathfrak{a} \circ T_z)$. Now $T_z(Y) \in \mathfrak{m} \oplus \mathfrak{n}_Q$. Using that $Y \in \overline{\mathfrak{n}}_Q$, it follows that $Y + T_z(Y) \in \overline{\mathfrak{n}}_Q \oplus \mathfrak{m} \oplus \mathfrak{n}_Q$. Therefore, $B\big(X, Y + T_z(Y)\big) = 0$. Moreover, as $T_z^\perp(X) \in \mathfrak{n}_Q$ and $T_z(Y) \in \mathfrak{m} \oplus \mathfrak{n}_Q$, we have $B\big(T_z^\perp(X), T_z(Y)\big) = 0$. It follows that

$$\begin{aligned} B\big(T_z^\perp(X), Y\big) &= B\big(T_z^\perp(X), Y\big) + B\big(T_z^\perp(X), T_z(Y)\big) + B\big(X, Y + T_z(Y)\big) \\ &= B\big(X + T_z^\perp(X), Y + T_z(Y)\big). \end{aligned}$$

The right-hand side vanishes as $X + T_z^\perp(X) \in \mathfrak{h}_z^\perp$ and $Y + T_z(Y) \in \mathfrak{h}_z$. It follows that $B\big(\mathrm{Im}\,(T_z^\perp), \ker(p_\mathfrak{a} \circ T_z)\big) = \{0\}$, and hence the claimed identity (8.7) follows.

We have

$$\big(\ker(p_\mathfrak{a} \circ T_z)\big)^\perp \cap \mathfrak{n}_Q \subseteq \Big(\bigoplus_{\substack{\alpha\in\Sigma(Q)\\ \mathfrak{a}\notin\mathrm{supp}_z(\mathfrak{g}_{-\alpha})}} \mathfrak{g}_{-\alpha}\Big)^\perp \cap \mathfrak{n}_Q = \bigoplus_{\substack{\alpha\in\Sigma(Q)\\ \mathfrak{a}\in\mathrm{supp}_z(\mathfrak{g}_{-\alpha})}} \mathfrak{g}_\alpha,$$

and hence

$$\mathrm{Im}\,(T_z^\perp) \subseteq \bigoplus_{\substack{\alpha\in\Sigma(Q)\\ \mathfrak{a}\in\mathrm{supp}_z(\mathfrak{g}_{-\alpha})}} \mathfrak{g}_\alpha$$

in view of (8.7). □

Recall the map $\Phi_z : \mathfrak{a}^\circ_{\mathrm{reg}} \to \mathfrak{n}_Q$ from (3.10).

Lemma 8.6. *Let $z \in Z$ be adapted. Then*

$$\mathrm{Im}\,(\Phi_z) \subseteq \bigoplus_{\substack{\alpha\in\Sigma(Q)\\ \alpha|_{\overline{\mathcal{C}}}\leq 0}} \mathfrak{g}_\alpha.$$

Proof. If $X \in \mathfrak{a}_{\mathrm{reg}}^{\circ}$, then X does not vanish on any root in $\Sigma(Q)$, and hence the map

$$\Psi : \mathfrak{n}_Q \to \mathfrak{n}_Q, \quad Y \mapsto \mathrm{Ad}\big(\exp(Y)\big)X - X$$

is a diffeomorphism. As

$$\mathfrak{n}_0 := \bigoplus_{\substack{\alpha\in\Sigma(Q)\\ \alpha|_{\overline{\mathcal{C}}}\leq 0}} \mathfrak{g}_\alpha$$

is an $\mathfrak{a}$-stable Lie subalgebra of $\mathfrak{n}_Q$, the restriction of Ψ to $\mathfrak{n}_0$ maps $\mathfrak{n}_0$ onto itself. Therefore, it suffices to prove that $\mathrm{Ad}\big(\exp\big(\Phi_z(X)\big)\big)X - X \in \mathfrak{n}_0$.

It follows from (3.9) that $\mathrm{Ad}\big(\exp\big(\Phi_z(X)\big)\big)X - X \in \mathrm{Im}\,(T_z^{\perp})$ for every $X \in \mathfrak{a}_{\mathrm{reg}}^{\circ}$. By Lemma 8.5 the image of $T_z^{\perp}$ is contained in the direct sum of all root spaces for roots $\alpha \in \Sigma(Q)$ with $\mathfrak{a} \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha})$. By Lemma 6.3 we have $\alpha\big|_{\overline{\mathcal{C}}} \leq 0$ for any such root. This proves the lemma. ☐

Proof of Proposition 8.4. Assume that $P\cdot z = P\cdot z'$ and let X be contained in the interior of $\mathcal{F}$. By Proposition 3.12 there exist $m \in M$, $a \in A$ and $n \in \exp\big(\mathrm{Im}\,(\Phi_z)\big)$ so that $z' = man\cdot z$. By Lemma 8.6

$$n \in \exp\Big(\bigoplus_{\substack{\alpha\in\Sigma(Q)\\ \alpha(X)\leq 0}} \mathfrak{g}_\alpha\Big).$$

It follows that the limit for $t \to \infty$ of $\exp(tX)man\exp(tX)^{-1} = ma\exp(tX)n\exp(tX)^{-1}$ exists in G. We write g for the limit. By Proposition 5.2 (iv) we now have

$$\mathfrak{h}_{z',\mathcal{F}} = \mathfrak{h}_{z',X} = \big(\mathrm{Ad}(man)\mathfrak{h}_z\big)_X = \mathrm{Ad}(g)\mathfrak{h}_{z,X} \in \mathrm{Ad}(G)\mathfrak{h}_{z,X} = \mathrm{Ad}(G)\mathfrak{h}_{z,\mathcal{F}}.$$

☐

We continue with a description of the closure of $\mathrm{Ad}(G)\mathfrak{h}_z$ in the Grassmannian. For this we need the so-called polar decomposition. The following proposition, describing the polar decomposition for Z, is an adaptation from [16, Theorem 5.13].

Proposition 8.7. *Let $\Xi \subseteq Z$ be a finite set of adapted points so that $P\cdot\Xi$ is the union of all open P-orbits in Z. Then there exists a compact subset $\Omega \subseteq G$ so that*

$$Z = \Omega\exp(\overline{\mathcal{C}})\cdot\Xi. \tag{8.8}$$

Proof. By [16, Theorem 5.13] there exists an adapted point $z_0 \in Z$, a finite set $F \subseteq G\cap\exp(i\mathfrak{a})N_{G_{\mathbb{C}}}(\mathfrak{h}_{z_0,\mathbb{C}})$ and a compact set $\Omega_0 \subseteq G$ so that

$$Z = \Omega_0\exp(\overline{\mathcal{C}})F\cdot z_0. \tag{8.9}$$

Moreover, for every open P-orbit $\mathcal{O}$ in Z there exists an $f \in F$ so that $f\cdot z_0 \in \mathcal{O}$. A priori it is possible that there exists $f, f' \in F$ with $f \neq f'$, but $Pf\cdot z_0 = Pf'\cdot z_0$.

We claim that for every $f \in F$ the point $f\cdot z_0$ is adapted and $\mathfrak{a}\cap\mathfrak{h}_{f\cdot z_0}^{\perp} = \mathfrak{a}\cap\mathfrak{h}_{z_0}^{\perp}$. The proof for the claim is the same as the proof for the analogous statements in Proposition 3.13.

Let $\mathcal{O}$ be an open P-orbit in Z and let $f \in F$ be so that $f \cdot z_0 \in \mathcal{O}$. By Proposition 3.13 we may choose a $f_{\mathcal{O}} \in G \cap \exp(i\mathfrak{a})H_{z,\mathbb{C}}$ so that $Pf_{\mathcal{O}} \cdot z_0 = \mathcal{O}$. Then

$$\mathfrak{a} \cap \mathfrak{h}_{f_{\mathcal{O}} \cdot z_0}^{\perp} = \mathfrak{a} \cap \mathfrak{h}_{z_0}^{\perp} = \mathfrak{a} \cap \mathfrak{h}_{f \cdot z_0}^{\perp}$$

In view of Lemma 3.8 and the decomposition (3.1) of L_Q with $z = z_0$, there exist for every $f \in F_{\mathcal{O}}$ elements $m_f \in M$ and $a_f \in A$ so that

$$f \cdot z_0 = m_f a_f f_{\mathcal{O}} \cdot z_0.$$

It follows from (8.9) that

$$Z = \Omega_1 \exp(\overline{\mathcal{C}}) F_1 \cdot z_0, \tag{8.10}$$

where

$$\Omega_1 := \Omega_0 \{m_f a_f : f \in F\} \quad \text{and} \quad F_1 := \{f_{\mathcal{O}} : \mathcal{O} \text{ is an open } P\text{-orbit}\}.$$

Note that Ω_1 is compact.

Let $\mathcal{O}$ be an open P-orbit and let $z \in \Xi \cap \mathcal{O}$. By Proposition 3.12 there exist $m_z \in M$, $a_z \in A$ and $n_z \in \operatorname{Im}(\exp \circ \Phi_{f_{\mathcal{O}} \cdot z_0})$ so that $f_{\mathcal{O}} \cdot z_0 = m_z a_z n_z \cdot z$. It follows from (8.10) that (8.8) holds with

$$\Omega := \Omega_2 \Big(\bigcup_{z \in \Xi} \overline{\{m_z a_z a n_z a^{-1} : a \in \exp(\overline{\mathcal{C}})\}} \Big).$$

In view of Lemma 8.6 the elements $\log(n_z)$ are sums of root vectors for roots that are non-positive on $\overline{\mathcal{C}}$. Therefore, the sets $\{m_z a_z a n_z a^{-1} : a \in \exp(\overline{\mathcal{C}})\}$ are bounded, and thus we conclude that Ω is compact. □

Proposition 8.8. *Let $z_0 \in Z$ and let $\Xi \subseteq Z$ be a finite set of adapted points so that $P \cdot \Xi$ is the union of all open P-orbits in Z. Then the following equality of subsets of the Grassmannian of $\dim(\mathfrak{h}_{z_0})$-dimensional subspaces of $\mathfrak{g}$ holds,*

$$\overline{\operatorname{Ad}(G)\mathfrak{h}_{z_0}} = \bigcup_{z \in \Xi, \mathcal{F} \text{ face of } \overline{\mathcal{C}}} \operatorname{Ad}(G)\mathfrak{h}_{z,\mathcal{F}}.$$

Proof. Let Ω be a compact subset of G so that (8.8) holds. Let $\mathfrak{s} \in \overline{\operatorname{Ad}(G)\mathfrak{h}_{z_0}}$ and let $(\omega_n)_{n \in \mathbb{N}}$, $(a_n)_{n \in \mathbb{N}}$ and $(z_n)_{n \in \mathbb{N}}$ be sequences in Ω, $\exp(\overline{\mathcal{C}})$ and Ξ, respectively, so that $\operatorname{Ad}(\omega_n a_n)\mathfrak{h}_{z_n}$ converges to $\mathfrak{s}$ for $n \to \infty$. By taking suitable subsequences we may assume that ω_n converges to an element $\omega \in \Omega$ for $n \to \infty$ and $z_n = z$ is constant.

Let I be the subset of $\mathcal{S}_z$ consisting of all $\alpha \in \mathcal{S}_z$ so that a_n^α is bounded away from 0. By taking a suitable subsequence we assume that there exists a convergent sequence $b_n \in A$ so that $(b_n^{-1} a_n)^\alpha$ is equal to 1 for all $\alpha \in I$ and converges to 0 as $n \to \infty$ for all $\alpha \in \mathcal{S}_z \setminus I$. Let $b \in A$ be the limit of the sequence $(b_n)_{n \in \mathbb{N}}$. Let $\mathcal{F}$ be the face of $\overline{\mathcal{C}}$ defined by I via the formula (8.2). Now

$$\lim_{n \to \infty} \operatorname{Ad}(b_n^{-1} a_n)\mathfrak{h}_z = \mathfrak{h}_{z,\mathcal{F}}$$

and thus

$$\mathfrak{s} = \lim_{n \to \infty} \operatorname{Ad}(\omega_n a_n)\mathfrak{h}_z = \operatorname{Ad}(\omega b)\mathfrak{h}_{z,\mathcal{F}} \in \operatorname{Ad}(G)\mathfrak{h}_{z,\mathcal{F}}.$$

□

9 Walls of the compression cone

For every wall $\mathcal{F}$ of $\overline{\mathcal{C}}$ there exists an $\alpha \in \Sigma + (\Sigma \cup \{0\})$ so that (8.3) holds. The main result in this section is the following proposition, which puts restriction on the elements α which can occur. The result will be needed for the proof of Lemma 12.4.

Proposition 9.1. *Let $\mathcal{F}$ be a wall of $\mathcal{C}$. Then either there exists a root $\alpha \in \Sigma(Q)$ so that $\mathfrak{a}_{\mathcal{F}} = \ker(\alpha)$, or there exist $\beta, \gamma \in \Sigma(Q)$ so that $\mathfrak{a}_{\mathcal{F}} = \ker(\beta + \gamma)$ and the following hold,*

(i) β *is a simple root,*

(ii) β *and* γ *are orthogonal*

(iii) $\mathrm{span}(\beta^\vee, \gamma^\vee) \cap \mathfrak{a}_{\mathfrak{h}} \neq \{0\}$.

Remark 9.2. Brion proved a stronger version of this lemma under the additional assumption that G and H are complex groups. See [4, Theorem 2.6]. The proof of Proposition 9.1 is heavily inspired by the proof of Brion.

Before we prove the proposition, we first prove a lemma. Recall the set $\mathcal{S}_z$ of indecomposable elements in the monoid $\mathcal{M}_z$, where $z \in Z$ is an adapted point.

Lemma 9.3. *Let $z \in Z$ be adapted. The set $\mathcal{S}_z$ consists of $\alpha + \beta \in S_z$, with α a simple root in $\Sigma(Q)$, and $\beta \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha}) \cap \Sigma(Q)$ or $\beta = 0$.*

Proof. We first choose a suitable linear order on Σ. For this let $X^\circ \in \mathfrak{a}^\circ_{\mathrm{reg}}$ and $X_{\mathfrak{h}} \in \mathfrak{a}_{\mathfrak{h}}$ be so that $X = X^\circ + X_{\mathfrak{h}}$ is order-regular and $\alpha(X) > 0$ for all $\alpha \in \Sigma(Q)$. By rescaling X°, we may assume that $\alpha(X) < \beta(X)$ whenever $\alpha, \beta \in \Sigma$ and $\alpha(X^\circ) < \beta(X^\circ)$. Let $>$ be the linear order on $\Sigma(Q)$ given by $\alpha > \beta$ if and only if $\alpha(X) > \beta(X)$.

For $\gamma \in \Sigma(Q) \cup \{\mathfrak{m}, \mathfrak{a}\}$ we define $\tilde{\gamma} \in \Sigma(Q) \cup \{0\}$ to be equal to γ if $\gamma \in \Sigma(Q)$ and 0 otherwise. Further, for a root $\alpha \in \Sigma(Q)$ we define $\mathcal{M}_{z,\alpha}$ to be the monoid generated by the set

$$\{\beta + \tilde{\gamma} : \beta \in \Sigma(Q), \beta \leq \alpha, \gamma \in \mathrm{supp}_z(\mathfrak{g}_{-\beta})\}.$$

Note that for the longest root $\alpha \in \Sigma(Q)$ we have $\mathcal{M}_{z,\alpha} = \mathcal{M}_z$.

To prove the lemma, we will show that a stronger assertion holds true, namely that for every $\gamma \in \Sigma(Q)$ each indecomposable element of $\mathcal{M}_{z,\gamma}$ is of the form $\alpha + \beta$ with α a simple root in $\Sigma(Q)$, and $\beta \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha}) \cap \Sigma(Q)$ or $\beta = 0$. This we will do by induction with respect to the length of the roots γ.

For simple roots $\gamma \in \Sigma(Q)$ the assertion is trivial. Now let $\alpha \in \Sigma$ be simple and $\beta \in \Sigma(Q)$ so that $\alpha + \beta \in \Sigma(Q)$. Assume that the assertion hold for all roots $\gamma \in \Sigma(Q)$ with $\gamma < \alpha + \beta$.

We have to consider two cases: the case that $\alpha \in \Sigma \backslash \Sigma(Q)$ and the case that $\alpha \in \Sigma(Q)$.

First we assume that $\alpha \in \Sigma \setminus \Sigma(Q)$ and that $\alpha + \beta$ is a root. We claim that

$$\mathcal{M}_{z,\alpha+\beta} = \mathcal{M}_{z,\beta}$$

Since the assertion is assumed to hold for β, it follows from the claim that the assertion also holds for $\alpha + \beta$. To prove the claim, we note that our choice of the linear order on

Σ guarantees that if $\delta \in \Sigma(Q)$ with $\beta < \delta \leq \alpha + \beta$, then $\delta - \beta \in \Sigma \setminus \Sigma(Q)$, and hence $\mathfrak{g}_{\beta-\delta} \in \mathfrak{l}_Q \cap \mathfrak{h}_z$. Since T_z is $(L_Q \cap H_z)$-equivariant by Lemma 4.1, we have

$$T_z([Y_{\beta-\delta}, Y]) = [Y_{\beta-\delta}, T_z(Y)] \qquad \big(Y_{\beta-\delta} \in \mathfrak{g}_{\beta-\delta}, Y \in \overline{\mathfrak{n}}_Q\big).$$

It follows that

$$\{\tilde{\gamma} : \gamma \in \mathrm{supp}_z(\mathfrak{g}_{-\delta})\} \subseteq \{\beta - \delta + \tilde{\gamma} : \gamma \in \mathrm{supp}_z(\mathfrak{g}_{-\beta})\},$$

and hence

$$\{\delta + \tilde{\gamma} : \gamma \in \mathrm{supp}_z(\mathfrak{g}_{-\delta})\} \subseteq \{\beta + \tilde{\gamma} : \gamma \in \mathrm{supp}_z(\mathfrak{g}_{-\beta})\}.$$

Therefore,

$$\begin{aligned}\mathcal{M}_{z,\alpha+\beta} &= \big\langle \delta + \tilde{\gamma} : \delta \in \Sigma(Q), \delta \leq \alpha + \beta, \gamma \in \mathrm{supp}_z(\mathfrak{g}_{-\delta})\big\rangle \\ &= \Big\langle \mathcal{M}_{z,\beta} \cup \big\{\delta + \tilde{\gamma} : \delta \in \Sigma(Q), \beta < \delta \leq \alpha + \beta, \gamma \in \mathrm{supp}_z(\mathfrak{g}_{-\delta})\big\}\Big\rangle \\ &\subseteq \Big\langle \mathcal{M}_{z,\beta} \cup \big\{\beta + \tilde{\gamma} : \gamma \in \mathrm{supp}_z(\mathfrak{g}_{-\beta})\big\}\Big\rangle = \mathcal{M}_{z,\beta}.\end{aligned}$$

The inclusion $\mathcal{M}_{z,\beta} \subseteq \mathcal{M}_{z,\alpha+\beta}$ is a consequence of the fact that $\beta < \alpha + \beta$. This proves the claim.

We now move on to the case that $\alpha \in \Sigma(Q)$. Let δ be the largest root so that $\delta < \alpha+\beta$. We claim that

$$\mathcal{M}_{z,\alpha+\beta} \subseteq \langle \mathcal{M}_{z,\alpha} \cup \mathcal{M}_{z,\delta}\rangle.$$

It follows from the claim that the indecomposable elements of $\mathcal{M}_{z,\alpha+\beta}$ are contained in the union of the sets of indecomposable elements of $\mathcal{M}_{z,\delta}$ and $\mathcal{M}_{z,\alpha}$. Since the assertion holds for α and is assumed to hold for δ, it follows that the assertion also holds for $\alpha + \beta$.

It remains to prove the claim. We first note that

$$\mathcal{M}_{z,\alpha+\beta} = \Big\langle \mathcal{M}_{z,\delta} \cup \big\{\alpha + \beta + \tilde{\gamma} : \gamma \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha-\beta})\big\}\Big\rangle$$

It thus suffices to prove that

$$\big\{\alpha + \beta + \tilde{\gamma} : \gamma \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha-\beta})\big\} \subseteq \langle \mathcal{M}_{z,\alpha} \cup \mathcal{M}_{z,\beta}\rangle.$$

Let $Y_{-\alpha} \in \mathfrak{g}_{-\alpha}$ and $Y_{-\beta} \in \mathfrak{g}_{-\beta}$. Let p_- be the projection onto $\overline{\mathfrak{n}}_Q$, respectively, along the decomposition $\mathfrak{g} = \overline{\mathfrak{n}}_Q \oplus \mathfrak{l}_Q \oplus \mathfrak{n}_Q$. From the uniqueness of the map T_z it follows that

$$\begin{aligned}&\big[Y_{-\alpha} + T_z(Y_{-\alpha}), Y_{-\beta} + T_z(Y_{-\beta})\big] \\ &\quad = [Y_{-\alpha}, Y_{-\beta}] + \big[Y_{-\alpha}, T_z(Y_{-\beta})\big] + \big[T_z(Y_{-\alpha}), Y_{-\beta}\big] + \big[T_z(Y_{-\alpha}), T_z(Y_{-\beta})\big] \\ &\quad = [Y_{-\alpha}, Y_{-\beta}] + T_z\big([Y_{-\alpha}, Y_{-\beta}]\big) + Y + T_z(Y),\end{aligned}$$

where

$$Y = p_-\Big(\big[Y_{-\alpha}, T_z(Y_{-\beta})\big] + \big[T_z(Y_{-\alpha}), Y_{-\beta}\big]\Big).$$

Therefore,

$$\begin{aligned}&T_z\big([Y_{-\alpha}, Y_{-\beta}]\big) \\ &\quad = \big[Y_{-\alpha}, T_z(Y_{-\beta})\big] + \big[T_z(Y_{-\alpha}), Y_{-\beta},\big] + \big[T_z(Y_{-\alpha}), T_z(Y_{-\beta})\big] - T_z(Y) - Y.\end{aligned} \tag{9.1}$$

Now let $\gamma \in S_{\alpha+\beta}$. Then $\gamma-\alpha-\beta \in \Sigma(Q)\cup\{0\}$ is a weight occurring in $T_z(\mathfrak{g}_{-\alpha-\beta}) = T_z\big([\mathfrak{g}_{-\alpha}, \mathfrak{g}_{-\beta}]\big)$. In view of (9.1) one of the following holds.

I. $\gamma-\alpha-\beta$ is a weight of $\mathfrak{a}$ occurring in $[\mathfrak{g}_{-\alpha}, T_z(\mathfrak{g}_{-\beta})]$. In this case $\gamma-\beta$ is a weight occurring in $T_z(\mathfrak{g}_{-\beta})$, and hence $\gamma \in \mathcal{M}_{z,\beta}$.

II. $\gamma-\alpha-\beta$ is a weight of $\mathfrak{a}$ occurring in $[T_z(\mathfrak{g}_{-\alpha}), \mathfrak{g}_{-\beta}]$. In this case $\gamma-\alpha$ is a weight occurring in $T_z(\mathfrak{g}_{-\alpha})$, and hence $\gamma \in \mathcal{M}_{z,\alpha}$.

III. $\gamma-\alpha-\beta$ is a weight of $\mathfrak{a}$ occurring in $[T_z(\mathfrak{g}_{-\alpha}), T_z(\mathfrak{g}_{-\beta})]$. In this case $\gamma-\alpha-\beta = \tilde{\delta}+\tilde{\epsilon}$ for some $\delta \in \operatorname{supp}_z(\mathfrak{g}_{-\alpha})$ and $\epsilon \in \operatorname{supp}_z(\mathfrak{g}_{-\beta})$. As $\alpha+\tilde{\delta} \in \mathcal{M}_{z,\alpha}$ and $\beta+\tilde{\epsilon} \in \mathcal{M}_{z,\beta}$, it follows that $\gamma = \alpha+\tilde{\delta}+\beta+\tilde{\epsilon} \in \mathcal{M}_{z,\alpha}+\mathcal{M}_{z,\beta} \subseteq \langle \mathcal{M}_{z,\alpha} \cup \mathcal{M}_{z,\beta}\rangle$.

IV. $\gamma-\alpha-\beta$ is a weight of $\mathfrak{a}$ occurring in $T_z \circ p_-\big([\mathfrak{g}_{-\alpha}, T_z(\mathfrak{g}_{-\beta})]\big)$. Since α is simple and $T_z(\mathfrak{g}_{-\beta}) \subseteq \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_Q$, the space $p_-\big([\mathfrak{g}_{-\alpha}, T_z(\mathfrak{g}_{-\beta})]\big)$ is non-trivial only if the weight 0 occurs in $T_z(\mathfrak{g}_{-\beta})$. In this case $\beta \in \mathcal{M}_\beta$ and $p_-\big([\mathfrak{g}_{-\alpha}, T_z(\mathfrak{g}_{-\beta})]\big) \subseteq \mathfrak{g}_{-\alpha}$. Now $\gamma-\alpha-\beta = \tilde{\delta}$ for some $\delta \in \operatorname{supp}_z(\mathfrak{g}_{-\alpha})$. As $\alpha+\tilde{\delta} \in \mathcal{M}_{z,\alpha}$, it follows that $\gamma = \alpha+\beta+\tilde{\delta} \in \mathcal{M}_\alpha+\mathcal{M}_\beta \subseteq \langle \mathcal{M}_\alpha \cup \mathcal{M}_\beta\rangle$.

V. $\gamma-\alpha-\beta$ is a weight of $\mathfrak{a}$ occurring in $T_z \circ p_-\big([T_z(\mathfrak{g}_{-\alpha}), \mathfrak{g}_{-\beta}]\big)$. In this case there occurs a weight δ in $T_z(\mathfrak{g}_{-\alpha})$ so that $\delta-\beta \in -\Sigma(Q)$ and the weight $\gamma-\alpha-\beta$ occurs in $T_z(\mathfrak{g}_{\delta-\beta})$. Now $\gamma-\alpha-\delta = (\beta-\delta)+(\gamma-\alpha-\beta) \in \mathcal{M}_{z,\beta-\delta}$ and $\alpha+\delta \in \mathcal{M}_{z,\alpha}$. Therefore, $\gamma \in \mathcal{M}_{z,\beta-\delta}+\mathcal{M}_{z,\alpha}$. The fact that $\delta-\beta$ is a negative root implies that $\delta < \beta$. It follows that $\mathcal{M}_{z,\beta-\delta} \subseteq \mathcal{M}_{z,\beta}$ and thus $\gamma \in \mathcal{M}_{z,\beta}+\mathcal{M}_{z,\alpha} \subseteq \langle \mathcal{M}_{z,\alpha} \cup \mathcal{M}_{z,\beta}\rangle$.

In each of the cases I–V we have $\gamma \in \langle \mathcal{M}_{z,\alpha} \cup \mathcal{M}_{z,\beta}\rangle$. This proves the lemma. $\square$

Proof of Proposition 9.1. Let $z \in Z$ be adapted. In the course of the proof we will need the existence of an element $X \in \mathfrak{a}^\circ_{\mathrm{reg}} \cap \mathfrak{h}_z^\perp$ so that $\beta(X) \neq -\gamma(X)$ for every pair of roots $\beta, \gamma \in \Sigma(Q)$. By Proposition 3.12 we may choose z so that such an element X exists.

Let $\alpha \in \mathcal{M}_z$ be an indecomposable element so that (8.3) holds. Note that $\alpha \in \Sigma(Q) + (\Sigma(Q) \cup \{0\})$. If $\alpha \in \Sigma(Q) \cup 2\Sigma(Q)$, then there is nothing left to prove. Therefore, assume that $\alpha \notin \Sigma(Q) \cup 2\Sigma(Q)$. In view of Lemma 9.3 there exists a simple root $\beta \in \Sigma(Q)$ so that $\gamma := \alpha-\beta$ is a root in $\Sigma(Q)$ and $\gamma \in \operatorname{supp}_z(\mathfrak{g}_{-\beta})$. Since $\alpha \notin \Sigma \cup 2\Sigma$, $\gamma \neq \beta$ and $\beta+\gamma$ is not a root. We will first show that β and γ are orthogonal. To do this, we will work towards a contradiction and we thus assume that $\langle \beta, \gamma\rangle > 0$. Note that $\gamma-\beta$ is a root and is positive.

Let $\delta \in \Sigma(Q) \cup \{\mathfrak{m}, \mathfrak{a}\}$. We define $\tilde{\delta} \in \Sigma(Q) \cup \{0\}$ to be equal to δ if $\delta \in \Sigma(Q)$ and 0 otherwise. We claim that

$$\tilde{\delta}-\beta \notin -\Sigma(Q) \quad \text{or} \quad \delta \notin \operatorname{supp}_z(\mathfrak{g}_{\beta-\gamma}) \quad \text{or} \quad \beta \notin \operatorname{supp}_z(\mathfrak{g}_{\tilde{\delta}-\beta}). \tag{9.2}$$

Indeed, otherwise $\gamma-\beta+\tilde{\delta} \in \mathcal{M}_z$ and $2\beta-\tilde{\delta} \in \mathcal{M}_z$, and hence $\alpha = (\gamma-\beta+\tilde{\delta})+(2\beta-\tilde{\delta})$ would be decomposable. Likewise,

$$\tilde{\delta}+\beta-\gamma \notin -\Sigma(Q) \quad \text{or} \quad \delta \notin \operatorname{supp}_z(\mathfrak{g}_{-\beta}) \quad \text{or} \quad \beta \notin \operatorname{supp}_z(\mathfrak{g}_{\tilde{\delta}+\beta-\gamma}) \tag{9.3}$$

since otherwise $\beta+\tilde{\delta} \in \mathcal{M}_z$ and $\gamma-\tilde{\delta} \in \mathcal{M}_z$ and thus $\alpha = (\beta+\tilde{\delta})+(\gamma-\tilde{\delta})$ would be decomposable.

Let $Y_{\beta-\gamma} \in \mathfrak{g}_{\beta-\gamma}$ and $Y_{-\beta} \in \mathfrak{g}_{-\beta}$. Then

$$\begin{aligned}
&[Y_{-\beta}, Y_{\beta-\gamma}] + T_z\Big([Y_{-\beta}, Y_{\beta-\gamma}]\Big)\\
&\qquad \in \big[Y_{-\beta} + T_z(Y_{-\beta}), Y_{\beta-\gamma} + T_z(Y_{\beta-\gamma})\big] + (\mathfrak{l}_Q \cap \mathfrak{h}_z)\\
&\qquad\quad + \sum_{\substack{\delta \in \mathrm{supp}_z(\mathfrak{g}_{-\beta})\\ \tilde{\delta}+\beta-\gamma \in -\Sigma(Q)}} \mathcal{G}\Big(T_z\big|_{\mathfrak{g}_{\tilde{\delta}+\beta-\gamma}}\Big) + \sum_{\substack{\delta \in \mathrm{supp}_z(\mathfrak{g}_{\beta-\gamma})\\ \tilde{\delta}-\beta \in -\Sigma(Q)}} \mathcal{G}\Big(T_z\big|_{\mathfrak{g}_{\tilde{\delta}-\beta}}\Big)
\end{aligned}$$

In view of (9.2) and (9.3) we have

$$\begin{aligned}
p_\beta T_z\Big([Y_{-\beta}, Y_{\beta-\gamma}]\Big) &= p_\beta\Big(\big[Y_{-\beta} + T_z(Y_{-\beta}), Y_{\beta-\gamma} + T_z(Y_{\beta-\gamma})\big]\Big)\\
&= \big[Y_{-\beta}, p_{2\beta}T_z(Y_{\beta-\gamma})\big] + \big[p_\gamma T_z(Y_{-\beta}), Y_{\beta-\gamma}\big]\\
&\qquad + \big[p_0 T_z(Y_{-\beta}), p_\beta T_z(Y_{\beta-\gamma})\big] + \big[p_\beta T_z(Y_{-\beta}), p_0 T_z(Y_{\beta-\gamma})\big].
\end{aligned}$$

For the second equality we used that β is simple, so that the only pairs of non-negative $\mathfrak{a}$-weights that add up to β are $(\beta, 0)$ and $(0, \beta)$. We claim that the last two terms on the right-hand side are equal to 0. Indeed, if $\big[p_0 T_z(Y_{-\beta}), p_\beta T_z(Y_{\beta-\gamma})\big] \neq 0$, then $\mathfrak{a} \in \mathrm{supp}_z(Y_{-\beta})$ or $\mathfrak{m} \in \mathrm{supp}_z(Y_{-\beta})$, and moreover $\beta \in \mathrm{supp}_z(Y_{\beta-\gamma})$. In particular $\beta \in \mathcal{M}_z$ and $\gamma \in \mathcal{M}_z$ and thus $\alpha = \beta + \gamma$ would be decomposable. Likewise, if $\big[p_\beta T_z(Y_{-\beta}), p_0 T_z(Y_{\beta-\gamma})\big] \neq 0$, then it would follow that $2\beta \in \mathcal{M}_z$ and $\gamma - \beta \in \mathcal{M}_z$, and hence $\alpha = 2\beta + (\gamma - \beta)$ would be decomposable. Therefore,

$$p_\beta T_z\Big([Y_{-\beta}, Y_{\beta-\gamma}]\Big) = \big[Y_{-\beta}, p_{2\beta}T_z(Y_{\beta-\gamma})\big] + \big[p_\gamma T_z(Y_{-\beta}), Y_{\beta-\gamma}\big] \tag{9.4}$$

for all $Y_{-\beta} \in \mathfrak{g}_{-\beta}$ and $Y_{\beta-\gamma} \in \mathfrak{g}_{\beta-\gamma}$.

Let $\tilde{Y}_{-\beta} \in \mathfrak{g}_{-\beta}$. Let further $X \in \mathfrak{h}^\perp \cap \mathfrak{a}$ be so that $\beta(X) \neq -\gamma(X)$. In view of (4.2) and (9.4)

$$\begin{aligned}
&B\big([Y_{-\beta}, Y_{\beta-\gamma}], p_\gamma T_z(\tilde{Y}_{-\beta})\big)\gamma(X) = B\Big(\tilde{Y}_{-\beta}, p_\beta T_z\big([Y_{-\beta}, Y_{\beta-\gamma}]\big)\Big)\beta(X)\\
&\qquad = B\Big(\tilde{Y}_{-\beta}, \big[Y_{-\beta}, p_{2\beta}T_z(Y_{\beta-\gamma})\big]\Big)\beta(X) + B\Big(\tilde{Y}_{-\beta}, \big[p_\gamma T_z(Y_{-\beta}), Y_{\beta-\gamma}\big]\Big)\beta(X).
\end{aligned}$$

Rearranging the terms we obtain

$$\begin{aligned}
&\frac{\gamma(X)}{2}B\big([Y_{-\beta}, Y_{\beta-\gamma}], p_\gamma T_z(\tilde{Y}_{-\beta})\big) - \frac{\beta(X)}{2}B\Big(\tilde{Y}_{-\beta}, \big[p_\gamma T_z(Y_{-\beta}), Y_{\beta-\gamma}\big]\Big)\\
&\qquad = -\frac{\gamma(X)}{2}B\big([Y_{-\beta}, Y_{\beta-\gamma}], p_\gamma T_z(\tilde{Y}_{-\beta})\big) + \frac{\beta(X)}{2}B\Big(\tilde{Y}_{-\beta}, \big[p_\gamma T_z(Y_{-\beta}), Y_{\beta-\gamma}\big]\Big)\\
&\qquad\quad + B\Big(\tilde{Y}_{-\beta}, \big[Y_{-\beta}, p_{2\beta}T_z(Y_{\beta-\gamma})\big]\Big)\beta(X).
\end{aligned}$$

We now apply the identities $B(U, [V, W]) = B([U, V], W)$ and $B(U, V) = B(V, U)$ for $U, V, W \in \mathfrak{g}$ to each of the terms on both sides of this identity. We thus find

$$\begin{aligned}
&\frac{\beta(X) + \gamma(X)}{2}B\Big([p_\gamma T_z(\tilde{Y}_{-\beta}), Y_{-\beta}] + [p_\gamma T_z(Y_{-\beta}), \tilde{Y}_{-\beta}], Y_{\beta-\gamma}\Big)\\
&\qquad = \frac{\beta(X) - \gamma(X)}{2}B\Big([p_\gamma T_z(\tilde{Y}_{-\beta}), Y_{-\beta}] - [p_\gamma T_z(Y_{-\beta}), \tilde{Y}_{-\beta}], Y_{\beta-\gamma}\Big)\\
&\qquad\quad + \beta(X)B\Big([\tilde{Y}_{-\beta}, Y_{-\beta}], p_{2\beta}T_z(Y_{\beta-\gamma})\Big).
\end{aligned}$$

The left-hand side is unchanged when swapping $\tilde{Y}_{-\beta}$ and $Y_{-\beta}$, while the right-hand side changes sign. Therefore, both sides equal 0 for all $Y_{-\beta}, \tilde{Y}_{-\beta} \in \mathfrak{g}_{-\alpha}$ and $Y_{\beta-\gamma} \in \mathfrak{g}_{\beta-\gamma}$. In particular

$$[p_\gamma T_z(\tilde{Y}_{-\beta}), Y_{-\beta}] + [p_\gamma T_z(Y_{-\beta}), \tilde{Y}_{-\beta}] = 0 \qquad \big(Y_{-\beta}, \tilde{Y}_{-\beta} \in \mathfrak{g}_{-\alpha}\big),$$

and hence

$$[p_\gamma T_z(Y_{-\beta}), Y_{-\beta}] = 0 \qquad \big(Y_{-\beta} \in \mathfrak{g}_{-\alpha}\big). \tag{9.5}$$

Taking commutators on both sides of (9.4) with $Y_{-\beta}$ and using the Jacobi-identity and (9.5) we obtain

$$\big[Y_{-\beta}, p_\beta T_z\Big([Y_{-\beta}, Y_{\beta-\gamma}]\Big)\big] = \big[p_\gamma T_z(Y_{-\beta}), [Y_{-\beta}, Y_{\beta-\gamma}]\big] + \big[Y_{-\beta}, [Y_{-\beta}, p_{2\beta} T_z(Y_{\beta-\gamma})]\big] \tag{9.6}$$

for all $Y_{-\beta} \in \mathfrak{g}_{-\beta}$, $Y_{\beta-\gamma} \in \mathfrak{g}_{\beta-\gamma}$. Note that the second term on the right-hand side is contained in $\mathfrak{m}$. We now pair both sides of (9.6) with X via the Killing form and obtain

$$\begin{aligned} -\beta(X)B\Big(Y_{-\beta}, p_\beta T_z\big([Y_{-\beta}, Y_{\beta-\gamma}]\big)\Big) &= B\Big(X, \big[Y_{-\beta}, p_\beta T_z\Big([Y_{-\beta}, Y_{\beta-\gamma}]\Big)\big]\Big) \\ &= B\Big(X, \big[p_\gamma T_z(Y_{-\beta}), [Y_{-\beta}, Y_{\beta-\gamma}]\big]\Big) = \gamma(X)B\Big(p_\gamma T_z(Y_{-\beta}), [Y_{-\beta}, Y_{\beta-\gamma}]\Big). \end{aligned} \tag{9.7}$$

We claim that $[Y_{-\beta}, \mathfrak{g}_{\beta-\gamma}] = \mathfrak{g}_{-\gamma}$ for every non-zero $Y_{-\beta} \in \mathfrak{g}_{-\beta}$. To see this, let $Y_{-\gamma} \in \mathfrak{g}_{-\gamma}$. Then $[\theta Y_{-\beta}, Y_{-\gamma}] \in \mathfrak{g}_{\beta-\gamma}$ and

$$\big[Y_{-\beta}, [\theta Y_{-\beta}, Y_{-\gamma}]\big] = -\big[\theta Y_{-\beta}, [Y_{-\gamma}, Y_{-\beta}]\big] - \big[Y_{-\gamma}, [Y_{-\beta}, \theta Y_{-\beta}]\big].$$

The first term on the right-hand side vanishes because $-\beta - \gamma$ is not a root, while the second term is equal to $\gamma\big([\theta Y_{-\beta}, Y_{-\beta}]\big)Y_{-\gamma}$, which is a non-zero multiple of $Y_{-\gamma}$ due to the assumption that $\langle \beta, \gamma \rangle > 0$. This proves the claim.

Because of the claim and (9.7) we have

$$-\beta(X)B\big(Y_{-\beta}, p_\beta T_z(Y_{-\gamma})\big) = \gamma(X)B\big(Y_{-\gamma}, p_\gamma T_z(Y_{-\beta})\big)$$

for every $Y_{-\beta} \in \mathfrak{g}_{-\beta}$ and $Y_{-\gamma} \in \mathfrak{g}_{-\gamma}$. However, in view of (4.2) we also have

$$\beta(X)B\big(Y_{-\beta}, p_\beta T_z(Y_{-\gamma})\big) = \gamma(X)B\big(Y_{-\gamma}, p_\gamma T_z(Y_{-\beta})\big).$$

It follows that $p_\gamma T_z(Y_{-\beta}) = 0$ for all $Y_{-\beta} \in \mathfrak{g}_{-\beta}$. This is in contradiction with the assumption that $\gamma \in \operatorname{supp}_z(\mathfrak{g}_{-\beta})$. We have thus proven that β and γ are orthogonal.

We move on to show that $\operatorname{span}(\beta^\vee, \gamma^\vee) \cap \mathfrak{h}_z \neq \{0\}$. Let $\delta \in \Sigma(Q) \cup \{\mathfrak{m}, \mathfrak{a}\}$. We have

$$\{\mathfrak{m}, \mathfrak{a}\} \cap \operatorname{supp}_z(\mathfrak{g}_{-\gamma}) = \emptyset \quad \text{or} \quad \{\mathfrak{m}, \mathfrak{a}\} \cap \operatorname{supp}_z(\mathfrak{g}_{-\beta}) = \emptyset. \tag{9.8}$$

Indeed, otherwise $\gamma \in \mathcal{M}_z$ and $\beta \in \mathcal{M}_z$, and hence $\alpha = \beta + \gamma$ would be decomposable. Likewise,

$$\tilde{\delta} - \gamma \notin -\Sigma(Q) \quad \text{or} \quad \delta \notin \operatorname{supp}_z(\mathfrak{g}_{-\beta}) \quad \text{or} \quad \{\mathfrak{m}, \mathfrak{a}\} \cap \operatorname{supp}_z(\mathfrak{g}_{\tilde{\delta}-\gamma}) = \emptyset \tag{9.9}$$

since otherwise $\beta+\tilde{\delta}\in\mathcal{M}_z$ and $\gamma-\tilde{\delta}\in\mathcal{M}_z$ and thus $\alpha=(\beta+\tilde{\delta})+(\gamma-\tilde{\delta})$ would be decomposable.

Let $Y_{-\gamma}\in\mathfrak{g}_{-\gamma}$ and $Y_{-\beta}\in\mathfrak{g}_{-\beta}$. Then

$$\begin{aligned}&[Y_{-\beta},Y_{-\gamma}]+T_z\Big([Y_{-\beta},Y_{-\gamma}]\Big)\\&\qquad\in\big[Y_{-\beta}+T_z(Y_{-\beta}),Y_{-\gamma}+T_z(Y_{-\gamma})\big]+(\mathfrak{l}_Q\cap\mathfrak{h}_z)\\&\qquad\quad+\sum_{\delta\in\operatorname{supp}_z(\mathfrak{g}_{-\gamma})\cap\{\mathfrak{m},\mathfrak{a}\}}\mathcal{G}\Big(T_z\big|_{\mathfrak{g}_{-\beta}}\Big)+\sum_{\substack{\delta\in\operatorname{supp}_z(\mathfrak{g}_{-\beta})\\ \tilde{\delta}-\gamma\in-\Sigma(Q)}}\mathcal{G}\Big(T_z\big|_{\mathfrak{g}_{\tilde{\delta}-\gamma}}\Big)\end{aligned}$$

In view of (9.8) and (9.9) we have

$$\begin{aligned}p_{\mathfrak{a}}T_z\Big([Y_{-\beta},Y_{-\gamma}]\Big)&\in p_{\mathfrak{a}}\Big(\big[Y_{-\beta}+T_z(Y_{-\beta}),Y_{-\gamma}+T_z(Y_{-\gamma})\big]\Big)+\mathfrak{a}_{\mathfrak{h}}\\&=\big[Y_{-\beta},p_\beta T_z(Y_{-\gamma})\big]+\big[p_\gamma T_z(Y_{-\beta}),Y_{-\gamma}\big]+\mathfrak{a}_{\mathfrak{h}}\\&=-\frac{\|\beta\|^2}{2}B\big(Y_{-\beta},p_\beta T_z(Y_{-\gamma})\big)\beta^\vee+\frac{\|\gamma\|^2}{2}B\big(p_\gamma T_z(Y_{-\beta}),Y_{-\gamma}\big)\gamma^\vee+\mathfrak{a}_{\mathfrak{h}}.\end{aligned}$$

Since $\alpha=\beta+\gamma$ is not a root, the left-hand side is equal to 0 and thus

$$-\frac{\|\beta\|^2}{2}B\big(Y_{-\beta},p_\beta T_z(Y_{-\gamma})\big)\beta^\vee+\frac{\|\gamma\|^2}{2}B\big(p_\gamma T_z(Y_{-\beta}),Y_{-\gamma}\big)\gamma^\vee\in\mathfrak{a}_{\mathfrak{h}}.$$

Moreover, since β and γ are linearly independent, the left-hand side is not equal to 0 if $Y_{-\beta}\in\mathfrak{g}_{-\beta}$ satisfies $p_\gamma T_z(Y_{-\beta})\neq 0$ and $Y_{-\gamma}=\theta p_\gamma T_z(Y_{-\beta})$. Such a $Y_{-\beta}$ exists because $\gamma\in\operatorname{supp}_z(\mathfrak{g}_{-\beta})$. □

10 Adapted points in boundary degenerations

The real spherical homogeneous spaces with stabilizer subgroup equal to the connected subgroup with Lie algebra $\mathfrak{h}_{z,\mathcal{F}}$, where $z\in Z$ is adapted and $\mathcal{F}$ is a face of $\overline{\mathcal{C}}$ are called boundary degenerations. In this section we establish a correspondence between adapted points in Z and adapted points in the boundary degenerations, and secondly, we give a comparison between the compression cones for Z and the boundary degenerations.

In view of Proposition 8.4 we may make the following definition.

Definition 10.1. Let $\mathcal{O}$ be an open P-orbit in Z and let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. We define the homogeneous space

$$Z_{\mathcal{O},\mathcal{F}}:=G/H_{z,\mathcal{F}},$$

where z is any adapted point in $\mathcal{O}$ and $H_{z,\mathcal{F}}$ is the connected subgroup of G with Lie algebra $\mathfrak{h}_{z,\mathcal{F}}$. The spaces $Z_{\mathcal{O},\mathcal{F}}$ are called the boundary degenerations of Z. If $z\in Z_{\mathcal{O},\mathcal{F}}$, then we write $\mathfrak{h}_z^{\mathcal{O},\mathcal{F}}$ for the stabilizer subalgebra of z.

We note that the spaces $Z_{\mathcal{O},\mathcal{F}}$ are quasi-affine real spherical spaces. We will now first explore the relation between adapted points in Z and in $Z_{\mathcal{O},\mathcal{F}}$.

Lemma 10.2. *Let $\mathcal{O}$ be an open P-orbit in Z and let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. Let $y \in Z_{\mathcal{O},\mathcal{F}}$. If there exists an adapted point $z \in \mathcal{O}$ so that $\mathfrak{h}_y^{\mathcal{O},\mathcal{F}} = \mathfrak{h}_{z,\mathcal{F}}$, then y is adapted and*

$$\mathfrak{a} \cap \mathfrak{h}_z^{\perp} \subseteq \mathfrak{a} \cap (\mathfrak{h}_y^{\mathcal{O},\mathcal{F}})^{\perp}. \tag{10.1}$$

Proof. Assume that $z \in \mathcal{O}$ is adapted and $\mathfrak{h}_y^{\mathcal{O},\mathcal{F}} = \mathfrak{h}_{z,\mathcal{F}}$. We will prove that y is adapted by verifying the conditions in Proposition 3.19. Let X be in the interior of $\mathcal{F}$ and $Y \in \mathcal{C}$. By Proposition 5.2 (iii) we have

$$(\mathfrak{h}_y^{\mathcal{O},\mathcal{F}})_Y = (\mathfrak{h}_{z,\mathcal{F}})_Y = (\mathfrak{h}_{z,X})_Y = \mathfrak{h}_{z,Y} = \mathrm{Ad}(m)\mathfrak{h}_{\emptyset}$$

for some $m \in M$. In view of Lemma 6.4 (ii) the P-orbit through y is open in $Z_{\mathcal{O},\mathcal{F}}$. Furthermore,

$$\mathfrak{a} \cap \mathfrak{h}_z^{\perp} = (\mathfrak{a} \cap \mathfrak{h}_z^{\perp})_X \subseteq \mathfrak{a} \cap (\mathfrak{h}_z^{\perp})_X = \mathfrak{a} \cap \mathfrak{h}_{z,X}^{\perp} = \mathfrak{a} \cap (\mathfrak{h}_y^{\mathcal{O},\mathcal{F}})^{\perp}.$$

This proves (10.1). Since z is adapted, we have $\mathfrak{a}^{\circ}_{\mathrm{reg}} \cap \mathfrak{h}_z^{\perp} \neq \emptyset$, and hence $\mathfrak{a}^{\circ}_{\mathrm{reg}} \cap (\mathfrak{h}_y^{\mathcal{O},\mathcal{F}})^{\perp} \neq \emptyset$. In view of Proposition 3.19 the point y is adapted. ☐

It follows from Lemma 8.3 and Lemma 10.2 that there exists an adapted point $y \in Z_{\mathcal{O},\mathcal{F}}$ so that $\mathfrak{a} \cap \mathfrak{h}_y^{\mathcal{O},\mathcal{F}} = \mathfrak{a}_{\mathfrak{h}}$. By Corollary 3.17 the same holds for all adapted points $y \in Z_{\mathcal{O},\mathcal{F}}$, and hence $\mathfrak{a}^{\circ}$ defined in Definition 3.18 equals $\mathfrak{a} \cap (\mathfrak{a} \cap \mathfrak{h}_y^{\mathcal{O},\mathcal{F}})^{\perp}$.

For an adapted point $y \in Z_{\mathcal{O},\mathcal{F}}$ we write $\Phi_y^{\mathcal{O},\mathcal{F}}$ for the unique smooth rational map

$$\Phi_y^{\mathcal{O},\mathcal{F}} : \mathfrak{a}^{\circ} \to \mathfrak{n}_Q$$

satisfying (i) and (iii) in Proposition 3.12 with Z replaced by $Z_{\mathcal{O},\mathcal{F}}$.

Lemma 10.3. *Let $\mathcal{O}$ be an open P-orbit in Z and let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. Let $z \in \mathcal{O}$ be adapted and let $y \in Z_{\mathcal{O},\mathcal{F}}$ satisfy $\mathfrak{h}_y^{\mathcal{O},\mathcal{F}} = \mathfrak{h}_{z,\mathcal{F}}$. Then*

$$\lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big) \circ \Phi_z = \Phi_y^{\mathcal{O},\mathcal{F}}$$

for every X in the interior of $\mathcal{F}$, where the convergence is pointwise.

Proof. Let X be an element from the interior of $\mathcal{F}$. By Lemma 8.6

$$\mathrm{Im}\big(\Phi_z\big) \subseteq \bigoplus_{\substack{\alpha\in\Sigma(Q)\\ \alpha|_{\overline{\mathcal{C}}}\leq 0}} \mathfrak{g}_{\alpha}.$$

Since $X \in \overline{\mathcal{C}}$, it follows that $\mathrm{Ad}\big(\exp(tX)\big) \circ \Phi_z$ converges pointwise. The limit is equal to

$$\Psi := \Big(\sum_{\substack{\alpha\in\Sigma(Q)\\ \alpha(X)=0}} p_{\alpha}\Big) \circ \Phi_z,$$

where p_{α} denotes the projection $\mathfrak{g} \to \mathfrak{g}_{\alpha}$ along the Bruhat decomposition. It remains to prove that $\Psi = \Phi_y^{\mathcal{O},\mathcal{F}}$.

Let $Y \in \mathfrak{a}^\circ_{\text{reg}}$. By Proposition 3.12 the point $\exp\big(\Phi_z(Y)\big) \cdot z$ is adapted. By Lemma 10.2 a point in $Z_{\mathcal{O},\mathcal{F}}$ with stabilizer subalgebra $\mathfrak{h}_{\exp(\Phi_z(Y))\cdot z,\mathcal{F}}$ is adapted. It follows from Proposition 5.2 (iv) that

$$\begin{aligned}\mathfrak{h}_{\exp(\Phi_z(Y))\cdot z,\mathcal{F}} &= \Big(\operatorname{Ad}\Big(\exp\big(\Phi_z(Y)\big)\Big)\mathfrak{h}_z\Big)_X = \operatorname{Ad}\Big(\exp\big(\Psi(Y)\big)\Big)\mathfrak{h}_{z,\mathcal{F}}\\ &= \operatorname{Ad}\Big(\exp\big(\Psi(Y)\big)\Big)\mathfrak{h}_y^{\mathcal{O},\mathcal{F}} = \mathfrak{h}^{\mathcal{O},\mathcal{F}}_{\exp(\Psi(Y))\cdot y}.\end{aligned}$$

We thus conclude that the point $\exp\big(\Psi(Y)\big) \cdot y$ is adapted.

By Proposition 3.12 we have $\mathbb{R}Y \subseteq \mathfrak{h}^\perp_{\exp(\Phi_z(Y))\cdot z}$, and hence applying (10.1) to the point $\exp(\Phi_z(Y)) \cdot z$ yields

$$\mathbb{R}Y \subseteq \big(\mathfrak{h}^{\mathcal{O},\mathcal{F}}_{\exp(\Psi(Y))\cdot y}\big)^\perp.$$

It follows from the final assertion in Proposition 3.12 that there exist $m \in M$ and $a \in A$ so that

$$\exp\big(\Psi(Y)\big) \cdot y = ma \exp\big(\Phi^{\mathcal{O},\mathcal{F}}_y(Y)\big) \cdot y.$$

By Proposition 3.6 the stabilizer of y is contained in L_Q. As Ψ and $\Phi^{\mathcal{O},\mathcal{F}}_y$ both map to N_Q, it follows that $\Psi(Y) = \Phi^{\mathcal{O},\mathcal{F}}_y(Y)$. □

Proposition 10.4. *Let $\mathcal{O}$ be an open P-orbit in Z and let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. Let further $z_0 \in \mathcal{O}$ be adapted and let $y_0 \in Z_{\mathcal{O},\mathcal{F}}$ be so that $\mathfrak{h}^{\mathcal{O},\mathcal{F}}_{y_0} = \mathfrak{h}_{z_0,\mathcal{F}}$. In view of Lemma 10.2 the point y_0 is adapted. Then a point $y \in P \cdot y_0$ is adapted if and only if there exists an adapted point $z \in \mathcal{O}$ so that $\mathfrak{h}^{\mathcal{O},\mathcal{F}}_y = \mathfrak{h}_{z,\mathcal{F}}$. Moreover, if $Y, Y' \in \mathfrak{a}^\circ_{\text{reg}}$ and $z \in \mathcal{O}$ and $y \in P \cdot y_0$ and satisfy*

$$z \in MA\exp\big(\Phi_{z_0}(Y)\big) \cdot z_0 \quad \textit{and} \quad y \in MA\exp\big(\Phi^{\mathcal{O},\mathcal{F}}_{y_0}(Y')\big) \cdot y_0 \tag{10.2}$$

(and hence are adapted), then $\mathfrak{h}^{\mathcal{O},\mathcal{F}}_y = \mathfrak{h}_{z,\mathcal{F}}$ if and only if $\Phi^{\mathcal{O},\mathcal{F}}_{y_0}(Y) = \Phi^{\mathcal{O},\mathcal{F}}_{y_0}(Y')$.

Proof. Assume that $y \in P \cdot y_0$ is adapted. By Proposition 3.12 there exists $m \in M$, $a \in A$ and $Y \in \mathfrak{a}^\circ$ so that $y = ma\exp\big(\Phi^{\mathcal{O},\mathcal{F}}_{y_0}(Y)\big) \cdot y_0$. Set $z = \exp\big(\Phi_{z_0}(Y)\big) \cdot z_0$ and let X be in the interior of $\mathcal{F}$. Then by Proposition 5.2 (iv) and Lemma 10.3

$$\mathfrak{h}_{z,\mathcal{F}} = \Big(\operatorname{Ad}\Big(ma\exp\big(\Phi_{z_0}(Y)\big)\Big)\mathfrak{h}_{z_0}\Big)_X = \operatorname{Ad}\Big(ma\exp\big(\Phi^{\mathcal{O},\mathcal{F}}_{y_0}(Y)\big)\Big)\mathfrak{h}_{z_0,X} = \mathfrak{h}^{\mathcal{O},\mathcal{F}}_y.$$

If $z \in \mathcal{O}$ is adapted and $y \in P \cdot y_0$ satisfies $\mathfrak{h}^{\mathcal{O},\mathcal{F}}_y = \mathfrak{h}_{z,\mathcal{F}}$, then y is adapted by Lemma 10.2. This proves the first assertion. We move on to the second.

Assume that (10.2) holds. If $\mathfrak{h}^{\mathcal{O},\mathcal{F}}_y = \mathfrak{h}_{z,\mathcal{F}}$, then $Y \in \mathfrak{h}_z^\perp$ and hence $Y \in \mathfrak{h}^\perp_{z,\mathcal{F}}$. This implies that $Y \in (\mathfrak{h}^{\mathcal{O},\mathcal{F}}_y)^\perp$. By Proposition 3.12

$$y \in MA\exp\big(\Phi^{\mathcal{O},\mathcal{F}}_{y_0}(Y)\big) \cdot y_0,$$

and hence $\Phi^{\mathcal{O},\mathcal{F}}_{y_0}(Y) = \Phi^{\mathcal{O},\mathcal{F}}_{y_0}(Y')$ in view of Proposition 3.6 (ii). The other implication is trivial. □

We end this section with a description of the compression cone of $Z_{\mathcal{O},\mathcal{F}}$.

Proposition 10.5. *Let $\mathcal{O}$ be an open P-orbit in Z and let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. The compression cone of $Z_{\mathcal{O},\mathcal{F}}$ is equal to $\mathcal{C} + \mathfrak{a}_{\mathcal{F}}$.*

Proof. The assertion follows directly from (8.6) and (8.4). □

In view of Proposition 10.5 the compression cone of $Z_{\mathcal{O},\mathcal{F}}$ does not depend on the open P-orbit $\mathcal{O}$. We therefore write $\mathcal{C}_{\mathcal{F}}$ for the compression cone of $Z_{\mathcal{O},\mathcal{F}}$, i.e.,

$$\mathcal{C}_{\mathcal{F}} := \mathcal{C} + \mathfrak{a}_{\mathcal{F}}.$$

11 Admissible points

Recall the group $\mathcal{N}$ from (7.1). Proposition 7.2 is most useful for points $z \in Z$ for which the limits $\mathfrak{h}_{z,X}$ for order-regular elements $X \in \mathfrak{a}$ are conjugates of $\mathfrak{h}_\emptyset$ by some element in $\mathcal{N}$. The purpose of the next definition is to single out those adapted points for which all such limits have this property.

Definition 11.1. We say that an adapted point $z \in Z$ is admissible if for every order-regular element $X \in \mathfrak{a}$, there exists an element $w \in \mathcal{N}$ so that

$$\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset.$$

In the remainder of this section we will prove the existence of admissible points. In the next section we will use the set of elements $w \in \mathcal{N}$ so that $\mathrm{Ad}(w)\mathfrak{h}_\emptyset$ occurs as a limit $\mathfrak{h}_{z,X}$ of $\mathfrak{h}_z$ for an admissible point z to construct the little Weyl group.

We begin with a few remarks.

Remark 11.2.

(a) The set of admissible points is L_Q-stable.

(b) If z is admissible and $v \in \mathcal{N}$ is such that $Pv^{-1} \cdot z$ is open, then $v^{-1} \cdot z$ is adapted by Lemma 7.3. Moreover, if $X \in \mathfrak{a}$, then

$$\mathfrak{h}_{v^{-1}\cdot z,X} = \mathrm{Ad}(v^{-1})\mathfrak{h}_{z,\mathrm{Ad}(v)X}.$$

From this it follows that $v^{-1} \cdot z$ is also admissible.

We define the subgroup $\mathcal{N}_\emptyset$ of $\mathcal{N}$ by

$$\mathcal{N}_\emptyset := \{w \in \mathcal{N} : \mathrm{Ad}(w)\mathfrak{h}_\emptyset = \mathrm{Ad}(m)\mathfrak{h}_\emptyset \text{ for some } m \in M\}.$$

Lemma 11.3. *We have $\mathcal{N}_\emptyset = N_{L_Q}(\mathfrak{a})$. Moreover, $\mathcal{N}_\emptyset$ is a normal subgroup of $\mathcal{N}$. Finally, the group $\mathcal{N}/\mathcal{N}_\emptyset$ is finite.*

Proof. Since the elements in $\mathcal{N}$ normalize $\mathfrak{a}$, they also normalize $\mathfrak{m}+\mathfrak{a}$. Note that

$$\mathfrak{h}_\emptyset + \mathfrak{m} + \mathfrak{a} = \overline{\mathfrak{q}}$$

is M-stable. Therefore, the elements in $\mathcal{N}_\emptyset$ normalize $\overline{\mathfrak{q}}$, and hence

$$\mathcal{N}_\emptyset \subseteq N_G(\mathfrak{a}) \cap \overline{Q} = N_{L_Q}(\mathfrak{a}).$$

To prove the other inclusion we first note that L_Q normalizes $\mathfrak{l}_{Q,\mathrm{nc}} + \mathfrak{a}_{\mathfrak{h}}$. Therefore, it follows directly from the definition (7.1) that $N_{L_Q}(\mathfrak{a}) \subseteq \mathcal{N}$. Now choose an adapted point $z \in Z$ so that $\mathfrak{h}_\emptyset = \mathfrak{h}_{z,X}$ for some $X \in \mathcal{C}$. Then

$$\mathfrak{h}_\emptyset = (\mathfrak{l}_Q \cap \mathfrak{h}_z) \oplus \overline{\mathfrak{n}}_Q.$$

We recall that $L_Q = MAL_{Q,\mathrm{nc}}$, see (3.1). The group $L_{Q,\mathrm{nc}}$ is contained in $\overline{Q}$, and hence normalizes $\overline{\mathfrak{n}}_Q$. As $L_{Q,\mathrm{nc}} \subseteq L_Q \cap H_z$, also $\mathfrak{l}_Q \cap \mathfrak{h}_z$ is normalized by $L_{Q,\mathrm{nc}}$. It follows that $L_{Q,\mathrm{nc}}$ normalizes $\mathfrak{h}_\emptyset$. Further, $\mathfrak{h}_\emptyset$ is A-stable. It thus follows that

$$\mathrm{Ad}(L_Q)\mathfrak{h}_\emptyset = \mathrm{Ad}(M)\mathfrak{h}_\emptyset.$$

In particular, $N_{L_Q}(\mathfrak{a}) \subseteq \mathcal{N} \cap L_Q \subseteq \mathcal{N}_\emptyset$. This proves the first assertion.

We move on to the second assertion. By definition $\mathcal{N}$ normalizes A and $L_{Q,\mathrm{nc}}$. Every element normalizing A also normalizes M. As $L_Q = MAL_{Q,\mathrm{nc}}$, it follows that $\mathcal{N}$ normalizes $\mathcal{N}_\emptyset$. This proves the second assertion.

For the final assertion we note that $\mathcal{N}$ and $\mathcal{N}_\emptyset$ both contain the group MA. As $\mathcal{N}/MA$ is a subgroup of the Weyl group of Σ, it is finite. This implies that $\mathcal{N}/\mathcal{N}_\emptyset$ is finite. □

We note that the quotient $\mathcal{N}/\mathcal{N}_\emptyset$ is a group in view of Lemma 11.3. The main result in this section is the following proposition.

Proposition 11.4.

(i) The set of admissible points is dense and has non-empty interior in the set of adapted points in Z (all with respect to the subspace topology). In particular, every open P-orbit in Z contains an admissible point.

(ii) For $z \in Z$ define

$$\mathcal{W}_z := \{w\mathcal{N}_\emptyset \in \mathcal{N}/\mathcal{N}_\emptyset : w \in \mathcal{N} \text{ and there exist } X \in \mathfrak{a} \text{ so that } \mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset\}.$$

Let $z \in Z$ be admissible and let $z' \in Z$. If z' is adapted, then $\mathcal{W}_{z'} \subseteq \mathcal{W}_z$. Moreover, if z' is admissible, then $\mathcal{W}_{z'} = \mathcal{W}_z$.

The remainder of this section is devoted to the proof of the proposition. We break the proof up into a sequence of lemmas.

Lemma 11.5. *Let $z \in Z$ be adapted. There exists an open neighborhood U of z in $P \cdot z$ so that $\mathcal{W}_z \subseteq \mathcal{W}_{z'}$ for all adapted points $z' \in U$.*

Proof. Let $w \in \mathcal{W}_z$ and let $v \in \mathcal{N}$ be so that $v\mathcal{N}_\emptyset = w$. By Proposition 7.2 the P-orbit $Pv^{-1} \cdot z$ is open. Therefore, there exists an open neighborhood U_v of e in G so that $v^{-1}U_v \cdot z \subseteq Pv^{-1} \cdot z$. It follows from the same proposition that $v \in \mathcal{W}_{z'}$ for every adapted point in $z' \in U_v \cdot z$. The assertion now follows with U equal to the intersection of the sets $U_v \cdot z$, where v runs over a set of representatives in $\mathcal{N}$ for $\mathcal{W}_z$. □

We now use Lemma 11.5 to prove a much stronger statement.

Lemma 11.6. *Let $z \in Z$ be adapted. There exists an open and dense subset U of the set of adapted points in Z (with respect to the subspace topology) so that $\mathcal{W}_z \subseteq \mathcal{W}_{z'}$ for all $z' \in U$.*

Proof. Let $w \in \mathcal{W}_z$. We will prove that there exists an open and dense subset U_w of the set of adapted points so that $w \in \mathcal{W}_{z'}$ for all $z' \in U_w$. Since $\mathcal{N}/\mathcal{N}_\emptyset$ is finite, the assertion in the lemma follows from this with U equal to the finite intersection $U = \bigcap_{w\in\mathcal{W}_z} U_w$.

Let $k = \dim(\mathfrak{h}_z)$ with $z \in Z$, and let $\iota : \mathrm{Gr}(\mathfrak{g}, k) \hookrightarrow \mathbb{P}(\bigwedge^k \mathfrak{g})$ be the Plücker embedding, i.e., ι is the map given by

$$\iota\big(\mathrm{span}(v_1, \dots, v_k)\big) = \mathbb{R}(v_1 \wedge \cdots \wedge v_k).$$

The map ι is a diffeomorphism onto a compact submanifold of $\mathbb{P}(\bigwedge^k \mathfrak{g})$. The image is in fact an algebraic subvariety of $\mathbb{P}(\bigwedge^k \mathfrak{g})$, as it is the intersection of a number of quadrics defined by the Plücker relations. See [9, p. 209–211].

Let $v \in \mathcal{N}$ be a representative of w and let $X \in \mathrm{Ad}(v)\mathcal{C}$. Let $e_1, \dots, e_m$ be a basis of $\bigwedge^k \mathfrak{g}$ consisting of eigenvectors of $\mathrm{ad}(X)$. We write $\mu_1, \dots, \mu_m \in \mathbb{R}$ for the corresponding eigenvalues. We may order the eigenvectors so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. Let $\xi \in \bigwedge^k \mathfrak{g}$ be the element so that $\iota(\mathfrak{h}_z) = \mathbb{R}\xi$ and let $c_1, \dots, c_m : \mathfrak{n}_Q \to \mathbb{R}$ be the functions determined by

$$\mathrm{Ad}\left(\exp(Y)\right)\xi = \sum_{i=1}^{m} c_i(Y)e_i.$$

Since $\mathfrak{n}_P$ is a nilpotent Lie algebra, the function

$$\mathfrak{n}_P \to \bigwedge^k \mathfrak{g}; \quad Y \mapsto \mathrm{Ad}\left(\exp(Y)\right)\xi$$

is polynomial, and hence also the functions c_i are polynomial. Since Φ_z is a rational function, the functions $c_i \circ \Phi_z : \mathfrak{a}^\circ \to \mathbb{R}$ are rational. Let j_0 be the smallest number so that $c_{j_0} \circ \Phi_z$ is not identically zero, and let j_1 be the largest number so that $\mu_{j_1} = \mu_{j_0}$. Define the rational map $\mathfrak{a}^\circ \to \mathbb{R}$

$$p := \sum_{i=j_0}^{j_1} (c_i \circ \Phi_z)^2.$$

Then for every Y in the open and dense subset $V := p^{-1}(\mathbb{R} \setminus \{0\})$ of $\mathfrak{a}^\circ$

$$\Big(\mathrm{Ad}\left(\exp \circ \Phi_z(Y)\right)\xi\Big)_X = \mathbb{R}\sum_{i=j_0}^{j_1} c_i \circ \Phi_z(Y)e_j.$$

By Lemma 11.5 there exists an open neighborhood U' of z so that $w \in \mathcal{W}_{z'}$ for all adapted points $z' \in U'$. Since $X \in \mathrm{Ad}(v)\mathcal{C}$ we have in view of Proposition 7.2 that for every adapted point $z' \in U'$ there exists a $m \in M$ so that $\mathfrak{h}_{z',X} = \mathrm{Ad}(vm)\mathfrak{h}_\emptyset$. It follows that

$$\mathbb{R}\sum_{i=j_0}^{j_1} c_i \circ \Phi_z(Y)e_j \in \mathrm{Ad}(vM)\iota(\mathfrak{h}_\emptyset) \tag{11.1}$$

for all $Y \in \mathfrak{a}^\circ$ so that $\exp\big(\Phi_z(Y)\big) \cdot z \in U'$. The set of elements Y for which this holds is open. Since the functions $c_i \circ \Phi_z$ are rational, we conclude that (11.1) holds for all $Y \in V$, i.e., for every $Y \in V$

$$\mathfrak{h}_{\exp\left(\Phi_z(Y)\right)\cdot z,X} = \mathrm{Ad}(vm)\mathfrak{h}_\emptyset$$

for some $m \in M$. In particular

$$w \in \mathcal{W}_{\exp\left(\Phi_z(Y)\right)\cdot z} \qquad \big(Y \in V\big).$$

Since $\mathfrak{h}_{ma\cdot z',X} = \mathrm{Ad}(ma)\mathfrak{h}_{z',X}$ for every $m \in M$, $a \in A$ and $z' \in Z$, it follows that

$$w \in \mathcal{W}_{z'} \qquad \big(z' \in MA\exp\big(\Phi_z(V)\big)\cdot z\big).$$

In view of Proposition 3.12 the set $MA\exp\big(\Phi_z(V)\big)\cdot z$ is open and dense in the set of adapted points in $P \cdot z$.

Finally it follows from Proposition 3.13 and Lemma 6.7 that for each open P-orbit $\mathcal{O}$ there exists a $z' \in \mathcal{O}$ so that $w \in \mathcal{W}_{z'}$. The argument above then shows that $w \in \mathcal{W}_{z'}$ for an open and dense subset of the set of adapted points in $\mathcal{O}$. □

By Lemma 11.3 we have $\mathcal{N}_\emptyset = N_{L_Q}(\mathfrak{a}) = N_{L_{Q,\mathrm{nc}}}(\mathfrak{a}) \times MA$. Every coroot $\alpha^\vee$ of a root $\alpha \in \Sigma(\mathfrak{a}, \mathfrak{l}_{Q,\mathrm{nc}})$ lies in $\mathfrak{a}_\mathfrak{h}$, and hence $\mathrm{Ad}(w)X - X \in \mathfrak{a}_\mathfrak{h}$ for every $w \in \mathcal{N}_\emptyset$ and $X \in \mathfrak{a}$. Therefore, $\mathcal{N}/\mathcal{N}_\emptyset$ acts naturally on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$. Note that the compression cone $\mathcal{C}$ is stable under translation by elements in $\mathfrak{a}_\mathfrak{h}$. We write $p_\mathfrak{h}$ for the projection $\mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_\mathfrak{h}$.

For an adapted point $z \in Z$ we define

$$\mathcal{A}_z := \{X + \mathfrak{a}_\mathfrak{h} \in \mathfrak{a}/\mathfrak{a}_\mathfrak{h} : \mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset \text{ for some } w \in \mathcal{N}\}.$$

Lemma 11.7. *Let $z \in Z$ be adapted. The following hold.*

(i) The point z is admissible if and only if $\mathcal{A}_z$ is dense in $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$.

(ii) $\mathcal{A}_z = \bigcup_{w\in\mathcal{W}_z} w \cdot p_\mathfrak{h}(\mathcal{C})$.

(iii) Let $w \in \mathcal{W}_z$ and let $v \in \mathcal{N}$ be so that $w = v\mathcal{N}_\emptyset$. By Proposition 7.2 and Lemma 7.3 the point $v^{-1}\cdot z$ is adapted. Then

$$\mathcal{A}_{v^{-1}\cdot z} = w^{-1} \cdot \mathcal{A}_z.$$

(iv) There exists an open and dense subset U of the set of adapted points in Z (with respect to the subspace topology) so that $\mathcal{A}_z \subseteq \mathcal{A}_{z'}$ for all $z' \in U$.

Proof. The identity in (ii) follows from Proposition 7.2. The identity shows that $\mathcal{A}_z$ is open. It follows from Proposition 5.2 (iii) that $\mathcal{A}_z$ is dense if and only if $p_{\mathfrak{h}}^{-1}(\mathcal{A}_z)$ contains all order-regular elements. The latter is true if and only if z is admissible. This proves (i).

We move on to prove (iii). Since $\mathfrak{h}_{v^{-1}\cdot z,X} = \mathrm{Ad}(v^{-1})\mathfrak{h}_{z,\mathrm{Ad}(v)X}$ for every $X \in \mathfrak{a}$, we have

$$\mathcal{W}_{v^{-1}\cdot z} = \{w'\mathcal{N}/\mathcal{N}_\emptyset : \text{ there exist } X \in \mathfrak{a} \text{ so that } \mathfrak{h}_{z,X} = \mathrm{Ad}(ww')\mathfrak{h}_\emptyset\} = w^{-1}\mathcal{W}_z.$$

The identity in (iii) now follows from (ii).

The assertion in (iv) follows from (ii) and Lemma 11.6. □

In view of the Lemmas 11.6 and 11.7 it suffices to prove the existence of one admissible point in Z. For this we need an alternative characterization of admissible points.

Lemma 11.8. *Let $z \in Z$ and let $X \in \mathfrak{a}$ be order regular. Then $\dim(\mathfrak{a}_{\mathfrak{h}}) = \dim(\mathfrak{h}_{z,X} \cap \mathfrak{a})$ if and only if there exists a $w \in N_G(\mathfrak{a})$ so that $\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$.*

Remark 11.9. It follows from Lemma 7.1 and Lemma 11.8 that an adapted point $z \in Z$ is admissible if and only if $\dim(\mathfrak{h}_{z,X} \cap \mathfrak{a}) = \dim(\mathfrak{a}_{\mathfrak{h}})$ for every order regular element $X \in \mathfrak{a}$.

Proof of Lemma 11.8. For every $w \in N_G(\mathfrak{a})$ we have $\mathrm{Ad}(w)\mathfrak{h}_\emptyset \cap \mathfrak{a} = \mathrm{Ad}(w)\mathfrak{a}_{\mathfrak{h}}$. Therefore, it trivially follows that $\mathfrak{h}_{z,X}\cap\mathfrak{a} = \mathrm{Ad}(w)\mathfrak{a}_{\mathfrak{h}}$ if $\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$ for some $w \in N_G(\mathfrak{a})$, and hence $\dim(\mathfrak{h}_{z,X} \cap \mathfrak{a}) = \dim(\mathfrak{a}_{\mathfrak{h}})$. It remains to prove the other implication.

Assume that $\dim(\mathfrak{h}_{z,X} \cap \mathfrak{a}) = \dim(\mathfrak{a}_{\mathfrak{h}})$. By Lemma 8.8 there exist an adapted point $y \in Z$, an element $g \in G$ and a face $\mathcal{F}$ of $\overline{\mathcal{C}}$ so that $\mathfrak{h}_{z,X} = \mathrm{Ad}(g)\mathfrak{h}_{y,\mathcal{F}}$. It follows from Lemma 8.3 that

$$N_{\mathfrak{g}}(\mathfrak{h}_{z,X}) = \mathfrak{h}_{z,X} + \mathrm{Ad}(g)\mathfrak{a}_{\mathcal{F}} + \mathrm{Ad}(g)N_{\mathfrak{m}}(\mathfrak{h}_{y,\mathcal{F}}).$$

Let $H_{z,X}$ be the connected subgroup of G with Lie algebra $\mathfrak{h}_{z,X}$ and let Γ be the open connected subgroup of $N_G(\mathfrak{h}_{z,X})/H_{z,X}$. The open connected subgroup of $N_{\mathfrak{g}}(\mathfrak{h}_{z,X})$ is equal to $\exp\big(\mathrm{Ad}(g)\mathfrak{a}_{\mathcal{F}}\big)M_0H_{z,X}$, where M_0 is the open connected subgroup of $gN_M(\mathfrak{h}_{y,\mathcal{F}})g^{-1}$. Like in (3.3) we have

$$\begin{aligned}\exp\big(\mathrm{Ad}(g)\mathfrak{a}_{\mathcal{F}}\big)M_0 \cap H_{z,X} &= \exp\big(\mathrm{Ad}(g)\mathfrak{a}_{\mathcal{F}} \cap \mathfrak{h}_{z,X}\big)\big(M_0 \cap H_{z,X}\big)\\ &= g\exp\big(\mathfrak{a}_{\mathcal{F}} \cap \mathfrak{h}_{y,\mathcal{F}}\big)g^{-1}\big(M_0 \cap H_{z,X}\big).\end{aligned}$$

In view of Lemma 8.3 we have $\mathfrak{a}_{\mathcal{F}} \cap \mathfrak{h}_{y,\mathcal{F}} = \mathfrak{a}_{\mathfrak{h}}$, and hence

$$\exp\big(\mathrm{Ad}(g)\mathfrak{a}_{\mathcal{F}}\big)M_0 \cap H_{z,X} = g\exp\big(\mathfrak{a}_{\mathfrak{h}}\big)g^{-1}\big(M_0 \cap H_{z,X}\big).$$

It follows that

$$\begin{aligned}\Gamma &\simeq \Big(\exp\big(\mathrm{Ad}(g)\mathfrak{a}_{\mathcal{F}}\big)M_0\Big)/\Big(\exp\big(\mathrm{Ad}(g)\mathfrak{a}_{\mathcal{F}}\big)M_0 \cap H_{z,X}\Big)\\ &\simeq g\exp\Big(\mathfrak{a}_{\mathcal{F}} \cap \mathfrak{a}_{\mathfrak{h}}^{\perp}\Big)g^{-1} \times M^\circ,\end{aligned}$$

where M^0 is compact. In view of Proposition 5.2 (ii) the subalgebra $\mathfrak{h}_{z,X}$ is $\mathfrak{a}$-stable, and hence $\mathfrak{a} \subseteq N_{\mathfrak{g}}(\mathfrak{h}_{z,X})$. Since $\dim(\mathfrak{h}_{z,X} \cap \mathfrak{a}) = \dim(\mathfrak{a}_{\mathfrak{h}})$, the group Γ contains a split abelian subgroup of dimension $\dim(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})$ and hence

$$\dim\left(\mathfrak{a}_{\mathcal{F}} \cap \mathfrak{a}_{\mathfrak{h}}^{\perp}\right) = \dim\left(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}\right).$$

This implies that $\mathcal{F} = \overline{\mathcal{C}}$, and hence $\mathfrak{h}_{z,X} = \mathrm{Ad}(g)\mathfrak{h}_{\emptyset}$.

Note that $\mathfrak{h}_{\emptyset}$ contains the subalgebra $\overline{\mathfrak{n}}_P$. By the Bruhat decomposition of G we may write $g = nw\overline{n}$ with $n \in N_P$, $w \in N_G(\mathfrak{a})$ and $\overline{n} \in \overline{N}_P$. Then $\overline{n}$ normalizes $\mathfrak{h}_{\emptyset}$ and thus $\mathfrak{h}_{z,X} = \mathrm{Ad}(nw)\mathfrak{h}_{\emptyset}$. Since both $\mathfrak{h}_{z,X}$ and $\mathrm{Ad}(w)\mathfrak{h}_{\emptyset}$ are normalized by A, we even have $\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_{\emptyset}$. ☐

Lemma 11.10. *Let $\alpha \in \Sigma$. The following hold.*

(i) If $U \in \mathfrak{g}_\alpha \setminus \{0\}$ and $V \in \mathfrak{g}_{-\alpha} \setminus \{0\}$, then $\mathrm{ad}^2(U)V \neq 0$.

(ii) If $U \in \mathfrak{g}_\alpha \setminus \{0\}$ and $V \in \mathfrak{g}_{-2\alpha} \setminus \{0\}$, then $\mathrm{ad}^4(U)V \neq 0$.

(iii) If $U \in \mathfrak{g}_{2\alpha} \setminus \{0\}$ and $V \in \mathfrak{g}_{-\alpha} \setminus \{0\}$, then $\mathrm{ad}(U)V \neq 0$.

Proof. Assume that $U \in \mathfrak{g}_\alpha$. Let $\mathfrak{f}$ be the Lie subalgebra of $\mathfrak{g}$ generated by U and θU and let

$$\mathcal{V} := \mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{m} \oplus \mathbb{R}\alpha^\vee \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}.$$

Note that $\mathfrak{f}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and that $\mathcal{V}$ is a representation of $\mathfrak{f}$. Now $\mathcal{V}$ decomposes as

$$\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_\alpha \oplus \mathcal{V}_{2\alpha},$$

where $\mathcal{V}_0$ is a finite sum of copies of the trivial representation, $\mathcal{V}_\alpha$ is a finite sum of copies of the highest weight representation with highest weight α (i.e., $\mathfrak{f}$), and $\mathcal{V}_{2\alpha}$ is a finite sum of copies of the highest weight-representation of $\mathfrak{f}$ with highest weight 2α.

The kernel of $\mathrm{ad}(U)$ in $\mathcal{V}_\alpha$ is equal to the space of highest weight vectors and hence is contained in $\mathfrak{g}_\alpha$. This implies that the kernel of $\mathrm{ad}^2(U)$ in $\mathcal{V}_\alpha$ is contained in $\mathfrak{m} \oplus \mathbb{R}\alpha^\vee \oplus \mathfrak{g}_\alpha$. In a similar fashion we deduce that the kernel of $\mathrm{ad}^2(U)$ in $\mathcal{V}_{2\alpha}$ is contained in $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. The assertion in (i) now follows as $\mathfrak{g}_{-\alpha} \subseteq \mathcal{V}_\alpha \oplus \mathcal{V}_{2\alpha}$.

For (ii) we continue the analysis and conclude in the same manner as before that the kernel of $\mathrm{ad}^4(U)$ in $\mathcal{V}_{2\alpha}$ is contained in $\mathfrak{g}_{-\alpha} \oplus \mathfrak{m} \oplus \mathbb{R}\alpha^\vee \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. The assertion now follows as $\mathfrak{g}_{-2\alpha} \subseteq \mathcal{V}_{2\alpha}$.

To prove (iii), assume that $U \in \mathfrak{g}_{2\alpha}$. Let $\mathfrak{e}$ be the subalgebra of $\mathfrak{g}$ generated by U and $\theta(U)$ and let

$$\mathcal{V}' := \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha.$$

Now $\mathfrak{e}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and $\mathcal{V}'$ is a representation of $\mathfrak{e}$. It is a sum of copies of the highest weight representation of $\mathfrak{e}$ with highest weight $\frac{1}{2}\alpha$. The kernel of $\mathrm{ad}(U)$ consists of highest weight-vectors and hence is contained in $\mathfrak{g}_\alpha$. This proves the final assertion. ☐

We now prove the existence of admissible points under a very restrictive assumption on Z.

Lemma 11.11. *Assume that the compression cone $\mathcal{C}$ of Z contains an open half-space. Then every open P-orbit in Z contains an admissible point.*

Proof. If $\mathcal{C} = \mathfrak{a}$, then every adapted point is admissible. Therefore, we assume that $\mathcal{C}$ is equal to a half-space. Let $z \in Z$ be adapted. If z is admissible, then we are done. Assume therefore that z is not admissible. In view of Lemma 11.8 there exists an order-regular element $X \in \mathfrak{a}$ so that $\mathfrak{a}_{\mathfrak{h}} \subsetneq \mathfrak{h}_{z,X} \cap \mathfrak{a}$. This implies that there exists a $Y \in \overline{\mathfrak{n}}_Q$ so that the limit $\mathbb{R}\big(Y + T_z(Y)\big)_X$ is a line in $\mathfrak{a}^\circ$. Now $\mathfrak{a} \in \mathrm{supp}_z(Y)$, and hence there exists a root $\alpha \in \Sigma(Q)$ so that $\mathfrak{a} \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha})$. It follows that $\alpha \in \mathcal{M}_z$. Since $\mathcal{C}$ is a half-space, the negative dual cone $-\mathcal{C}^\vee$ is a half-line. As $-\mathcal{C}^\vee$ is generated by $\mathcal{M}_z$, it follows that

$$\mathcal{M}_z \subseteq \mathbb{R}_{>0}\alpha. \tag{11.2}$$

Note that α vanishes on $\mathfrak{a}_{\mathfrak{h}}$. The root α may not be reduced. Without loss of generality we may however assume that α is the shortest element in $\mathbb{R}\alpha \cap \Sigma(Q)$ so that $\mathfrak{a} \in \mathrm{supp}_z(\mathfrak{g}_{-\alpha})$.

The fact that $\mathfrak{a}$ occurs in the support of some element Y implies that there exist $X \in \mathfrak{a}^\circ$ so that $X \notin \mathfrak{h}_z^\perp$. It follows from Proposition 3.12 that the function Φ_z defined in that proposition is non-trivial. The only roots in $\Sigma(Q)$ that are non-positive on $\overline{\mathcal{C}}$ are multiplies of α, and hence by Lemma 8.6

$$\mathrm{Im}\,(\Phi_z) \subseteq \bigoplus_{\beta \in \Sigma(Q) \cap \mathbb{R}\alpha} \mathfrak{g}_\beta$$

We claim that in fact

$$\mathrm{Im}\,(\Phi_z) \subseteq \mathfrak{g}_\alpha + \oplus \mathfrak{g}_{2\alpha}.$$

To prove the claim we use Lemma 8.5, from which it follows that

$$\mathrm{Im}\,\big(T_z^\perp\big) \subseteq \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}.$$

The claim now follows from (3.9). We define the maps

$$\phi_k : \mathfrak{a}^\circ_{\mathrm{reg}} \to \mathfrak{g}_{k\alpha} \qquad (k = 1, 2)$$

to be determined by $\Phi_z = \phi_1 + \phi_2$. By assumption ϕ_1 is not identically equal to 0. We can derive explicit expressions for ϕ_1 and ϕ_2 from (3.9). Using that

$$\mathrm{Ad}\,\big(\exp(Y)\big)X = \sum_{k=0}^{\infty} \frac{1}{k!}\,\mathrm{ad}^k(Y)X \qquad (X \in \mathfrak{a}, Y \in \mathfrak{n}_Q),$$

we obtain for $X \in \mathfrak{a}^\circ$

$$\begin{aligned}
T_z^\perp(X) &= \mathrm{Ad}\Big(\exp\big(-\phi_1(X) - \phi_2(X)\big)\Big)X - X \\
&= -[\phi_1(X) + \phi_2(X), X] + \frac{1}{2}[\phi_1(X) + \phi_2(X), [\phi_1(X) + \phi_2(X), X]] \\
&= \alpha(X)\Big(\phi_1(X) + 2\phi_2(X)\Big) - \alpha(X)[\phi_1(X) + \phi_2(X), \phi_1(X) + 2\phi_2(X)] \\
&= \alpha(X)\Big(\phi_1(X) + 2\phi_2(X)\Big).
\end{aligned}$$

It follows that

$$\phi_k(X) = \frac{1}{k\alpha(X)} p_{k\alpha} T_z^\perp(X) \qquad \big(k \in \{1,2\}, X \in \mathfrak{a}^\circ, \alpha(X) \neq 0\big), \tag{11.3}$$

where $p_{k\alpha} : \mathfrak{g} \to \mathfrak{g}_{k\alpha}$ is the projection along the Bruhat decomposition.

We claim that there exists an element $X \in \ker(\alpha) \cap \mathfrak{a}^\circ$ so that $p_\alpha\big(T_z^\perp(X)\big) \neq 0$. To prove the claim we aim at a contradiction and assume that $\ker(\alpha) \cap \mathfrak{a}^\circ \subseteq \ker(p_\alpha \circ T_z^\perp)$. Since ϕ_1 is not identically equal to 0 it follows from Lemma 8.5 that $\mathfrak{a} \in \operatorname{supp}_z(\mathfrak{g}_{-\alpha})$. Therefore, not all of $\mathfrak{a}^\circ$ is contained in $\big(\mathcal{G}(T_z\big|_{\mathfrak{g}_{-\alpha}})\big)^\perp$. Moreover, if $X \in \mathfrak{a}^\circ$ is not contained in $\big(\mathcal{G}(T_z\big|_{\mathfrak{g}_{-\alpha}})\big)^\perp$ then there exists a $Y_{-\alpha} \in \mathfrak{g}_{-\alpha}$ so that

$$B\big(X, Y_{-\alpha} + T_z(Y_{-\alpha})\big) \neq 0.$$

However,

$$B\big(X + T_z^\perp(X), Y_{-\alpha} + T_z(Y_{-\alpha})\big) = 0.$$

It follows that

$$\begin{aligned} B\big(T_z^\perp(X), Y_{-\alpha}\big) &= B\big(T_z^\perp(X), Y_{-\alpha} + T_z(Y_{-\alpha})\big) \\ &= B\big(X + T_z^\perp(X), Y_{-\alpha} + T_z(Y_{-\alpha})\big) - B\big(X, Y_{-\alpha} + T_z(Y_{-\alpha})\big) \neq 0 \end{aligned}$$

For the first equality we used that $T_z^\perp(X) \in \mathfrak{n}_Q$ and $T_z(Y_{-\alpha}) \in \mathfrak{q}$, so that

$$B\big(T_z^\perp(X), T_z(Y_{-\alpha})\big) = 0.$$

It follows that $p_\alpha\big(T_z^\perp(X)\big) \neq 0$ and thus the map $p_\alpha \circ T_z^\perp$ is not identically equal to 0. As $\ker(\alpha) \cap \mathfrak{a}^\circ$ has codimension 1 in $\mathfrak{a}^\circ$, it follows that $\ker(\alpha) \cap \mathfrak{a}^\circ = \ker(p_\alpha \circ T_z^\perp)$. Now

$$\mathfrak{a}^\circ \cap \mathfrak{h}_z^\perp = \ker(T_z^\perp) \subseteq \ker(p_\alpha \circ T_z^\perp) = \ker(\alpha) \cap \mathfrak{a}^\circ.$$

This implies that α vanishes on $\mathfrak{a}^\circ \cap \mathfrak{h}_z^\perp$ and hence $\mathfrak{a}^\circ_{\text{reg}} \cap \mathfrak{h}_z^\perp = \emptyset$. This is in contradiction with the assumption that z is adapted, and hence the claim is proven.

We now fix an element $X \in \ker(\alpha) \cap \mathfrak{a}^\circ$ so that $p_\alpha\big(T_z^\perp(X)\big) \neq 0$. Let $U_\alpha, C_\alpha \in \mathfrak{g}_\alpha$ and $U_{2\alpha}, C_{2\alpha} \in \mathfrak{g}_{2\alpha}$ be so that

$$T_z^\perp(X) = 2U_\alpha + 4U_{2\alpha} \quad \text{and} \quad T_z^\perp(\alpha^\vee) = 2C_\alpha + 4C_{2\alpha}.$$

If $t > 0$ is sufficiently large, then $X + \frac{1}{t}\alpha^\vee \in \mathfrak{a}^\circ_{\text{reg}}$. By (11.3) we then have for $t \gg 1$

$$\phi_k\big(X + \frac{1}{t}\alpha^\vee\big) = tU_{k\alpha} + C_{k\alpha}.$$

For $t \in \mathbb{R}$ define

$$n_t := \exp\big(C_\alpha + C_{2\alpha} + tU_\alpha + tU_{2\alpha}\big). \tag{11.4}$$

Note that for sufficiently large $t > 0$

$$n_t = \exp\Big(\Phi_z\big(X + \frac{1}{t}\alpha^\vee\big)\Big),$$

and hence $n_t \cdot z$ is adapted. We claim that $n_t \cdot z$ is admissible for sufficiently large $t > 0$. Let $X \in \mathfrak{a}$ be order-regular. We will show that $\mathfrak{h}_{n_t \cdot z, X} \cap \mathfrak{a} = \mathfrak{a}_{\mathfrak{h}}$ for sufficiently large $t > 0$. The claim then follows from Lemma 11.8 and Lemma 7.1.

Define

$$\mathfrak{f} := \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\beta \in \mathbb{R}\alpha \cap \Sigma} \mathfrak{g}_\beta \quad \text{and} \quad \mathcal{E} := \bigoplus_{\beta \in \Sigma(Q) \setminus \mathbb{R}\alpha} \mathfrak{g}_{-\beta} \oplus \mathfrak{g}_\beta.$$

It follows from (11.2) that

$$T_z(\mathfrak{g}_{-\beta}) \subseteq \bigoplus_{\gamma \in \big(\beta + \mathbb{R}\alpha\big) \cap \big(\Sigma(Q) \cup \{0\}\big)} \mathfrak{g}_\gamma \qquad \big(\beta \in \Sigma(Q)\big),$$

where $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$. In particular, $\mathcal{G}\big(T_z\big|_{\mathfrak{g}_{-\beta}}\big)$ is contained in $\mathfrak{f}$ if and only if $\beta \in \mathbb{R}_{>0}\alpha$. It follows that $\mathfrak{h}_z$ decomposes as

$$\mathfrak{h}_z = (\mathfrak{l}_Q \cap \mathfrak{h}_z) \oplus \mathcal{G}(T_z) = (\mathfrak{l}_Q \cap \mathfrak{h}_z) \oplus \big(\mathfrak{f} \cap \mathcal{G}(T_z)\big) \oplus \big(\mathcal{E} \cap \mathcal{G}(T_z)\big).$$

Now $\mathfrak{f}$ is a Lie subalgebra of $\mathfrak{g}$, which normalizes $\mathcal{E}$ and centralizes $\mathfrak{l}_Q \cap \mathfrak{h}_z$. Therefore, as $n_t \in \exp(\mathfrak{f})$, we have

$$\mathfrak{h}_{n_t \cdot z} = \mathrm{Ad}(n_t)\mathfrak{h}_z = (\mathfrak{l}_Q \cap \mathfrak{h}_z) \oplus \big(\mathfrak{f} \cap \mathrm{Ad}(n_t)\mathcal{G}(T_z)\big) \oplus \big(\mathcal{E} \cap \mathrm{Ad}(n_t)\mathcal{G}(T_z)\big).$$

Since $\mathfrak{l}_Q$, $\mathfrak{f}$ and $\mathcal{E}$ are $\mathfrak{a}$-stable

$$\mathfrak{h}_{n_t \cdot z, X} \cap \mathfrak{a} = \mathfrak{a}_{\mathfrak{h}} \oplus \Big(\mathfrak{a} \cap \big(\mathfrak{f} \cap \mathrm{Ad}(n_t)\mathcal{G}(T_z)\big)_X\Big).$$

It remains to prove that

$$\mathfrak{a} \cap \big(\mathfrak{f} \cap \mathrm{Ad}(n_t)\mathcal{G}(T_z)\big)_X = \{0\}. \tag{11.5}$$

For the proof of (11.5) we distinguish between two cases: the case that $\frac{1}{2}\alpha$ is not a root, and the case that $\frac{1}{2}\alpha$ is a root.

We first assume that $\frac{1}{2}\alpha$ is not a root. For $Y = Y_{-\alpha} + Y_{-2\alpha} \in \mathfrak{g}_{-\alpha}$ and $t \in \mathbb{R}$ we set

$$P_1(Y,t) := p_\alpha\Big(\mathrm{Ad}(n_t)\big(Y + T_z(Y)\big)\Big) - \frac{t^3}{6}\,\mathrm{ad}(U_\alpha)^3 Y_{-2\alpha} \in \mathfrak{g}_\alpha,$$
$$P_2(Y,t) := p_{2\alpha}\Big(\mathrm{Ad}(n_t)\big(Y + T_z(Y)\big)\Big) - \frac{t^4}{24}\,\mathrm{ad}(U_\alpha)^4 Y_{-2\alpha} \in \mathfrak{g}_{2\alpha}.$$

Both P_1 and P_2 depend linearly on the first variable and are vector valued polynomial functions in the second. The degrees of $P_1(Y,\,\cdot\,)$ and $P_2(Y,\,\cdot\,)$ are at most 2 and 3 respectively. By Lemma 11.10 we have $\mathrm{ad}(U_\alpha)^4 Y_{-2\alpha} \neq 0$ if $Y_{-2\alpha} \neq 0$ and $\mathrm{ad}(U_\alpha)^2 Y_{-\alpha} \neq 0$ if $Y_{-\alpha}$. Therefore, for every $Y \neq 0$ the polynomial function

$$P_Y : t \mapsto \frac{t^3}{6}\,\mathrm{ad}(U_\alpha)^3 Y_{-2\alpha} + P_1(Y,t) + \frac{t^4}{24}\,\mathrm{ad}(U_\alpha)^4 Y_{-2\alpha} + P_2(Y,t)$$

is non-constant. Moreover, if we restrict to Y in the sphere

$$S := \{Y \in \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha} : -B(Y, \theta Y) = 1\},$$

then the vector-valued coefficients of P_Y are uniformly bounded. Therefore, there exists an $r > 0$ so that $P_Y(t) \neq 0$ for every $Y \in S$ and $t > r$. We claim that (11.5) holds for $t > r$.

To prove the claim, we note for every non-zero $Y \in \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$ we have

$$(p_\alpha + p_{2\alpha})\Big(\operatorname{Ad}(n_t)\big(Y + T_z(Y)\big)\Big) = P_Y(t) \neq 0.$$

Therefore, if $\alpha(X) > 0$, then the limit

$$\Big(\mathbb{R}\operatorname{Ad}(n_t)\big(Y + Tz(Y)\big)\Big)_X \tag{11.6}$$

is contained in $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. If $\alpha(X) < 0$, then (11.6) is equal to $\mathbb{R}Y_{-2\alpha}$ if $Y_{-2\alpha} \neq 0$ and $\mathbb{R}Y_{-\alpha}$ otherwise. In particular, the limit (11.6) is not contained in $\mathfrak{a}$. It is easily seen from (5.1) that a limit of a subspace is spanned by the limits of all lines in the subspace. Therefore,

$$\Big(\operatorname{Ad}(n_t)\big(\mathfrak{f} \cap \mathcal{G}(Tz)\big)\Big)_X$$

is spanned by the lines (11.6) with $Y \in S$. It follows that (11.5) holds, and thus we have proven that $n_t \cdot z$ is admissible for $t \gg 1$ in case $\frac{1}{2}\alpha$ is not a root.

We move on to the second case and assume that $\frac{1}{2}\alpha$ is a root. Now 2α is not a root and therefore (11.4) simplifies to

$$n_t = \exp\big(C_\alpha + tU_\alpha\big).$$

For every $Y = Y_{-\alpha/2} + Y_{-\alpha} \in \mathfrak{g}_{-\frac{1}{2}\alpha} \oplus \mathfrak{g}_{-\alpha}$ and $t \in \mathbb{R}$

$$\begin{aligned} P_{\frac{1}{2}}(Y, t) &:= p_{\frac{1}{2}\alpha}\Big(\operatorname{Ad}(n_t)\big(Y + T_z(Y)\big)\Big) - t\operatorname{ad}(U_\alpha)Y_{-\alpha/2} \in \mathfrak{g}_{\frac{1}{2}\alpha}, \\ P_1(Y, t) &:= p_\alpha\Big(\operatorname{Ad}(n_t)\big(Y + T_z(Y)\big)\Big) - \frac{t^2}{2}\operatorname{ad}(U_\alpha)^2 Y_{-\alpha} \end{aligned}$$

define functions that are linear in the first and polynomial in the second variable. In fact $P_{\frac{1}{2}}(Y, \cdot)$ is constant and the degree of $P_1(Y, \cdot)$ is at most 1. By Lemma 11.10 we have that $\operatorname{ad}(U_\alpha)Y_{-\alpha/2} \neq 0$ if $Y_{-\alpha/2} \neq 0$, and $\operatorname{ad}(U_\alpha)^2 Y_{-\alpha} \neq 0$ if $Y_{-\alpha} \neq 0$. It follows that the polynomial function

$$\begin{aligned} P_Y : t \mapsto &(p_{\frac{1}{2}\alpha} + p_\alpha)\Big(\operatorname{Ad}(n_t)\big(Y + T_z(Y)\big)\Big) \\ &= t\operatorname{ad}(U_\alpha)Y_{-\alpha/2} + P_{\frac{1}{2}}(Y, t) + \frac{t^2}{2}\operatorname{ad}(U_\alpha)^2 Y_{-\alpha} + P_1(Y, t) \end{aligned}$$

is non-constant. The same reasoning as in the previous case now shows that (11.5) holds if $\frac{1}{2}\alpha$ is a root as well. $\square$

Proof of Proposition 11.4. In view of the Lemmas 11.6 and 11.7 it suffices to prove the existence of one admissible point in Z.

Let $z \in Z$ be adapted. If z is admissible, then there is nothing left to prove. Thus we assume that z is not admissible and use it to construct an admissible point.

Recall that $p_{\mathfrak{h}}$ is the projection $\mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}$. Now $\mathcal{A}_z$ is not dense in $\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}$ by Lemma 11.7 (i). It follows from Lemma 11.7 (ii) that there exist a $w \in \mathcal{W}_z$ and a wall $\mathcal{F}$ of $\overline{\mathcal{C}}$ so that $w \cdot p_{\mathfrak{h}}(\mathcal{F})$ is contained in the boundary of $\overline{\mathcal{A}}$. Let $v \in \mathcal{N}$ be so that $v\mathcal{N}_\emptyset = w$. By Lemma 11.7 (iii) the wall $p_{\mathfrak{h}}(\mathcal{F})$ is contained in the boundary of $\overline{\mathcal{A}_{v^{-1}\cdot z}}$. By replacing z by $v^{-1} \cdot z$, we may therefore assume that $p_{\mathfrak{h}}(\mathcal{F})$ is contained in the boundary of $\mathcal{A}_z$.

Let $\mathcal{O} = P \cdot z$. The compression cone of $Z_{\mathcal{O},\mathcal{F}}$ contains the half-space $\mathfrak{a}^- + \mathfrak{a}_{\mathcal{F}}$. (In fact the compression cone is equal to this half-space.) Therefore we may apply Lemma 11.11 to the space $Z_{\mathcal{O},\mathcal{F}}$. Let $y \in Z_{\mathcal{O},\mathcal{F}}$ satisfy $\mathfrak{h}_{z,\mathcal{F}} = \mathfrak{h}_y^{\mathcal{O},\mathcal{F}}$. The point y is adapted by Proposition 10.4, and hence $P \cdot y$ is open. By Lemma 11.11 there exists an admissible point $y' \in P \cdot y$. In view of Proposition 3.12 there exist $m \in M$, $a \in A$ and $Y \in \mathfrak{a}^\circ_{\mathrm{reg}}$ so that $y' = ma \exp\big(\Phi_y^{\mathcal{O},\mathcal{F}}(Y)\big) \cdot y$.

Since the set of order-regular elements is dense in $\mathfrak{a}$, the complement of $p_{\mathfrak{h}}^{-1}(\mathcal{A}_z)$ is equal to the closure of the set of order-regular elements in the complement of $p_{\mathfrak{h}}^{-1}(\mathcal{A}_z)$. The boundary of $p_{\mathfrak{h}}^{-1}(\mathcal{A}_z)$ consists of elements $X \in \mathfrak{a}$ that are not order-regular. Therefore, the set of order-order regular elements in the complement of $p_{\mathfrak{h}}^{-1}(\mathcal{A}_z)$ is a union of connected components of the set of order-regular elements. Note that there are only finitely many such connected components. It follows that there exists a connected component $\mathcal{R}$ of the set of order-regular elements, so that $p_{\mathfrak{h}}(\mathcal{R})$ is contained in the complement of $\mathcal{A}_z$ and $\overline{\mathcal{R}}$ intersects with the interior of $\mathcal{F}$.

Let $z' := ma \exp\big(\Phi_z(Y)\big) \cdot z$. We claim that $p_{\mathfrak{h}}(\mathcal{R}) \subseteq \mathcal{A}_{z'}$. By Proposition 10.4 we have $\mathfrak{h}_{y'}^{\mathcal{O},\mathcal{F}} = \mathfrak{h}_{z',\mathcal{F}}$. Then, in view of Proposition 5.2 (iii),

$$\mathfrak{h}_{z',X} = (\mathfrak{h}_{z',\mathcal{F}})_X = (\mathfrak{h}_{y'}^{\mathcal{O},\mathcal{F}})_X$$

for every $X \in \mathcal{R}$. Since y' is an admissible point in $Z_{\mathcal{O},\mathcal{F}}$, there exists an element $v' \in \mathcal{N}$ so that $(\mathfrak{h}_{y'}^{\mathcal{O},\mathcal{F}})_X = \mathrm{Ad}(v')\mathfrak{h}_\emptyset$. It follows that $p_{\mathfrak{h}}(\mathcal{R}) \subseteq \mathcal{A}_{z'}$. This proves the claim.

In view of Lemma 11.7 (iv) there exists a dense and open subset U of the set of adapted points so that

$$\mathcal{A}_z \cup p_{\mathfrak{h}}(\mathcal{R}) \subseteq \mathcal{A}_{z''} \qquad (z'' \in U).$$

Let $z'' \in U$. If z'' is admissible, then we are done. If not, we replace z by z'' and repeat the above procedure to find another adapted point z' with $\mathcal{A}_z \subsetneq \mathcal{A}_{z'}$. It follows from Lemma 11.7 (ii) that after finitely many iterations this process ends, and thus we find an admissible point in Z. □

12 The little Weyl group

In this section we construct the little Weyl group of Z.

We define

$$\begin{aligned}\mathcal{W} &:= \mathcal{W}_z \qquad (12.1)\\ &= \{w\mathcal{N}_\emptyset \in \mathcal{N}/\mathcal{N}_\emptyset : w \in \mathcal{N} \text{ and there exist } X \in \mathfrak{a} \text{ so that } \mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset\},\end{aligned}$$

where $z \in Z$ is any admissible point. This set does not depend on the choice of z by Proposition 11.4. We recall that $p_\mathfrak{h}$ is the projection $\mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_\mathfrak{h}$.

Theorem 12.1. *The set $\mathcal{W}$ is a subgroup of $\mathcal{N}/\mathcal{N}_\emptyset$. Moreover, $\mathcal{W}$ acts faithfully on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$ as a reflection group and is as such generated by the simple reflections in the walls of $p_\mathfrak{h}(\overline{\mathcal{C}})$. Moreover, $p_\mathfrak{h}(\overline{\mathcal{C}})$ is a fundamental domain for the action of $\mathcal{W}$ on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$. Finally, $\mathcal{W}$ is equal to the little Weyl group of Z as defined in [15, Section 9].*

We will prove the theorem in a number of steps. We begin with the first assertion in the theorem.

Proposition 12.2. *$\mathcal{W}$ is a subgroup of $\mathcal{N}/\mathcal{N}_\emptyset$.*

Proof. Let $z \in Z$ be admissible. Let $w \in \mathcal{W}$ and let $v \in \mathcal{N}$ be so that $w = v\mathcal{N}_\emptyset$. By Proposition 7.2 the P-orbit $Pv^{-1} \cdot z$ is open, and hence $v^{-1} \cdot z$ is admissible; see Remark 11.2 (b). Let $w' \in \mathcal{W}$ and let $v' \in \mathcal{N}$ be so that $w' = v'\mathcal{N}_\emptyset$. In view of Proposition 7.2 there exists a $m \in M$ so that for every $X \in \mathrm{Ad}(vv')\mathcal{C}$

$$\mathrm{Ad}(v^{-1})\mathfrak{h}_{z,X} = \mathfrak{h}_{v^{-1}\cdot z,\mathrm{Ad}(v^{-1})X} = \mathrm{Ad}(v'm)\mathfrak{h}_\emptyset,$$

and hence $\mathfrak{h}_{z,X} = \mathrm{Ad}(vv'm)\mathfrak{h}_\emptyset$. Therefore, $ww' = vv'\mathcal{N}_\emptyset \in \mathcal{W}$. It follows that $w\mathcal{W} \subseteq \mathcal{W}$, and hence, since $\mathcal{W}$ is finite,

$$w\mathcal{W} = \mathcal{W}.$$

We thus see that $\mathcal{W}$ is closed under multiplication. As $\mathcal{W}$ is finite, it is a subgroup of $\mathcal{N}/\mathcal{N}_\emptyset$. □

It follows from Proposition 7.2, Proposition 11.4 and Lemma 11.7 that

$$\mathfrak{a}/\mathfrak{a}_\mathfrak{h} = \bigcup_{w\in\mathcal{W}} w \cdot p_\mathfrak{h}(\overline{\mathcal{C}}) \qquad (12.2)$$

and

$$w \cdot p_\mathfrak{h}(\mathcal{C}) \cap w' \cdot p_\mathfrak{h}(\mathcal{C}) = \emptyset \qquad (w, w' \in \mathcal{W}, w \neq w'). \qquad (12.3)$$

For an open P-orbit $\mathcal{O}$ in Z and a face $\mathcal{F}$ of $\overline{\mathcal{C}}$ we write $\mathcal{W}_{\mathcal{O},\mathcal{F}}$ for the subgroup (12.1) of $\mathcal{N}/\mathcal{N}_\emptyset$ for the spherical space $Z_{\mathcal{O},\mathcal{F}}$.

Lemma 12.3. *Let $\mathcal{O}$ be an open P-orbit in Z and let $\mathcal{F}$ be a wall of $\overline{\mathcal{C}}$. Then $\mathcal{W}_{\mathcal{O},\mathcal{F}}$ is a subgroup of $\mathcal{W}$ of order 2. Moreover, $\mathcal{W}_{\mathcal{O},\mathcal{F}}$ stabilizes $p_\mathfrak{h}(\mathcal{F})$. Finally, $\mathcal{W}_{\mathcal{O},\mathcal{F}}$ does not depend on the open P-orbit $\mathcal{O}$.*

Proof. Let $\mathcal{R}$ be a connected component of the set of order-regular elements in $\mathfrak{a}$ so that $\overline{\mathcal{R}}$ intersects with the relative interior of $\mathcal{F}$ and $\mathcal{R}\cap\mathcal{C}=\emptyset$. Let $w\in\mathcal{W}$ be the element so that $p_{\mathfrak{h}}(\mathcal{R})\subseteq w\cdot p_{\mathfrak{h}}(\mathcal{C})$ and let $v\in\mathcal{N}$ be a representative of w.

Let $z\in Z$ be admissible and let $y\in Z_{\mathcal{O},\mathcal{F}}$ be so that $\mathfrak{h}_y^{\mathcal{O},\mathcal{F}}=\mathfrak{h}_{z,\mathcal{F}}$. By Proposition 10.4 the point y is adapted. It follows from Proposition 5.2 (iii) and Proposition 7.2 that there exists an $m\in M$ so that for all $X\in\mathcal{R}$

$$(\mathfrak{h}_y^{\mathcal{O},\mathcal{F}})_X=(\mathfrak{h}_{z,\mathcal{F}})_X=\mathfrak{h}_{z,X}=\mathrm{Ad}(vm)\mathfrak{h}_\emptyset,$$

and hence $w=v\mathcal{N}_\emptyset\in\mathcal{W}_{\mathcal{O},\mathcal{F}}$.

The compression cone of $Z_{\mathcal{O},\mathcal{F}}$ is given by $\mathcal{C}_\mathcal{F}=\mathcal{C}+\mathfrak{a}_\mathcal{F}$, see Proposition 10.5. Since $\mathcal{F}$ is a wall, the space $p_{\mathfrak{h}}(\mathfrak{a}_\mathcal{F})$ has codimension 1 in $\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}$, and hence $p_{\mathfrak{h}}(\mathcal{C}_\mathcal{F})$ is an open half-space. Therefore, also $w\cdot p_{\mathfrak{h}}(\mathcal{C}_\mathcal{F})$ is an open half-space. Moreover, $p_{\mathfrak{h}}(\mathcal{C}_\mathcal{F})$ and $w\cdot p_{\mathfrak{h}}(\mathcal{C}_\mathcal{F})$ are disjoint, and thus

$$\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}=p_{\mathfrak{h}}(\overline{\mathcal{C}_\mathcal{F}})\cup w\cdot p_{\mathfrak{h}}(\overline{\mathcal{C}_\mathcal{F}})\quad\text{and}\quad p_{\mathfrak{h}}(\mathcal{C}_\mathcal{F})\cap w\cdot p_{\mathfrak{h}}(\mathcal{C}_\mathcal{F})=\emptyset.$$

It follows that the group $\mathcal{W}_{\mathcal{O},\mathcal{F}}$ is of order 2. Since w is non-trivial, we have $\mathcal{W}_{\mathcal{O},\mathcal{F}}=\{1,w\}$

If $\mathcal{R}'$ is another connected component of the set of order-regular elements in $\mathfrak{a}$ so that $\overline{\mathcal{R}'}$ intersects with the relative interior of $\mathcal{F}$ and $\overline{\mathcal{R}'}\cap\mathcal{C}=\emptyset$, then there exists a $w'\in\mathcal{W}$ so that $p_{\mathfrak{h}}(\mathcal{R}')\subseteq w'\cdot p_{\mathfrak{h}}(\mathcal{C})$. The arguments above show that $\mathcal{W}_{\mathcal{O},\mathcal{F}}=\{1,w'\}$ and it follows that $w=w'$. Therefore, all connected components $\mathcal{R}'$ of the set of order-regular elements in $\mathfrak{a}$ so that $\overline{\mathcal{R}'}$ intersects with the relative interior of $\mathcal{F}$ and $\overline{\mathcal{R}'}\cap\mathcal{C}=\emptyset$ have the property that $p_{\mathfrak{h}}(\mathcal{R}')\subseteq w\cdot p_{\mathfrak{h}}(\mathcal{C})$. This shows that the relative interior of $p_{\mathfrak{h}}(\mathcal{F})$ is contained in $w\cdot p_{\mathfrak{h}}(\overline{\mathcal{C}})$ and hence $p_{\mathfrak{h}}(\mathcal{F})$ is a wall of $w\cdot p_{\mathfrak{h}}(\overline{\mathcal{C}})$. The element w stabilizes $p_{\mathfrak{h}}(\overline{\mathcal{C}})\cap w\cdot p_{\mathfrak{h}}(\overline{\mathcal{C}})$. The latter set is equal to the common wall $p_{\mathfrak{h}}(\mathcal{F})$.

Finally, if $\mathcal{O}'$ is another open P-orbit in Z, then the arguments above yield an element $w'\in\mathcal{W}$ so that $w'\cdot p_{\mathfrak{h}}(\overline{\mathcal{C}})\cap p_{\mathfrak{h}}(\overline{\mathcal{C}})=p_{\mathfrak{h}}(\mathcal{F})$. Now both $\overline{w\cdot\mathcal{C}}$ and $\overline{w'\cdot\mathcal{C}}$ share the wall $\mathcal{F}$ with $\overline{\mathcal{C}}$. It follows that $w\cdot\mathcal{C}=w'\cdot\mathcal{C}$, and hence $w'=w$. □

In view of Lemma 12.3 we may for a wall $\mathcal{F}$ of $\overline{\mathcal{C}}$ define

$$\mathcal{W}_\mathcal{F}:=\mathcal{W}_{\mathcal{O},\mathcal{F}},$$

where $\mathcal{O}$ is any open P-orbit in Z. In the following lemma we identify the non-trivial element in $\mathcal{W}_\mathcal{F}$. The lemma heavily relies on Proposition 9.1.

Lemma 12.4. *For every wall $\mathcal{F}$ of $\overline{\mathcal{C}}$ there exists a $s_\mathcal{F}\in\mathcal{N}$ that acts on $\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}$ as the reflection in the hyperplane $\mathfrak{a}_\mathcal{F}/\mathfrak{a}_{\mathfrak{h}}$. Moreover,*

$$\mathcal{W}_\mathcal{F}=\{e\mathcal{N}_\emptyset,s_\mathcal{F}\mathcal{N}_\emptyset\}.$$

Proof. Let $z\in Z$ be an admissible point and let α be an indecomposable element in $\mathcal{M}_z$ so that (8.3) holds.

If $\alpha\in\Sigma\cup2\Sigma$, then the simple reflection s in α is contained in the Weyl group W of Σ and normalizes $\mathfrak{a}_{\mathfrak{h}}$ as $\alpha\big|_{\mathfrak{a}_{\mathfrak{h}}}=0$. Note that s acts on $\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}$ by reflecting in $\mathfrak{a}_\mathcal{F}/\mathfrak{a}_{\mathfrak{h}}$.

If $\alpha \notin \Sigma \cup 2\Sigma$, then by Proposition 9.1 there exist $\beta, \gamma \in \Sigma(Q)$ so that $\alpha = \beta + \gamma$, β and γ are orthogonal and $\operatorname{span}(\beta^\vee, \gamma^\vee) \cap \mathfrak{a}_\mathfrak{h} \neq \{0\}$. Let $\sigma_\beta \in W$ and $\sigma_\gamma \in W$ be the simple reflections in β and γ respectively. Then $s := \sigma_\beta\sigma_\gamma$ acts on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$ by reflecting in $\ker \alpha/\mathfrak{a}_\mathfrak{h} = \mathfrak{a}_\mathcal{F}/\mathfrak{a}_\mathfrak{h}$.

Let $s_\mathcal{F} \in \mathcal{N}$ be a representative of s. It remains to prove that $s_\mathcal{F}\mathcal{N}_\emptyset \in \mathcal{W}_\mathcal{F}$. Let $v \in \mathcal{N}$ be a representative of the non-trivial element in $\mathcal{W}_\mathcal{F}$. Let $\mathcal{O} = P \cdot z$ and let y be an admissible point in $Z_{\mathcal{O},\mathcal{F}}$. The compression cone $\mathcal{C}_\mathcal{F}$ is an open half-space. Therefore, for every $X \in \mathcal{C}_\mathcal{F}$ we have

$$(\mathfrak{h}_y^{\mathcal{O},\mathcal{F}})_X = \operatorname{Ad}(m)\mathfrak{h}_\emptyset \quad \text{and} \quad (\mathfrak{h}_y^{\mathcal{O},\mathcal{F}})_{-X} = \operatorname{Ad}(m'v)\mathfrak{h}_\emptyset$$

for some elements $m, m' \in M$. Both $\mathfrak{h}_\emptyset$ and $\operatorname{Ad}(v)\mathfrak{h}_\emptyset$ are $\mathfrak{a}$-stable. Let ι be the Plücker embedding. If $v_1, \dots, v_n \in \mathfrak{g}$ is a basis of $\mathfrak{h}_\emptyset$, then $\iota(\mathfrak{h}_\emptyset) = \mathbb{R}(v_1 \wedge \cdots \wedge v_n)$. Since $\mathfrak{h}_\emptyset$ is $\mathfrak{a}$-stable, we may assume that every v_i is a joint eigenvector for $\operatorname{ad}(\mathfrak{a})$. Now $\iota\big(\mathfrak{h}_\emptyset\big)$ is a joint eigenspace for $\operatorname{ad}(\mathfrak{a})$ with weight equal to the sum of the weights of $v_1, \dots, v_n$. From (6.1) it follows that this weight is equal to $-2\rho_Q$, where ρ_Q is the half-sum of the roots in $\Sigma(Q)$ counted with multiplicity. Likewise, $\operatorname{ad}(\mathfrak{a})$ acts on the line $\iota\big(\operatorname{Ad}(v)\mathfrak{h}_\emptyset\big)$ with weight $-2\operatorname{Ad}^*(v)\rho_Q$.

Let $\mathcal{M}_y^{\mathcal{O},\mathcal{F}}$ be the monoid (8.1) for the space $Z_{\mathcal{O},\mathcal{F}}$ and the adapted point y. Then

$$\mathcal{M}_y^{\mathcal{O},\mathcal{F}} \subseteq \mathbb{R}_{>0}\alpha.$$

Therefore, if $X \in \mathcal{C}_\mathcal{F}$ and $Y \in \mathfrak{h}_y^{\mathcal{O},\mathcal{F}} \setminus \{0\}$, then $(\mathbb{R}Y)_{-X}$ is a line with eigenweight differing by a non-zero multiple of α from an eigenweight of a line $(\mathbb{R}Y')_X$ with $Y' \in \mathfrak{h}_y^{\mathcal{O},\mathcal{F}} \setminus \{0\}$. Hence, every $\mathfrak{a}$-weight that occurs in $\operatorname{Ad}(v)\mathfrak{h}_\emptyset$ differs by a multiple of α from an $\mathfrak{a}$-weight in $\mathfrak{h}_\emptyset$. It follows that $\operatorname{Ad}^*(v)\rho_Q = \rho_Q + r\alpha$ for some $r \in \mathbb{R} \setminus \{0\}$. Since the lengths of $\operatorname{Ad}^*(v)\rho_Q$ and ρ_Q are equal, it follows that

$$\|\rho_Q\|^2 = \|\rho_Q\|^2 + 2r\langle\rho_Q, \alpha\rangle + r^2\|\alpha\|^2. \tag{12.4}$$

Because α is either a root in $\Sigma(Q)$ or a sum of roots in $\Sigma(Q)$, we have $\langle\rho_Q, \alpha\rangle > 0$. Therefore, the equation (12.4) has precisely one non-zero solution r. As $\operatorname{Ad}^*(s_\mathcal{F})\rho_Q \in \rho_Q + \mathbb{R}\alpha$ and $\operatorname{Ad}^*(s_\mathcal{F})\rho_Q \neq \rho_Q$, it follows that

$$\operatorname{Ad}^*(v)\rho_Q = \operatorname{Ad}^*(s_\mathcal{F})\rho_Q.$$

Therefore, $s_\mathcal{F}v^{-1} \in N_{L_Q}(\mathfrak{a})$. By Lemma 11.3 the latter group is equal to $\mathcal{N}_\emptyset$, and hence $s_\mathcal{F}\mathcal{N}_\emptyset = v\mathcal{N}_\emptyset$. □

Proof of Theorem 12.1. In view of Lemmas 12.3 and 12.4 the group $\mathcal{W}_{\mathrm{refl}}$ generated by the simple reflections in the walls of $p_\mathfrak{h}\big(\overline{\mathcal{C}}\big)$ is a subgroup of $\mathcal{W}$. It follows from (12.2) and (12.3) that in fact $\mathcal{W}_{\mathrm{refl}} = \mathcal{W}$. In particular, $p_\mathfrak{h}(\overline{\mathcal{C}})$ is a fundamental domain for the action of $\mathcal{W}$ on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$. Comparison to [15, Section 9] shows that $\mathcal{W}$ is equal to the little Weyl group. Indeed, in view of [15, Theorem 9.5, Corollary 9.6 & Corollary 12.5] the little Weyl group is a reflection group acting on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$ and is generated by the simple reflections in the walls of the cone $p_\mathfrak{h}\big(\overline{\mathcal{C}}\big)$. □

13 Spherical root system

In this section we attach a root system Σ_Z to Z of which $\mathcal{W}$ is the Weyl group.

We recall the edge $\mathfrak{a}_E$ of the compression from (6.5).

Lemma 13.1. *$\mathcal{W}$ acts trivially on the subspace $\mathfrak{a}_E/\mathfrak{a}_{\mathfrak{h}}$ in $\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}$.*

Proof. The little Weyl group $\mathcal{W}$ is generated by simple reflections in the walls of $p_{\mathfrak{h}}(\overline{\mathcal{C}})$. Since $\mathfrak{a}_E/\mathfrak{a}_{\mathfrak{h}}$ is contained in each of these walls, the simple reflections act trivially on $\mathfrak{a}_E/\mathfrak{a}_{\mathfrak{h}}$. It follows that $\mathcal{W}$ acts trivially on $\mathfrak{a}_E/\mathfrak{a}_{\mathfrak{h}}$. $\square$

It follows from Lemma 13.1 that $\mathcal{W}$ acts on $\mathfrak{a}/\mathfrak{a}_E$ in a natural manner. We write p_E for the projection $\mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_E$. If $w \in \mathcal{W}$, then the action of w commutes with p_E. It follows from (12.3) that $w \cdot p_E(\mathcal{C}) = p_E(\mathcal{C})$ if and only if $w = e$. Therefore, w acts trivially on $\mathfrak{a}/\mathfrak{a}_E$ if and only if $w = e$.

We now come to the main result in this section.

Theorem 13.2. *The group $\mathcal{W}$ is a crystallographic group, i.e., it is the Weyl group of a root system Σ_Z in $(\mathfrak{a}/\mathfrak{a}_E)^*$.*

Proof. We will verify the criterion in Section VI. 2. 5 of [3]. For this we define

$$\Lambda := (\mathfrak{a}/\mathfrak{a}_E)^* \cap \mathbb{Z}\Sigma(\mathfrak{a}).$$

Let $z \in Z$ be adapted. We recall the monoid $\mathcal{M}_z$ from (8.1) and note that $\mathcal{M}_z \subseteq \Lambda$. It follows from Proposition 6.9 (i) and (8.1) that Λ has full rank in $(\mathfrak{a}/\mathfrak{a}_E)^*$.

It follows from Theorem 12.1 and Lemma 13.1 that $\mathcal{W}$ acts faithfully as a finite reflection group on $\mathfrak{a}/\mathfrak{a}_E$. Moreover, since $\mathcal{W}$ is a subquotient of $N_G(\mathfrak{a})$, it preserves the lattice Λ. Thus by [3, Proposition 9 in Section VI.2.5] there exist a root system Σ_Z in $(\mathfrak{a}/\mathfrak{a}_E)^*$ for which $\mathcal{W}$ is the Weyl group. $\square$

The proof of Proposition 9 in Section VI. 2. 5 of [3] provides a construction of Σ_Z. Each reflection s in $\mathcal{W}$ determines a root as follows. Let D_s be the -1-eigenspace of s in $(\mathfrak{a}/\mathfrak{a}_E)^*$. Then the primitive elements $\alpha, -\alpha$ in $D_s \cap \Lambda$ belong to Σ_Z. All roots in Σ_Z are obtained in this manner. This root system is called the spherical root system of the real spherical homogeneous space Z.

Remark 13.3.

(i) In the complex case, the root system constructed here is identical to the one in [14, Section 6]. If Z is symmetric, then Theorem 6.7 in *loc. cit.* makes a comparison between Σ_Z and the restricted roots system Σ_Z^r of the complex symmetric space Z. Namely, Σ_Z is the reduced root system associated to $2\Sigma_Z^r$.

(ii) Similarly to each real reductive symmetric space Z, a restricted root system Σ_Z^r is attached in [22, Theorem 5]. This root system is in general not reduced. The root system Σ_Z is the reduced root system associated to $2\Sigma_Z^r$.

14 Reduction to quasi-affine spaces

Many results for quasi-affine real spherical homogeneous spaces hold true also for real spherical spaces that are not quasi-affine. These results can be proven by a simple reduction to the quasi-affine case. In this section we drop the standing assumption that Z is quasi-affine.

By Chevalley's theorem there exists a real rational representation (π, V) of G and a vector $v_H \in V$ so that H is equal to the stabilizer of $\mathbb{R}v_H$. Let χ be the character with which H acts on $\mathbb{R}v_H$. Set

$$G' := G \times \mathbb{R}^\times \quad \text{and} \quad H' := \{\big(h, \chi(h)^{-1}\big) \in G' : h \in H\}.$$

Then

$$Z' := G'/H'$$

is a quasi-affine real spherical homogeneous space. We denote the natural projection $Z' \to Z$ by π.

The results in the previous sections apply to the space Z'. Many of them imply the analogous assertions for Z. We will list here the most relevant.

We define P' to be the minimal parabolic subgroup $P \times \mathbb{R}^\times$ of G'. Define

$$M' := M \times \{1\}, \quad A' = A \times \mathbb{R}^\times \quad \text{and} \quad N_P' := N_P \times \{1\}.$$

Then $P' = M'A'N_P'$ is a Langlands decomposition of P'.

A point $z \in Z$ is called adapted (with respect to $P = MAN_P$) if there exists an adapted point $z' \in Z'$ (with respect to $P' = M'A'N_P'$) so that $\pi(z') = z$. Since

$$\{e\} \times \mathbb{R}^\times \subseteq A'$$

and the sets of adapted points in Z' are stable under multiplication by elements in A', the set $\pi^{-1}(z)$ consists of adapted points if and only if z is adapted.

For an adapted point $z' \in Z'$ let $L_Q' = Z_{G'}\big(\mathfrak{a}' \cap \mathfrak{h}_{z'}^\perp\big)$ and let $Q' = L_Q'P'$. Define

$$Q := \pi(Q') \quad \text{and} \quad L_Q := \pi(L_Q').$$

Then Q is a parabolic subgroup containing the minimal parabolic subgroup P, and (i) and (ii) in Proposition 3.6 hold true for all adapted points $z \in Z$.

We set

$$\mathfrak{h}_\emptyset := (\mathfrak{l}_Q \cap \mathfrak{h}_{z_0}) + \overline{\mathfrak{n}}_Q$$

for some adapted point $z_0 \in Z$. We further define the compression cone of Z to be

$$\mathcal{C} := \{X \in \mathfrak{a} : \mathfrak{h}_{z,X} = \mathrm{Ad}(m)\mathfrak{h}_\emptyset \text{ for some } m \in M\}$$

where $z \in Z$ is an adapted point. The compression cone $\mathcal{C}'$ for Z' is related to $\mathcal{C}$ by the identity

$$\mathcal{C}' = \mathcal{C} \times \mathbb{R} \subseteq \mathfrak{a} \times \mathbb{R}.$$

It follows from Proposition 6.5 that $\mathcal{C}$ does not depend on the adapted point $z \in Z$ chosen to define it.

We call an adapted point $z \in Z$ admissible if for every order-regular element $X \in \mathfrak{a}$ there exists a $w \in N_G(\mathfrak{a})$ so that $\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$. Then z is admissible if and only if $\pi^{-1}(z)$ consists of admissible points in Z'. It follows from Proposition 11.4 (i) that the set of admissible points is open and dense in the set of adapted points in Z with respect to the subspace topology. Define

$$\mathcal{N}_\emptyset := \{w \in N_G(\mathfrak{a}) : \mathrm{Ad}(w)\mathfrak{h}_\emptyset = \mathrm{Ad}(m)\mathfrak{h}_\emptyset \text{ for some } m \in M\},$$

and

$$\mathcal{W} := \{w\mathcal{N}_\emptyset \in N_G(\mathfrak{a})/\mathcal{N}_\emptyset : w \in \mathcal{N} \text{ and there exist } X \in \mathfrak{a} \text{ so that } \mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset\},$$

where z is an admissible point in Z. Then π induces a bijection between $\mathcal{W}$ and the little Weyl group $\mathcal{W}'$ of Z'. In particular $\mathcal{W}$ is a finite group and acts on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$ as a reflection group. Let $p_\mathfrak{h}$ be the projection $\mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_\mathfrak{h}$. Then $\mathcal{W}$ is generated by the simple reflections in the walls of $p_\mathfrak{h}(\overline{\mathcal{C}})$ and $p_\mathfrak{h}(\overline{\mathcal{C}})$ is a fundamental domain for the action of $\mathcal{W}$ on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$.

If $\mathfrak{a}_E$ denotes the edge of $\overline{\mathcal{C}}$, then the edge of $\overline{\mathcal{C}'}$ is given by $\mathfrak{a}'_E = \mathfrak{a}_E \times \mathbb{R}$. Therefore, there is a canonical identification $\phi : \mathfrak{a}/\mathfrak{a}_E \to (\mathfrak{a} \times \mathbb{R})/\mathfrak{a}'_E$. The map ϕ intertwines the action of $\mathcal{W}$ and $\mathcal{W}'$. Finally, if $\Sigma_{Z'}$ is the spherical root system of Z', then

$$\Sigma_Z := \{\alpha \circ \phi : \alpha \in \Sigma_{Z'}\}$$

is a root system, which is called the spherical root system of Z.

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Chapter IV

The most continuous part of the Plancherel decomposition for a real spherical space

Joint with Eitan Sayag.

Abstract

In this article we give a precise description of the Plancherel decomposition of the most continuous part of $L^2(Z)$ for a real spherical homogeneous space Z. Our starting point is the recent construction of Bernstein morphisms by Delorme, Knop, Krötz and Schlichtkrull. The most continuous part decomposes into a direct integral of unitary principal series representations. We give an explicit construction of the H-invariant functionals on these principal series. We show that for generic induction data the multiplicity space equals the full space of H-invariant functionals. Finally, we determine the inner products on the multiplicity spaces by refining the Maaß-Selberg relations.

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1 Introduction

In this paper we provide a complete description of the most continuous part of the Plancherel decomposition for a unimodular real spherical homogeneous space.

Let $Z := G/H$, where $G = \underline{G}(\mathbb{R})$ is the group of real points of a connected reductive algebraic group $\underline{G}$ defined over $\mathbb{R}$ and $H = \underline{H}(\mathbb{R})$ the set of real points of an algebraic subgroup $\underline{H}$ of $\underline{G}$. We assume that Z is unimodular and hence admits a positive G-invariant Radon measure μ_Z. The basic problem in harmonic analysis on Z is to obtain an explicit description of the Plancherel decomposition of the regular representation of G on $L^2(Z, \mu_Z)$ into a direct integral of irreducible unitary representations of G.

In the case that the group G is considered as a homogeneous space of $G \times G$ such an explicit description of the Plancherel decomposition is found in the celebrated work of Harish-Chandra [21], [22], [23]. For real reductive symmetric spaces it was obtained by Delorme in [16] and independently by Van den Ban and Schlichtkrull in [8], [9].

Recall that the space $Z = G/H$ is called symmetric in case H is an open subgroup of the fixed point subgroup G^σ for an involutive automorphism $\sigma : G \to G$. The representations of G occurring in the Plancherel decomposition of a reductive symmetric space Z split into finitely many *series* according to the (class of) parabolic subgroup $P \subseteq G$ from which they are induced. The relevant parabolic subgroups of G are the so-called σ-parabolics, namely those parabolic subgroups P for which P and $\sigma(P)$ are opposite to each other. With a Langlands decomposition $P = M_P A_P N_P$, with $\sigma(A_P) = A_P$, the part attached to P has the form of a direct integral of generalized principal series representations. More specifically, these are induced representations $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes \mathbf{1})$, where ξ is in the discrete series of representations for the symmetric space $M_P/M_P \cap H$ and λ is a unitary character of $\mathfrak{a}_P = \mathrm{Lie}(A_P)$ that vanishes on $\mathfrak{a}_P \cap \mathfrak{h}$. The part corresponding to the minimal σ-parabolic subgroup Q is called the most continuous part of $L^2(Z)$. The Plancherel decomposition of the most continuous part was determined for real reductive symmetric spaces in the work of Van den Ban and Schlichtkrull in [7]. This was based on the earlier works of Van den Ban on invariant linear functionals [1], [2]. The most continuous part of $L^2(Z)$ decomposes as

$$L^2_{\mathrm{mc}}(Z) \simeq \widehat{\bigoplus_{\xi \in \widehat{M}_Q}} \int^{\oplus}_{i(\mathfrak{a}_Q/\mathfrak{a}_Q \cap \mathfrak{h})^*_+} V^*(\xi) \otimes \mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes \mathbf{1})\, d\lambda,$$

where $d\lambda$ is the Lebesgue measure on $i(\mathfrak{a}_Q/\mathfrak{a}_Q \cap \mathfrak{h})^*$ and $i(\mathfrak{a}_Q/\mathfrak{a}_Q \cap \mathfrak{h})^*_+$ is a fundamental domain for the stabilizer of $(\mathfrak{a}_Q/\mathfrak{a}_Q \cap \mathfrak{h})^*$ in the Weyl group. The multiplicity spaces $V^*(\xi)$ are independent of λ. Moreover, $V^*(\xi)$ is non-zero only for finite dimensional unitary representations ξ of M_Q.

A homogeneous space Z is called real spherical if a minimal parabolic subgroup P of G admits an open orbit in Z. All real reductive symmetric spaces are real spherical. A remarkable property of the class of real spherical spaces is the fact that all irreducible smooth representations of G admit a finite dimensional space of H-invariant functionals by [33, Theorem C] and [37]. This property makes harmonic analysis on real spherical spaces suitable for developing Plancherel theory.

In this paper we provide an explicit Plancherel decomposition for the most continuous part of $L^2(Z)$ for a real spherical space Z, thus generalizing the main result of [7].

Our starting point is the recent work of Delorme, Knop, Krötz and Schlichtkrull [17]. Their construction of Bernstein morphisms allows to decompose $L^2(Z)$ into finitely many *blocks* of representations, each attached to a so-called boundary degeneration of Z. The block for the most degenerate of these boundary degenerations we call the most continuous part of $L^2(Z)$. We show that, as in the symmetric case, the most continuous part decomposes into a direct integral of principal series representations. To determine the Plancherel decomposition of the most continuous part we construct linear functionals on these principal series. For generic parameters our construction provides a basis for the space of H-invariant linear functionals. We then show the key result that all H-invariant functionals are tempered and the wave packets constructed using these functionals are square integrable. Finally, by refining the Maaß-Selberg relations of [17], we obtain a complete description of the inner product on the multiplicity spaces. This yields the full description of the most continuous part of $L^2(Z)$.

Assuming that the twisted discrete series conjecture from [35, (1.3)] holds, the most continuous part of $L^2(Z)$ exhausts $L^2(Z)$ in case Z is a complex spherical space, i.e., in case G and H are both complex. Thus, our construction is expected to yield the full Plancherel formula for complex spherical spaces.

1.1 The most continuous part via Bernstein morphisms

To describe the most continuous part $L^2(Z)_{\mathrm{mc}}$ of $L^2(Z)$ and the results in this article we recall some important invariants of the real spherical homogeneous space Z, boundary degenerations of Z, twisted discrete series representations and finally the Bernstein morphism, relating $L^2(Z)$ to twisted discrete series representations for boundary degenerations of Z.

We fix a minimal parabolic subgroup P and a well chosen (with respect to H) Langlands decomposition $P = MAN$. Inside the Lie algebra $\mathfrak{a}$ of A one finds the compression cone, which is an open cone $\mathcal{C}$ whose closure is finitely generated and contains $\mathfrak{a}_{\mathfrak{h}} := \mathfrak{a} \cap \mathfrak{h}$. The cone $\overline{\mathcal{C}}/\mathfrak{a}_{\mathfrak{h}} \subseteq \mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}$ serves as a fundamental domain for a finite reflection group W_Z, called the little Weyl group of Z. Attached to the little Weyl group is a root system Σ_Z, called the spherical root system. The faces of the cone $\overline{\mathcal{C}}$ are parameterized by subsets of a simple system Π_Z of Σ_Z, i.e., the sets

$$\mathcal{F}_I := \overline{\mathcal{C}} \cap \bigcap_{\alpha \in I} \ker(\alpha) \qquad (I \subseteq \Pi_Z)$$

are precisely the faces of $\overline{\mathcal{C}}$.

In [28] a smooth G-equivariant compactification $\widehat{\underline{Z}}(\mathbb{R})$ of $\underline{Z}(\mathbb{R})$ was constructed. For every $I \subseteq \Pi_Z$ and X contained in the relative interior of $\mathcal{F}_I$ the limit

$$z_I := \lim_{t \to \infty} \exp(tX)H \in \widehat{\underline{Z}}(\mathbb{R})$$

exists and does not depend on the choice of X. The stabilizer of z_I is a real algebraic subgroup of G, and hence equals the set of real points of an algebraic subgroup $\widehat{\underline{H}}_I$ of

$\underline{G}$ defined over $\mathbb{R}$. We note that $A_I := \exp(\mathrm{span}\mathcal{F}_I)$ is a subgroup of $\widehat{\underline{H}}_I(\mathbb{R})$. We set $\widehat{\underline{Z}}_I := \underline{G}/\widehat{\underline{H}}_I$. Now $\widehat{\underline{Z}}(\mathbb{R})$ admits a stratification in G-manifolds of the form $\widehat{\underline{Z}}_I(\mathbb{R})$ where $I \subseteq \Pi_Z$. In the case where Z admits a wonderful compactification $\widehat{\underline{Z}}$, one has

$$\widehat{\underline{Z}}(\mathbb{R}) = \underline{Z}(\mathbb{R}) \cup \bigcup_{I \subseteq \Pi_Z} \widehat{\underline{Z}}_I(\mathbb{R}).$$

In the general case there is a need to use multiple copies of G-manifolds of the form $\widehat{\underline{Z}}_I(\mathbb{R})$.

We use these spaces to define the boundary degenerations. The group $\widehat{H}_I(\mathbb{R})$ acts on the normal space of $\widehat{\underline{Z}}_I(\mathbb{R})$ at the point z_I. The kernel of this representation on the normal space is a normal real algebraic subgroup of $\widehat{H}_I(\mathbb{R})$, i.e., there exists a normal algebraic subgroup $\underline{H}_I$ of $\widehat{\underline{H}}_I$ so that the kernel of the isotropy action of $\widehat{H}_I(\mathbb{R})$ on the normal space of $\widehat{\underline{Z}}_I(\mathbb{R})$ at z_I is equal to $\underline{H}_I(\mathbb{R})$. The quotient $\widehat{\underline{H}}_I(\mathbb{R})/\underline{H}_I(\mathbb{R})$ is abelian. Its identity component is equal to $A_I/(A_I \cap H)$.

We define the algebraic varieties

$$\underline{Z}_I := \underline{G} \cdot z_I \qquad (I \subseteq \Pi_z).$$

These varieties are called boundary degenerations of $\underline{Z}$. The manifold $\underline{Z}_I(\mathbb{R})$ is a finite union of homogeneous spaces for G, each of which is real spherical and is unimodular. The group A_I acts from the right on $\underline{Z}_I(\mathbb{R})$. The kernel of this action is $A_H := \exp(\mathfrak{a}_{\mathfrak{h}})$.

The right-action of A_I on $\underline{Z}_I(\mathbb{R})$ induces a right-action on $L^2\big(\underline{Z}_I(\mathbb{R})\big)$, which commutes with the left-regular representation of G. The decomposition of $L^2\big(\underline{Z}_I(\mathbb{R})\big)$ with respect to the right-action of A_I yields a disintegration in unitary G-modules

$$L^2\big(\underline{Z}_I(\mathbb{R})\big) = \int_{\rho+i(\mathfrak{a}_I/\mathfrak{a}_{\mathfrak{h}})^*}^{\oplus} L^2\big(\underline{Z}_I(\mathbb{R}), \chi\big)\, d\chi.$$

Here $\rho \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ is an element so that the sections of the line bundle

$$\underline{Z}_I(\mathbb{R}) \times_{A_I} \mathbb{C}_\chi \to \underline{Z}_I(\mathbb{R})/A_I.$$

with $\chi \in \rho + i(\mathfrak{a}_I/\mathfrak{a}_{\mathfrak{h}})^*$ are half-densities, $L^2\big(\underline{Z}_I(\mathbb{R}), \chi\big)$ is the space of square integrable sections of this line bundle and $d\chi$ is the Lebesgue measure on $\rho_Q + i(\mathfrak{a}_I/\mathfrak{a}_{\mathfrak{h}})^*$.

The irreducible subrepresentations of $L^2_{\mathrm{tds}}\big(\underline{Z}_I(\mathbb{R}), \chi\big)$ for any $\chi \in \rho_Q + i(\mathfrak{a}_I/\mathfrak{a}_{\mathfrak{h}})^*$ are called twisted discrete series representations. Let $L^2_{\mathrm{tds}}\big(\underline{Z}_I(\mathbb{R}), \chi\big)$ be the closure of the span of all irreducible subrepresentations of $L^2\big(\underline{Z}_I(\mathbb{R}), \chi\big)$. The spaces $L^2_{\mathrm{tds}}\big(\underline{Z}_I(\mathbb{R}), \chi\big)$ depend measurably on the character χ. We define

$$L^2_{\mathrm{tds}}\big(\underline{Z}_I(\mathbb{R})\big) := \int_{\rho_Q+i(\mathfrak{a}_I/\mathfrak{a}_{\mathfrak{h}})^*}^{\oplus} L^2_{\mathrm{tds}}\big(\underline{Z}_I(\mathbb{R}), \chi\big)\, d\chi.$$

The main result of [17] is the construction of a map

$$B : \bigoplus_{I \subseteq \Pi_Z} L^2_{\mathrm{tds}}\big(\underline{Z}_I(\mathbb{R})\big) \to L^2\big(\underline{Z}(\mathbb{R})\big) \tag{1.1}$$

with the following properties: B is G-equivariant, surjective, isospectral, and for every $I \subseteq \Pi_Z$ the restriction

$$B_I := B|_{L^2_{\mathrm{tds}}\left(\underline{Z}_I(\mathbb{R})\right)} \tag{1.2}$$

is a sum of partial isometries. The existence of such a map goes back to ideas of Bernstein and hence B is called the Bernstein morphism. The Bernstein morphism was first constructed by Sakellaridis and Venkatesh for p-adic spherical spaces in [46].

In the case that G is split and under the assumption of a conjecture on the nature of twisted discrete series representations, Delorme determined the kernel of the Bernstein morphism in [15]. The kernel is described by so-called scattering operators. Even with this description of the kernel of the Bernstein morphism, the decomposition of $L^2(Z)$ remains very abstract. In fact, for general $I \subseteq \Pi_Z$ very little is known about the nature of the twisted discrete series of representations for $\underline{Z}_I(\mathbb{R})$. However, for $I = \emptyset$ the representation belonging to the twisted discrete series for $\underline{Z}_\emptyset(\mathbb{R})$ can be determined explicitly. This allows for an explicit Plancherel decomposition of the subspace

$$L^2_{\mathrm{mc}}(Z) := \mathrm{Im}\left(B_\emptyset\right) \cap L^2(Z).$$

The space $L^2_{\mathrm{mc}}(Z)$ decomposes in the largest continuous families of representations. Therefore, $L^2_{\mathrm{mc}}(Z)$ is called the most continuous part of $L^2(Z)$.

The boundary degeneration $\underline{Z}_\emptyset(\mathbb{R})$ equals a finite union of copies of one real spherical homogeneous space for G which we denote by $Z_\emptyset = G/H_\emptyset$. To be more precise, the copies of $Z_\emptyset$ in $\underline{Z}_\emptyset(\mathbb{R})$ are parameterized by the open P-orbits in $\underline{Z}(\mathbb{R})$.

The local structure theorem of [31] applied to the spherical space Z, provides an adapted parabolic subgroup $Q \subseteq G$ and Langlands decomposition $Q = M_Q A_Q N_Q$ with $A_Q \subseteq A$. Let $\overline{Q} = M_Q A_Q \overline{N}_Q$ be the opposite parabolic. For a reductive symmetric space Q is the minimal σ-parabolic subgroup. Now the space $Z_\emptyset$ can be explicitly described as

$$Z_\emptyset = G/H_\emptyset, \qquad H_\emptyset = (M_Q \cap H)(A \cap H)\overline{N}_Q.$$

In this case $A_\emptyset = A$. The fact that the subgroup $H_\emptyset$ satisfies

$$\overline{N}_Q \subseteq H_\emptyset \subseteq \overline{Q}$$

makes decomposing $L^2(Z_\emptyset)$ into a direct integral of irreducible unitary representation of G easy. Indeed employing induction by stages we obtain

$$L^2(Z_\emptyset) = \mathrm{Ind}_{H_\emptyset}^G(\mathbf{1}) = \mathrm{Ind}_{\overline{Q}}^G\Big(\mathrm{Ind}_{H_\emptyset}^{\overline{Q}}(\mathbf{1})\Big).$$

Moreover,

$$\mathrm{Ind}_{H_\emptyset}^{\overline{Q}}(\mathbf{1}) \simeq L^2\big(M_Q/M_Q \cap H\big)\widehat{\otimes} L^2\big(A/A \cap H\big).$$

The space $L^2\big(M_Q/M_Q \cap H\big)$ decomposes discretely as

$$L^2\big(M_Q/M_Q \cap H\big) \simeq \widehat{\bigoplus_{\xi \in \widehat{M}_Q}} (V_\xi^*)^{M_Q \cap H} \otimes \xi$$

Moreover, the multiplicity space $(V_\xi^*)^{M_Q\cap H}$ can only be non-zero for finite dimensional unitary representations ξ of M_Q. The space $L^2\big(A/A\cap H\big)$ decomposes as

$$L^2\big(A/A\cap H\big)\simeq\int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*}^{\oplus}\mathbb{C}_\lambda\,d\lambda,$$

where $\mathbb{C}_\lambda$ is the 1-dimensional representation of A corresponding to $\lambda\in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ and $d\lambda$ is the Lebesgue measure on $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. It follows that $L^2(Z_\emptyset)$ decomposes as a direct integral of principal series representations $\mathrm{Ind}_Q^G(\xi\otimes\lambda\otimes\mathbf{1})$ with ξ an irreducible finite dimensional unitary representation of M_Q and $\lambda\in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. Two such representations $\mathrm{Ind}_Q^G(\xi\otimes\lambda\otimes\mathbf{1})$ and $\mathrm{Ind}_Q^G(\xi'\otimes\lambda'\otimes\mathbf{1})$ are isomorphic if and only if there exists an element w of the Weyl group so that $\lambda'=w\cdot\lambda$ and $\xi'=w\cdot\xi$. We thus arrive at the decomposition

$$L^2(Z_\emptyset)\simeq\widehat{\bigoplus_{\xi\in\widehat{M}_{Q,\mathrm{fu}}}}\int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+}^{\oplus}\mathcal{M}_{\emptyset,\xi}\otimes\mathrm{Ind}_Q^G(\xi\otimes\lambda\otimes\mathbf{1})\,d\lambda,$$

where $\widehat{M}_{Q,\mathrm{fu}}$ denotes the set of equivalence classes of irreducible finite dimensional unitary representations of M_Q and $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ is a fundamental domain for the stabilizer of $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ in the Weyl group. The space $\mathcal{M}_{\emptyset,\xi}$ is the so-called multiplicity space attached to the representation $\mathrm{Ind}_Q^G(\xi\otimes\lambda\otimes\mathbf{1})$. It turns out to be independent of λ. It follows from this description of the Plancherel decomposition that all irreducible unitary representations occurring in $L^2(Z_\emptyset)$ belong to the twisted discrete series of representation for $Z_\emptyset$. Furthermore, the twisted discrete series for $Z_\emptyset$ consists of the principal series representations of the form $\mathrm{Ind}_Q^G(\xi\otimes\lambda\otimes\mathbf{1})$ with $\lambda\in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ and (ξ,V_ξ) a finite dimensional unitary representation of M_Q.

Invoking the formal properties of the Bernstein maps described above, we obtain a decomposition of $L^2_{\mathrm{mc}}(Z)$ as

$$L^2_{\mathrm{mc}}(Z)\simeq\widehat{\bigoplus_{\xi\in\widehat{M}_{Q,\mathrm{fu}}}}\int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+}^{\oplus}\mathcal{M}_{\xi,\lambda}\otimes\mathrm{Ind}_Q^G(\xi\otimes\lambda\otimes\mathbf{1})\,d\lambda. \tag{1.3}$$

In this article we give a precise description of the Plancherel decomposition of $L^2_{\mathrm{mc}}(Z)$, which amounts to the explicit determination of the multiplicity spaces $\mathcal{M}_{\xi,\lambda}$ with their inner product structure and the Fourier transform that realizes the unitary equivalence (1.3).

The elements of the multiplicity spaces can be interpreted as H-invariant continuous linear functionals on the space of smooth vectors for principal series representations $\mathrm{Ind}_Q^G(\xi\otimes\lambda\otimes\mathbf{1})$. As such, they can be studied as V_ξ^*-valued distributions on Z.

1.2 Main results

To formulate our results concerning the multiplicity spaces for the induced representation $\mathrm{Ind}_Q^G(\xi\otimes\lambda\otimes\mathbf{1})$ and the Plancherel decomposition of $L^2_{\mathrm{mc}}(Z)$, we need some preparation.

For the proofs in the text it is more convenient to work with V_ξ^*-valued distributions rather than functionals. However, for clarity of exposition we state our results here in terms of continuous linear functionals.

More is known about P-orbits in Z than about $\overline{Q}$ or Q-orbits. Therefore, instead of representations induced from $\overline{Q}$, we rather first consider representations induced from the minimal parabolic subgroup P. To describe the connection between the relevant representations induced from $\overline{Q}$ and representations induced from P, we fix a finite dimensional unitary representation (ξ, V_ξ) of M_Q. Such a representation is necessarily trivial on the connected subgroup of M_Q with Lie algebra equal to the sum of all non-compact simple ideals in the Lie algebra of M_Q. Therefore, for $\lambda \in \mathfrak{a}_{Q,\mathbb{C}}^*$ the representation $\mathrm{Ind}_Q^G(\xi\otimes\lambda\otimes\mathbf{1})$ is a subrepresentation of $\mathrm{Ind}_P^G(\xi\big|_M \otimes \lambda + \rho_{P,Q} \otimes \mathbf{1})$, where $\rho_{P,Q}$ is the half sum of all roots of $\mathfrak{a}$ that occur in $P \cap M_Q$. Moreover, for generic $\lambda \in \mathfrak{a}_{Q,\mathbb{C}}^*$ the representations $\mathrm{Ind}_{\overline{Q}}^G(\xi \otimes \lambda \otimes \mathbf{1})$ and $\mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes \mathbf{1})$ are equivalent.

We write $\mathcal{H}_{\overline{Q},\xi,\lambda}$, $\mathcal{H}_{Q,\xi,\lambda}$ and $\mathcal{H}_{P,\xi,\lambda}$ for the spaces of smooth vectors for the representations $\mathrm{Ind}_{\overline{Q}}^G(\xi \otimes \lambda \otimes \mathbf{1})$, $\mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes \mathbf{1})$ and $\mathrm{Ind}_P^G(\xi\big|_M \otimes \lambda + \rho_{P,Q} \otimes \mathbf{1})$, respectively. Now for generic $\lambda \in \mathfrak{a}_{Q,\mathbb{C}}^*$

$$\mathcal{H}_{\overline{Q},\xi,\lambda} \simeq \mathcal{H}_{Q,\xi,\lambda} \subseteq \mathcal{H}_{P,\xi,\lambda}.$$

Our concern is with H-invariant continuous linear functionals on $\mathcal{H}_{\overline{Q},\xi,\lambda} \simeq \mathcal{H}_{Q,\xi,\lambda}$. It is a remarkable fact that every such functional on $\mathcal{H}_{Q,\xi,\lambda}$ is obtained by restricting an H-fixed continuous linear functional on $\mathcal{H}_{P,\xi,\lambda}$. The geometry of the orbits makes it more convenient to first determine the H-fixed continuous functionals on $\mathcal{H}_{P,\xi,\lambda}$ and with that those on the $\mathcal{H}_{\overline{Q},\xi,\lambda}$, rather than considering functionals on $\mathcal{H}_{\overline{Q},\xi,\lambda}$ directly.

The analysis of H-fixed continuous linear functionals on $\mathcal{H}_{P,\xi,\lambda}$ requires a closer study of the P-orbits in Z. We now discuss some aspects of this. For $z \in Z$ we denote by H_z the stabilizer of z in G and by $\mathfrak{h}_z = \mathrm{Lie}(H_z)$ its Lie algebra. For every element $X \in \mathfrak{a}$ the limit

$$\mathfrak{h}_{z,X} := \lim_{t\to\infty} \mathrm{Ad}\left(\exp(tX)\right)\mathfrak{h}_z \tag{1.1}$$

exists in the Grassmannian manifold. Let $\mathcal{O}$ be a P-orbit in Z. The subspace $\mathfrak{a}_{\mathcal{O}} := \mathfrak{a} \cap \mathfrak{h}_{z,X}$ with $z \in \mathcal{O}$ and $X \in \mathfrak{a}^-$ is an invariant of $\mathcal{O}$ as it is independent of the choices of z and X. This allows us to define the rank of $\mathcal{O}$ by

$$\mathrm{rank}(\mathcal{O}) = \dim(\mathfrak{a}/\mathfrak{a}_{\mathcal{O}}).$$

For every open P-orbit $\mathcal{O}$ we have $\mathfrak{a}_{\mathcal{O}} = \mathfrak{a}_{\mathfrak{h}}$. The rank of each open orbit is therefore the same; this is an invariant of Z called the rank of Z. The rank of any P-orbit is bounded by $\mathrm{rank}(Z)$ and an orbit is called of maximal rank if $\mathrm{rank}(\mathcal{O}) = \mathrm{rank}(Z)$. The set of maximal rank orbits is in general strictly larger than the set of open orbits. For example, in the group case every P-orbit is of maximal rank. See Example 3.3. For our purposes the set of maximal rank orbits $\mathcal{O}$ with $\mathfrak{a}_{\mathcal{O}} = \mathfrak{a}_{\mathfrak{h}}$ is of great importance. We denote this set by $(P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$. For many real spherical spaces the set of open P-orbits does not exhaust $(P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$. This is for example the case for $Z = G/\overline{N}_P$ and $Z = \mathrm{SO}(5,\mathbb{C})/\mathrm{GL}(2,\mathbb{C})$.

For any H-fixed continuous linear functional ℓ on $\mathcal{H}_{P,\xi,\lambda}$ one can naturally attach a V_ξ^*-valued distribution μ_ℓ on Z that is left-P equivariant. For such distributions on Z we denote by $(P\backslash Z)_\ell$ the set of P-orbits in $\mathrm{supp}(\mu_\ell)$ that are open in the relative topology of $\operatorname{supp}(\mu_\ell)$. Our first result is a strong restriction on the support of the distributions μ_ℓ when the induction parameter λ is generic. Furthermore, we show that these distributions do not admit transversal derivatives. More precisely, we prove the following:

Theorem A (Theorems 5.4, 5.2 & 6.4)**.** *There exists a finite union $\mathcal{S}$ of hyperplanes in $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ so that for $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ with $\operatorname{Im}\lambda \notin \mathcal{S}$ the H-fixed continuous linear functionals ℓ on $\mathcal{H}_{P,\xi,\lambda}$ satisfy the following.*

(i) Only maximal rank orbits with $\mathfrak{a}_\mathcal{O} = \mathfrak{a}_\mathfrak{h}$ contribute to $(P\backslash Z)_\ell$, i.e.,

$$(P\backslash Z)_\ell \subseteq (P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}.$$

(ii) For each orbit $\mathcal{O} \in (P\backslash Z)_\ell$ there exists a representative $x_\mathcal{O} \in G$ and an $\eta_\mathcal{O} \in (V_\xi^*)^{M_Q \cap x_\mathcal{O} H x_\mathcal{O}^{-1}}$ so that for every $f \in \mathcal{H}_{P,\xi,\lambda}$ with

$$\operatorname{supp}(f) \cap \bigcup_{\mathcal{O}\in(P\backslash Z)_\mu} \partial\mathcal{O} = \emptyset$$

we have the formula

$$\ell(f) = \sum_{\mathcal{O}\in(P\backslash Z)_\ell} \int_{(x_\mathcal{O}^{-1}Px_\mathcal{O}\cap H)\backslash H} \big\langle \eta_\mathcal{O}, f(x_\mathcal{O}h)\big\rangle dh,$$

where dh denotes an H-invariant Radon measure on $(x_\mathcal{O}^{-1}Px_\mathcal{O} \cap H)\backslash H$. In particular, the distribution μ_ℓ attached to the functional ℓ does not admit any transversal derivatives.

(iii) Every non-zero H-fixed continuous linear functional ℓ on $\mathcal{H}_{P,\xi,\lambda}$ restricts to a non-zero H-fixed continuous linear functional on $\mathcal{H}_{Q,\xi,\lambda}$. In fact, the restriction map

$$\operatorname{Hom}_H\big(\mathcal{H}_{P,\xi,\lambda}, \mathbb{C}\big) \to \operatorname{Hom}_H\big(\mathcal{H}_{Q,\xi,\lambda}, \mathbb{C}\big)$$

is an isomorphism.

The next result concerns the actual construction of H-invariant continuous functionals attached to maximal rank orbits. First, for each $\mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}$ we carefully choose a representative $x_\mathcal{O} \in G$, see Section 6.4. Given an orbit $\mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}$ there exists a shifted open cone $\Gamma_\mathcal{O} \subseteq (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ so that for all $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ with $\operatorname{Re}\lambda + \rho_{P,Q} \in \Gamma_\mathcal{O}$ the integrals

$$\ell_{\xi,\lambda,\eta}(f) := \int_{(x_\mathcal{O}^{-1}Px_\mathcal{O}\cap H)\backslash H} \big\langle \eta, f(x_\mathcal{O}h)\big\rangle dh \tag{1.2}$$

are absolutely convergent for all $\eta \in (V_\xi^*)^{M_Q \cap x_\mathcal{O} H x_\mathcal{O}^{-1}}$ and $f \in \mathcal{H}_{P,\xi,\lambda}$. Moreover, when viewed as V_ξ^*-valued distributions on Z, each family $\lambda \mapsto \ell_{\xi,\lambda,\eta}$ extends to a meromorphic family with parameter $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$. We set

$$V^*(\xi) := \bigoplus_{\mathcal{O}\in(P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}} (V_\xi^*)^{M_Q \cap x_\mathcal{O} H x_\mathcal{O}^{-1}}.$$

Note that $V^*(\xi)$ is finite dimensional. We thus obtain a map

$$V^*(\xi) \to \mathrm{Hom}_H\big(\mathcal{H}_{P,\xi,\lambda}, \mathbb{C}\big); \quad \eta \mapsto \ell_{\xi,\lambda,\eta}$$

with meromorphic dependence on $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$. After suitably normalizing these functionals using amongst other things the long intertwining operator, see (6.1), we arrive at a map

$$\ell^\circ_{\xi,\lambda} : V^*(\xi) \to \mathrm{Hom}_H\big(\mathcal{H}_{\overline{Q},\xi,\lambda}, \mathbb{C}\big),$$

which is an isomorphism for generic λ. More precisely, the following hold.

Theorem B (Theorem 6.1 & Corollary 8.1)**.**

(i) For every $\eta \in V^*(\xi)$ the map $\lambda \mapsto \ell^\circ_{\xi,\lambda}(\eta)$, considered as a family of V^*_ξ-valued distributions on Z, is meromorphic on $(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$.

(ii) For every $\eta \in V^*(\xi)$ the map $\lambda \mapsto \ell^\circ_{\xi,\lambda}(\eta)$ is holomorphic on an open neighborhood of $i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$.

(iii) There exists a finite union $\mathcal{S}$ of proper subspaces of $(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ so that $\ell^\circ_{\xi,\lambda}$ is an isomorphism for $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$ with $\mathrm{Im}\,\lambda \notin \mathcal{S}$.

We now come to the determination of the multiplicity spaces. Each multiplicity space $\mathcal{M}_{\xi,\lambda}$ is naturally identified with a subspace of $\mathrm{Hom}_H(\mathcal{H}_{\overline{Q},\xi,\lambda}, \mathbb{C})$. However, an H-fixed continuous linear functional ℓ on $\mathcal{H}_{\overline{Q},\xi,\lambda}$ can only be contained in $\mathcal{M}_{\xi,\lambda}$ if the generalized matrix coefficients with ℓ are almost contained in $L^2(Z)$. To be more precise, a functional ℓ can only contribute if it is tempered, i.e., if all generalized matrix coefficients with ℓ define tempered functions on Z.

Theorem C (Theorem 7.2 & Theorem 7.1 and its Corollary 8.2)**.** *For $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ outside of a finite union of proper subspaces of $i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ every H-fixed continuous linear functional on $\mathcal{H}_{\overline{Q},\xi,\lambda}$ is tempered. In fact, for almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ we have*

$$\mathcal{M}_{\xi,\lambda} = \mathrm{Hom}_H\big(\mathcal{H}_{\overline{Q},\xi,\lambda}, \mathbb{C}\big).$$

To state the main result of the article, Theorem 8.1, we define the Fourier transform for a smooth function ϕ with compact support on Z

$$\mathscr{F}(\phi)(\xi,\lambda) \in V^*(\xi) \otimes \mathcal{H}_{\overline{Q},\xi,\lambda} \simeq \mathrm{Hom}_{\mathbb{C}}\big(V^*(\xi^\vee), \mathcal{H}_{\overline{Q},\xi,\lambda}\big)$$

by

$$\mathscr{F}(\phi)(\xi,\lambda)\eta := \int_Z \phi(gH)\big(g \cdot \ell^\circ_{\xi^\vee,-\lambda}\big)(\eta)\, dgH \qquad \big(\eta \in V^*(\xi^\vee)\big).$$

On $V^*(\xi)$ there is a natural inner product induced by the inner product on V_ξ. We normalize this inner product by a factor of $\dim(V_\xi)$.

Theorem D (Theorem 8.1)**.** *Let $i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_+$ be a fundamental domain for the stabilizer of $i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ in the Weyl group. Then the Fourier transform $f \mapsto \mathscr{F} f$ extends to a unitary isomorphism*

$$L^2_{\mathrm{mc}}(Z) \to \widehat{\bigoplus_{\xi \in \widehat{M}_{Q,\mathrm{fu}}}} \int^\oplus_{i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_+} V^*(\xi) \otimes \mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes \mathbf{1})\, d\lambda.$$

As mentioned before, the representations $\operatorname{Ind}_Q^G(\xi \otimes \lambda \otimes \mathbf{1})$ are irreducible for generic $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. Moreover, if $\xi, \xi' \in \widehat{M_{Q,\mathrm{fu}}}$ and $\lambda, \lambda' \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ are generic the representations $\operatorname{Ind}_Q^G(\xi \otimes \lambda \otimes \mathbf{1})$ and $\operatorname{Ind}_Q^G(\xi' \otimes \lambda' \otimes \mathbf{1})$ are not equivalent if $(\xi, \lambda) \neq (\xi', \lambda')$. Therefore, the above decomposition of $L^2_{\mathrm{mc}}(Z)$ is the Plancherel decomposition.

1.3 Methods of Proof and structure of the article

After setting up our notation in Section 2, we begin in Section 3 with the study of P-orbits in Z. There are two main results. The first is Theorem 3.2, which is a structure theorem for maximal rank orbits. It is a generalization of a structure theorem of Brion, [11, Proposition 6 & Theorem 3], for complex spherical spaces. Theorem 3.2 is of crucial importance for our construction of H-fixed distributions. The second main result in Section 3 is Theorem 3.3. We define an equivalence relation on the P-orbits of maximal rank. We then show that the Weyl group W of the root system of $\mathfrak{a}$ in $\mathfrak{g}$ naturally acts on the set of equivalence classes. This action is transitive. The set of open orbits forms one equivalence class; its stabilizer is the little Weyl group W_Z. This result was first obtained by Knop in [27] for complex spaces and by Knop and Zhgoon in [32] for spherical spaces defined over a field of characteristic 0. Their results are more general than ours, but our description of the action of W is tailor made for the way we apply it. The W-action is applied at several places, most notably for the precise choice of the representatives $x_\mathcal{O}$ for the P-orbits in $(P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}$. Our approach to P-orbits on Z differs substantially from the techniques used by Brion, Knop and Zhgoon. The main tool for our considerations is the limit subalgebra $\mathfrak{h}_{z,X}$ from (1.1). Previously we used an analysis of these limit subalgebras to give a construction of the little Weyl group in [39]. We heavily rely on the results from that article for the two main results in Section 3.

In Section 4 we set up a dictionary between invariant functionals on the smooth vectors of a principal series representation $\operatorname{Ind}_S^G(\xi\otimes\lambda\otimes\mathbf{1})$ induced from a parabolic subgroup S and a space $\mathcal{D}'(S : \xi : \lambda)$ of S-equivariant V_ξ^*-valued distributions. Even though the exposition of the results is easier in the language of functionals, as in Section 1.2, we find it easier to work with distributions and that is what we chose to do throughout the article. In Section 4.3 we describe the action of the intertwining operators on those distributions as we were not able to locate such a description in the literature. The proofs for these results are relegated to Appendix B. As was explained in Section 1.2, it is easier to work with the minimal parabolic subgroup P, rather than directly with $\overline{Q}$ or Q. To facilitate this, we make a comparison between inductions from different parabolic subgroups, when the induction data allows, in Sections 4.4 – 4.6.

In Section 5 we prove the first two assertions in Theorem A. For reductive symmetric spaces the restrictions on the support and transversal derivatives of H-fixed distributions in $\mathcal{D}'(P : \xi : \lambda)$ are obtained using Bruhat's theory which he developed in his thesis. See [1, Theorem 5.1] and [14, Théorème 1 in Section 3.3]. This approach relies heavily on precise knowledge of all P-orbits in Z. For reductive symmetric spaces the P-orbits are very well understood; a complete description of the P-orbits has been given by Matsuki in [40] and [41]. Unfortunately, for real spherical spaces such a description is not available. We therefore resort to a different method, namely principal asymptotics, which is a technical tool from [35, Theorem 5.1]. The method of principal asymptotics can be considered as the analogue of the limit subalgebras (1.1) for H-fixed distributions in $\mathcal{D}'(P : \xi : \lambda)$. Given such a distribution μ, an orbit $\mathcal{O} \in (P\backslash Z)_\mu$, a point $z \in \mathcal{O}$ and a sufficiently regular $X \in \mathfrak{a}^-$, the principal asymptotics of μ is a distribution $\mu_{z,X}$ defined on a left-P invariant open neighborhood of e in G that is left P-equivariant and right invariant under the limit subalgebra $\mathfrak{h}_{z,X}$. These last distributions are easier to analyse. An immediate corollary is that the imaginary part of λ must vanish on $\mathfrak{a}_\mathcal{O}$, which implies the first assertion in Theorem A, see Theorem 5.4. Moreover, μ has transversal derivatives on $\mathcal{O}$ if and only if $\mu_{z,X}$ has transversal derivatives. The proof for the second assertion is now essentially reduced to the case of $\overline{N}_P$-invariant distributions in $\mathcal{D}'(P : \xi : \lambda)$. For the latter distributions the absence of transversal derivatives for generic λ is proved by an analysis of the action of the center of $\mathcal{U}(\mathfrak{g})$ in Theorem 5.2.

In Section 6 we construct the H-invariant distributions in $\mathcal{D}'(P : \xi : \lambda)$. By considering powers of matrix coefficients of finite dimensional H-spherical representations, one easily sees that the integrals (1.2) are absolutely convergent if $\mathrm{Re}\,\lambda$ is in a certain shifted cone. We thus find holomorphic families of H-fixed distributions with family parameter λ. We then use the technique of Bernstein and Sato to extend these families to meromorphic families. This method is well known; it was for example used before by Olafsson in [42, Theorem 5.1], Brylinski and Delorme [13, Proposition 4] and Frahm [20, Theorem 3.3].

For symmetric spaces there are other ways to obtain the meromorphic extension. We mention here two methods of Van den Ban: [1, Theorem 5.10] using intertwining operators and [2, Theorem 9.1] using translation functors. The second method of Van den Ban is arguably the best since it provides a rather explicit functional equation. See also [14, Théorème 2] where this method was used by Carmona and Delorme. In our setting neither the method based on intertwining operators, nor the method based on translation functors is straightforwardly applicable since both require that the only orbits contributing in Theorem A (i) are the open orbits. For real spherical spaces with the wavefront property, e.g. reductive symmetric spaces, only the open orbits contribute. We give a short proof of this in Appendix A.

To construct the H-invariant distributions in $\mathcal{D}'(P : \xi : \lambda)$ on non-open P-orbits of maximal rank we use an idea from [4, Theorem 7.1]. We mention here that a similar construction for p-adic spherical spaces was done by Sakellaridis in [45, Section 4]. The applicability of the idea heavily relies on the structure of maximal rank orbits. For reductive symmetric spaces this is readily obtained from the rich structure theory that exists for these spaces. For real spherical spaces the necessary assertions were proven using our methods concerning limits subalgebras in Theorem 3.2. Every maximal rank orbit $\mathcal{O}$ is contained in an open P'-orbit $\mathcal{O}'$ for a certain minimal parabolic subgroup P'. Moreover, $\mathcal{O}'$ decomposes as a family of orbits of a unipotent subgroup of P' parameterized by the points in $\mathcal{O}$. This geometric decomposition translates on the level of distributions to a decomposition of the distributions we constructed before on open orbits into the application of a standard intertwining operator on a distribution supported on $\overline{\mathcal{O}}$. The outcome of this analysis is a construction of H-invariant distributions μ in $\mathcal{D}'(P : \xi : \lambda)$ with $(P\backslash Z)_\mu = \{\mathcal{O}\}$ by applying the inverse of a standard intertwining operator to a H-invariant distribution in $\mathcal{D}'(P' : \xi : \lambda)$ constructed on an open orbit. As a corollary we find that the H-invariant distributions in $\mathcal{D}'(P:\xi:\lambda)$ fit into meromorphic families with family parameter λ. All this is described in Proposition 6.2. With the rather explicit formulas for the distributions we obtain from Proposition 6.2 it is then shown that the distributions constructed on P-orbits in $(P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}$ actually are Q-equivariant, which establishes assertion (iii) in Theorem A. See Theorem 6.4.

By combining Theorem 5.4, Theorem 5.2 and Theorem 6.4 we obtain in Theorem 6.3 a full description of $\mathcal{D}'(Q:\xi:\lambda)^H$ for generic $\lambda\in(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$. The remainder of Section 6 is devoted to a description of the action of the normalizer of H in A on $\mathcal{D}'(Q:\xi:\lambda)$ and a proper normalization of the families of distributions we constructed. The latter results in the assertions (i) and (iii) in Theorem B.

In section 7 we prove Theorem C. There are two main results. The first is Theorem 7.2, which asserts that for generic $\lambda\in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ all H-fixed distributions in $\mathcal{D}'(\overline{Q}:\xi:\lambda)$ are tempered. The proof begins with an a priori estimate, which is then improved in a recursive process. For reductive symmetric spaces this was done by Van den Ban in [2, Section 18] using a technique of Wallach from [49, Theorem 4.3.5]. For real spherical spaces this method is not easily applicable. This is due to the lack of a good polar decomposition. Instead we adapt the techniques developed in [18] for the construction of the constant term map. In [18] only tempered eigenfunctions are considered. However the techniques can be applied to non-tempered eigenfunctions as well and then used to improve estimates and prove temperedness.

Once we have established the temperedness of the distributions, we move on to the second main result in section 7: the square integrability of wave packets in Theorem 7.1. The proof is similar to the analogous result for reductive symmetric spaces by Van den Ban, Carmona and Delorme in [3]. Also this result relies heavily on the constant term map. An important consequence, Corollary 8.2, is that for almost every $\lambda\in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ the multiplicity space $\mathcal{M}_{\xi,\lambda}$ is identical to $\mathcal{D}'(\overline{Q}:\xi:\lambda)^H$, and hence can be identified with $V^*(\xi)$ in view of Theorem B (iii).

In Section 8 we prove the Plancherel decomposition of $L^2_{\mathrm{mc}}(Z)$. The abstract Plancherel decomposition provides $V^*(\xi)$ for almost every $\lambda\in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ with an inner product. To prove Theorem D it remains to show that this inner product is independent of λ and up to a factor of $\dim(V_\xi)$ equal to the inner product induced from the one on V_ξ. This we do in Section 8. We first prove the required identity for the space $Z=Z_\emptyset$ by a direct computation in Theorem 8.1. The result for $Z_\emptyset$ can in view of the Maaß-Selberg relations [17, Theorem 9.6] be used to determine the inner products for Z itself. In order to apply the Maaß-Selberg relations we have to determine the constant terms of all distributions. We give explicit formulas in Proposition 8.1 and 8.2. If Z has the wavefront property, then the Maaß-Selberg relations from [17, Theorem 9.6] suffice to determine the inner product on $V^*(\xi)$. For the group case, and more generally for reductive symmetric spaces, this was done in [17, Sections 14 & 15]. For general real spherical spaces a refinement of the Maaß-Selberg relations is needed. This refinement is obtained in Corollaries 8.2 and 8.1. For the proof we construct suitable G-invariant differential operators on Z using Knop's Harish-Chandra homomorphism from [26]. We then determine the Plancherel decomposition of $L^2_{\mathrm{mc}}(Z)$ in Theorem 8.1.

Assertion (ii) in Theorem A is an easy corollary to Theorem 8.1. For reductive symmetric spaces this was proven in [6, Theorem 1]. Finally, we provide in Corollary 8.4 an explicit form for the so-called scattering operators introduced by [15] in the case G is a split real reductive group. Our formulas are written in terms of the standard intertwining operators acting on H-fixed linear functionals.

1.4 Acknowledgments

We thank Bernhard Krötz, Henrik Schlichtkrull and Erik van den Ban for various discussions on the subject matter of this paper. This article was written over several years. While working on the sections 3 to 6 the first author was funded by the Deutsche Forschungsgemeinschaft Grant 262362164.

2 Setup and notation

Groups are indicated by capital roman letters. Their Lie algebras are denoted by the corresponding lower-case fraktur letter.

Let $\underline{G}$ be a connected reductive algebraic group defined over $\mathbb{R}$ and set $G := \underline{G}(\mathbb{R})$. Let $\underline{H}$ be an algebraic subgroup of $\underline{G}$ defined over $\mathbb{R}$ and set $H := \underline{H}(\mathbb{R})$. We write $Z = G/H$. If $z \in Z$, then the stabilizer subgroup of Z is indicated by H_z and its Lie algebra by $\mathfrak{h}_z$.

We set $G_\mathbb{C} := \underline{G}(\mathbb{C})$ and $H_\mathbb{C} := \underline{H}(\mathbb{C})$. If E is a real vector space, then we write $E_\mathbb{C}$ for its complexification.

Throughout the article we fix a minimal parabolic subgroup P of G and a Langlands decomposition $P = MAN$. We assume that Z is real spherical, i.e., there exists an open P-orbit in Z. We further assume that Z is unimodular. In view of [17, Lemma 12.7] the space Z is quasi-affine.

Let θ be a Cartan involution of G so that A is θ-stable. We denote the corresponding involution on the Lie algebra $\mathfrak{g}$ by θ as well and write K for the fixed point subgroup of θ. Note that K is a maximal compact subgroup of G.

If Q is a parabolic subgroup of G, then we write N_Q for the unipotent radical of Q and $\overline{N}_Q$ for the unipotent radical θN_Q of the opposite parabolic subgroup θQ.

We write Σ for the root system of $\mathfrak{a}$ in $\mathfrak{g}$. If Q is a parabolic subgroup containing A, then we define $\Sigma(Q)$ to be subset of Σ of roots α so that the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{n}_Q$. We define ρ_Q to be the element of $\mathfrak{a}^*$ given by

$$\rho_Q(X) = \frac{1}{2} \operatorname{tr} \operatorname{ad}(X)\big|_{\mathfrak{n}_Q}.$$

Further, we write $\mathfrak{a}^-$ for the open negative Weyl chamber with respect to $\Sigma(P)$.

We fix an $\operatorname{Ad}(G)$-invariant bilinear form B on $\mathfrak{g}$ so that $-B(\,\cdot\,, \theta\,\cdot\,)$ is positive definite. For $E \subseteq \mathfrak{g}$, we define

$$E^\perp = \big\{X \in \mathfrak{g} : B(X, E) = \{0\}\big\}.$$

For the notation for function spaces we follow the book of Schwartz [48]. In particular, spaces of compactly supported smooth, smooth and Schwartz functions are denoted by $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{S}$ respectively. Their strong duals are as usual indicated by a $'$.

If N is a connected and simply connected subgroup of G so that its Lie algebra $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$, then we equip N with the Haar-measure dn given by the pull-back of the Lebesgue measure on $\mathfrak{n}$ along the exponential map. This we do in particular for the group A and the unipotent radicals N_Q of parabolic subgroups Q. Every compact subgroup we equip with the normalized Haar measure. We do this in particular for the groups K and M. We normalize the Haar measure dg on G so that

$$\int_G \phi(g)\,dg = \int_K \int_A \int_{N_P} a^{2\rho_P} \phi(kan)\,dn\,da\,dk \qquad \big(\phi \in \mathcal{D}(G)\big).$$

In view of the Local structure theorem, see Proposition 3.1, there exists a parabolic subgroup Q containing A and a point $z \in Z$, so that $P \cdot z$ is open and

$$N_Q \times M/(M \cap H_z) \times A/(A \cap H_z) \to P \cdot z; \quad (n, m, a) \mapsto nma \cdot z$$

is a diffeomorphism. Let $\mathfrak{a}_0 = \mathfrak{a} \cap (\mathfrak{a} \cap \mathfrak{h}_z)^\perp$ and $A_0 = \exp(\mathfrak{a}_0)$. Then the group MA_0N_Q is unimodular. We normalize the invariant Radon measure on Z by

$$\int_Z \phi(z)\,dz = \int_{N_Q} \int_M \int_{A_0} \phi(nma \cdot z)\,da\,dm\,dn \qquad \big(\phi \in \mathcal{D}(Z)\big).$$

The Haar measure on H we normalize by requiring that

$$\int_G \phi(g)\,dg = \int_Z \int_H \phi(gh)\,dh\,dgH \qquad \big(\phi \in \mathcal{D}(G)\big).$$

Finally, we normalize the Lebesgue measure $i(\mathfrak{a}/\mathfrak{a} \cap \mathfrak{h}_z)^*$ so that

$$\phi(e) = \int_{i(\mathfrak{a}/\mathfrak{a}\cap\mathfrak{h}_z)^*} \int_{A/(A \cap H_z)} \phi(a) a^\lambda\,da\,d\lambda \qquad \Big(\phi \in \mathcal{D}\big(A/(A \cap H_z)\big)\Big).$$

3 P-Orbits of maximal rank

3.1 The local structure theorem

In this section we give a reformulation of the local structure theorem, which follows from [31, Theorem 2.3] and its constructive proof.

Proposition 3.1. *There exists a parabolic subgroup Q with $P \subseteq Q$, a Levi decomposition $Q = L_Q N_Q$ with $A \subseteq L_Q$, and for every open P-orbit $\mathcal{O}$ in Z a point $z \in \mathcal{O}$ so that the following assertions hold.*

(i) $Q \cdot \mathcal{O} = \mathcal{O}$.

(ii) $Q \cap H_z = L_Q \cap H_z$.

(iii) The map

$$N_Q \times L_Q/L_Q \cap H_z \to Z; \quad \big(n, l(L_Q \cap H_z)\big) \mapsto nl \cdot z$$

is a diffeomorphism onto $\mathcal{O}$.

(iv) *The sum* $\mathfrak{l}_{Q,\mathrm{nc}}$ *of all non-compact simple ideals in* $\mathfrak{l}_Q$ *is contained in* $\mathfrak{h}_z$.

(v) *There exists an* $X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp$ *so that* $L_Q = Z_G(X)$ *and* $\alpha(X) > 0$ *for all* $\alpha \in \Sigma(Q)$.

Remark 3.2. The existence of an $X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp$ with $L_Q = Z_G(X)$ has the following consequence. Let $\alpha \in \Sigma$. Then $\mathfrak{g}_\alpha \subseteq \mathfrak{l}_Q$ if and only if $\alpha^\vee \in \mathfrak{a} \cap \mathfrak{h}_z$.

3.2 Adapted points

We now recall the notion of adapted points and some relevant results from [39].

Following [39] we say that a point $z \in Z$ is adapted (to the Langlands decomposition $P = MAN$) if the following two conditions are satisfied.

(i) $P \cdot z$ is open in Z, i.e., $\mathfrak{p} + \mathfrak{h}_z = \mathfrak{g}$,

(ii) There exists an $X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp$ so that $Z_\mathfrak{g}(X) = \mathfrak{l}_Q$.

See Definition 3.3 and Remark 3.4 (b) in [39]. It follows from Proposition 3.1 that every open P-orbit in Z contains an adapted point. Moreover, the set of adapted points is MA-stable.

The Lie subalgebra $\mathfrak{a} \cap \mathfrak{h}_z$ is the same for all adapted points z by [39, Corollary 3.17]. We denote this subalgebra by $\mathfrak{a}_\mathfrak{h}$ and refer to the dimension of $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$ as the rank of Z.

Adapted points have several of the properties that are listed in the local structure theorem, Proposition 3.1. The following proposition is a combination of Proposition 3.6 and Remark 3.7 (b) in [39].

Proposition 3.1. *Let* $z \in Z$ *be adapted. Then the following hold.*

(i) $Q \cap H_z = L_Q \cap H_z$,

(ii) $\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathfrak{h}_z$

(iii) *The map*

$$M/(M \cap H_z) \times \mathfrak{a}/\mathfrak{a}_\mathfrak{h} \to L_Q/(L_Q \cap H_z);$$
$$\big(m(M \cap H_z), X + \mathfrak{a}_\mathfrak{h}\big) \mapsto m\exp(X)(L_Q \cap H_z)$$

is a diffeomorphism.

(iv) *The map*

$$N_Q \times L_Q/(L_Q \cap H_z) \to Z; \quad \big(n, l(L_Q \cap H_z)\big) \mapsto nl \cdot z$$

is a diffeomorphism onto $P \cdot z$.

The adapted points in a given open P-orbit are up to MA-translation parameterized by Q-regular elements in $\mathfrak{a} \cap \mathfrak{a}_\mathfrak{h}^\perp$, i.e., by elements $X \in \mathfrak{a} \cap \mathfrak{a}_\mathfrak{h}^\perp$ so that $Z_\mathfrak{g}(X) = \mathfrak{l}_Q$. The following proposition follows directly from [39, Proposition 3.12].

Proposition 3.2. *Let* $\mathcal{O}$ *be an open* P*-orbit in* Z*. Let* $X \in \mathfrak{a} \cap \mathfrak{a}_\mathfrak{h}^\perp$*. If* $Z_\mathfrak{g}(X) = \mathfrak{l}_Q$*, then there exists an adapted point* $z \in \mathcal{O}$ *so that* $X \in \mathfrak{h}_z^\perp$*. Moreover, if* $z' \in \mathcal{O}$ *is another adapted point so that* $X \in \mathfrak{h}_{z'}^\perp$*, then there exist* $m \in M$ *and* $a \in A$ *so that* $z' = ma \cdot z$.

3.3 Limits of subspaces

In this section we discuss limits of subspaces of $\mathfrak{g}$ in the Grassmannian and their properties.

For $k \in \mathbb{N}$ let $\mathrm{Gr}(\mathfrak{g}, k)$ be the Grassmannian of k-dimensional subspaces of the Lie algebra $\mathfrak{g}$.

We say that an element $X \in \mathfrak{a}$ is *order-regular* if

$$\alpha(X) \neq \beta(X)$$

for all $\alpha, \beta \in \Sigma$ with $\alpha \neq \beta$.

If $X \in \mathfrak{a}$ is order-regular, then in particular $\alpha(X) \neq -\alpha(X)$ and therefore $\alpha(X) \neq 0$ for every $\alpha \in \Sigma$. This implies that order-regular elements in $\mathfrak{a}$ are regular. The name order-regular refers to the fact that every order-regular element $X \in \mathfrak{a}$ determines a linear order $\geq$ on Σ by setting

$$\alpha \geq \beta \quad \text{if and only if} \quad \alpha(X) \geq \beta(X)$$

for $\alpha, \beta \in \Sigma$.

The following proposition is taken from [35, Lemma 4.1] and [39, Proposition 5.2].

Proposition 3.1. *Let $E \in \mathrm{Gr}(\mathfrak{g}, k)$ and let $X \in \mathfrak{a}$. The limit*

$$E_X := \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)E,$$

exists in the Grassmannian $\mathrm{Gr}(\mathfrak{g}, k)$. If $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ are the eigenvalues and $p_1, \ldots, p_n$ the corresponding projections onto the eigenspaces V_i of $\mathrm{ad}(X)$, then E_X is given by

$$E_X = \bigoplus_{i=1}^{n} p_i\Big(E \cap \bigoplus_{j=1}^{i} V_j\Big). \tag{3.1}$$

The following hold.

(i) If E is a Lie subalgebra of $\mathfrak{g}$, then E_X is a Lie subalgebra of $\mathfrak{g}$.

(ii) If $X \in \mathfrak{a}$ is order-regular, then E_X is $\mathfrak{a}$-stable.

(iii) Let $\mathcal{R} \subseteq \mathfrak{a}$ be a connected component of the set of order-regular elements in $\mathfrak{a}$. If $X \in \overline{\mathcal{R}}$ and $Y \in \mathcal{R}$, then $\big(E_X\big)_Y = E_Y$. In particular, if $X, Y \in \mathcal{R}$, then $E_X = E_Y$.

(iv) If $g, g' \in G$ and

$$\lim_{t\to\infty} \exp(tX) g \exp(-tX) = g',$$

then

$$\big(\mathrm{Ad}(g)E\big)_X = \mathrm{Ad}(g')E_X$$

(v) Let $E_{\mathbb{C},X}$ be the limit of $\mathrm{Ad}\big(\exp(tX)\big)E_{\mathbb{C}}$ for $t \to \infty$ in the Grassmannian of k-dimensional complex subspaces in the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Then

$$E_{\mathbb{C},X} = (E_X)_{\mathbb{C}}.$$

We note that if X is not order-regular, then E_X need not be stable under the action of $\mathfrak{a}$, even if X is regular.

For $z \in Z$ and $X \in \mathfrak{a}$ we define

$$\mathfrak{h}_{z,X} := (\mathfrak{h}_z)_X.$$

3.4 Compression cone

We may and will assume that the point $eH \in G/H = Z$ is adapted. We define

$$\mathfrak{h}_\emptyset := (\mathfrak{l}_Q \cap \mathfrak{h}) \oplus \overline{\mathfrak{n}}_Q.$$

For $z \in Z$, we define the cone

$$\mathcal{C}_z := \{X \in \mathfrak{a} : \mathfrak{h}_{z,X} = \mathrm{Ad}(m)\mathfrak{h}_\emptyset \text{ for some } m \in M\}.$$

By [39, Proposition 6.5] the cones $\mathcal{C}_z$ are the same for all adapted points $z \in Z$. We therefore may define

$$\mathcal{C} := \mathcal{C}_z,$$

where z is any adapted point in Z. We call $\mathcal{C}$ the compression cone of Z.

In the following proposition we list some of the properties of the compression cone from [39, Section 6].

Proposition 3.1.

(i) Let $z \in Z$. If $P \cdot z$ is not open, then $\mathcal{C}_z = \emptyset$. If $P \cdot z$ is open, then $\mathfrak{a}^- \subseteq \mathcal{C}_z \subseteq \mathcal{C}$.

(ii) $\mathcal{C} = \mathcal{C} + \mathfrak{a}_\mathfrak{h}$.

(iii) $\overline{\mathcal{C}}$ is a finitely generated cone.

The edge of $\overline{\mathcal{C}}$ we denote by $\mathfrak{a}_E$, i.e.,

$$\mathfrak{a}_E := \overline{\mathcal{C}} \cap -\overline{\mathcal{C}}. \tag{3.1}$$

Note that $\mathfrak{a}_\mathfrak{h} \subseteq \mathfrak{a}_E$, but that in general $\mathfrak{a}_E$ may be strictly larger. We recall from [29, Section 6] that Z is called wavefront if

$$\mathcal{C} = \mathfrak{a}^- + \mathfrak{a}_\mathfrak{h}.$$

For wavefront spaces Z we have $\mathfrak{a}_E = \mathfrak{a}_\mathfrak{h}$. All reductive symmetric spaces are wavefront, i.e., all spaces G/H with H an open subgroup of the fixed point subgroup of an involutive automorphism of G.

3.5 Rank of a P-orbit

In this section we define the rank of a P-orbit in Z. We begin with a lemma.

Lemma 3.1. *Let $z \in Z$. The set $\mathfrak{a} \cap \mathfrak{h}_{p\cdot z,X}$ is the same for all $p \in P$ and $X \in \mathfrak{a}^-$.*

Proof. Let $X \in \mathfrak{a}^-$ and $p \in P$. Further, let $p_\mathfrak{a} : \mathfrak{p} \to \mathfrak{a}$ be the projection along the decomposition $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_P$. In view of (3.1)

$$\mathfrak{a} \cap \mathfrak{h}_{p\cdot z,X} = p_\mathfrak{a}\big(\mathfrak{h}_{p\cdot z} \cap \mathfrak{p}\big).$$

Note that $p_\mathfrak{a}$ is invariant under the adjoint action of P on $\mathfrak{p}$. As

$$\mathfrak{h}_{p\cdot z} \cap \mathfrak{p} = \mathrm{Ad}(p)\mathfrak{h}_z \cap \mathfrak{p} = \mathrm{Ad}(p)\big(\mathfrak{h}_z \cap \mathfrak{p}\big),$$

it follows that

$$\mathfrak{a} \cap \mathfrak{h}_{p\cdot z,X} = p_\mathfrak{a}\big(\mathfrak{h}_z \cap \mathfrak{p}\big).$$

The right-hand side is independent of p and X. □

Let $\mathcal{O}$ be a P-orbit in Z. Lemma 3.1 allows us to define the set

$$\mathfrak{a}_\mathcal{O} := \mathfrak{a} \cap \mathfrak{h}_{z,X}$$

where z is any point $\mathcal{O}$ and X is any element in $\mathfrak{a}^-$. We call the dimension of $\mathfrak{a}/\mathfrak{a}_\mathcal{O}$ the rank of the orbit $\mathcal{O}$.

Remark 3.2. If $\mathcal{O}$ is an open P-orbit, then it follows from Proposition 3.1 (ii) that

$$\mathfrak{a}_\mathcal{O} = \mathfrak{a}_\mathfrak{h}.$$

3.6 P-Orbits of maximal rank

The main result in this section is the following proposition, which will be crucial in this article.

Proposition 3.1. *Let $\mathcal{O} \in P\backslash Z$. Then*

$$\mathrm{rank}(\mathcal{O}) \leq \mathrm{rank}(Z).$$

Let $X \in \mathfrak{a}^-$ be order-regular and let $z \in \mathcal{O}$. Then $\mathrm{rank}(\mathcal{O}) = \mathrm{rank}(Z)$ if and only if there exists a $w \in N_G(\mathfrak{a})$ so that

$$\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset. \tag{3.1}$$

If (3.1) holds, then

$$\mathfrak{a}_\mathcal{O} = \mathrm{Ad}(w)\mathfrak{a}_\mathfrak{h} \tag{3.2}$$

and there exists an open P-orbit $\mathcal{O}'$ in Z so that

$$w^{-1} \cdot \mathcal{O} \subseteq \mathcal{O}'.$$

We say that a P-orbit $\mathcal{O}$ in Z is of maximal rank if $\mathrm{rank}(\mathcal{O}) = \mathrm{rank}(Z)$.

Remark 3.2.

(a) Fix a P-orbit $\mathcal{O}$ of maximal rank, a point $z \in \mathcal{O}$ and an order-regular element $X \in \mathfrak{a}^-$. The element $w \in N_G(\mathfrak{a})$ in (3.1) is not unique. It follows from [39, Lemma 10.3] that the stabilizer of $\mathfrak{h}_\emptyset$ in $N_G(\mathfrak{a})$ is equal to $N_{L_Q}(\mathfrak{a})$. Therefore, the equality (3.1) only determines the coset $wN_{L_Q}(\mathfrak{a}) \in N_G(\mathfrak{a})/N_{L_Q}(\mathfrak{a})$. The element w may be chosen so that

$$\mathrm{Ad}^*(w)\Sigma(P) \cap \big(-\Sigma(P)\big) = \mathrm{Ad}(w)^*\Sigma(Q) \cap \big(-\Sigma(P)\big). \tag{3.3}$$

To see this, consider the group $L_Q' = (L_Q \cap H)A$. L_Q' is reductive and normalizes $\mathfrak{h}_\emptyset$. Since $P \cap wL_Q'w^{-1}$ and $w(P \cap L_Q')w^{-1}$ are both minimal parabolic subgroups of $wL_Q'w^{-1}$ containing A, there exists a $v \in N_{wL_Q'w^{-1}}(\mathfrak{a})$ so that

$$vw(P \cap L_Q')w^{-1}v^{-1} = P \cap wL_Qw^{-1} = P \cap vwL_Q'w^{-1}v^{-1}.$$

Let $w' = vw$. Then

$$w'N_Pw'^{-1} = w'\big((N_P \cap L_Q')N_Q\big)w'^{-1} = (N_P \cap w'L_Qw'^{-1})w'N_Qw'^{-1}.$$

If now w is replaced by w', then it follows that both (3.1) and (3.3) hold.

(b) Fix an order-regular element $X \in \mathfrak{a}^-$ and a P-orbit $\mathcal{O}$ in Z of maximal rank. The element $w \in N_G(\mathfrak{a})$ in (3.1) depends on the choice of the point $z \in \mathcal{O}$. Indeed, it follows from Proposition 3.1 (iv) that

$$\mathfrak{h}_{man\cdot z,X} = \mathrm{Ad}(m)\mathfrak{h}_{z,X} \qquad (m \in M, a \in A, n \in N_P).$$

Therefore,

$$\mathfrak{h}_{man\cdot z,X} = \mathrm{Ad}(mw)\mathfrak{h}_\emptyset \qquad (m \in M, a \in A, n \in N_P)$$

if $z \in \mathcal{O}$ satisfies (3.1). Note that the coset $wZ_G(\mathfrak{a}) \in N_G(\mathfrak{a})/Z_G(\mathfrak{a}) = W$ is independent of $z \in \mathcal{O}$.

(c) Fix a P-orbit $\mathcal{O}$ in Z of maximal rank and a point $z \in \mathcal{O}$. The element $w \in N_G(\mathfrak{a})$ in (3.1) depends on the choice of the order-regular element $X \in \mathfrak{a}^-$, as can be seen in the following example.

Example 3.3. Assume that $G = {}^\backprime G \times {}^\backprime G$ for an reductive group ${}^\backprime G$, and $H = \mathrm{diag}({}^\backprime G)$. Let ${}^\backprime P$ be a minimal parabolic subgroup of ${}^\backprime G$ with Langlands decomposition ${}^\backprime P = {}^\backprime M{}^\backprime A{}^\backprime N$ and let ${}^\backprime\overline{P} = {}^\backprime M{}^\backprime A{}^\backprime\overline{N}$ be opposite to ${}^\backprime P$. We write P for the minimal parabolic subgroup ${}^\backprime P \times {}^\backprime\overline{P}$ of G. Let $\mathcal{R}$ be a set of representatives for the Weyl group of ${}^\backprime G$ in $N_{{}^\backprime G}({}^\backprime\mathfrak{a})$. Then $\mathcal{R}$ is in bijection with $P\backslash G/H$ via the map

$$\mathcal{R} \to P\backslash G/H; \quad w \mapsto \mathcal{O}(w) := P(e, w)H.$$

Now fix $w \in \mathcal{R}$ and let z be the point $(e, w)H$ in $\mathcal{O}(w)$. Let $X_1, X_2 \in {}^{\backprime}\mathfrak{a}^-({}^{\backprime}P)$. We assume that $X := (X_1, -X_2) \in \mathfrak{a}^-$ is order-regular. Then for all $\alpha, \beta \in \Sigma({}^{\backprime}\mathfrak{g}, {}^{\backprime}\mathfrak{a})$

$$\alpha(X_1) \neq \beta(X_2).$$

The limit subalgebra $\mathfrak{h}_{z,X}$ is equal to

$$\{(Y, \mathrm{Ad}(w)Y) : Y \in {}^{\backprime}\mathfrak{m} \oplus {}^{\backprime}\mathfrak{a}\} \oplus \bigoplus_{\substack{\alpha \in \Sigma({}^{\backprime}\mathfrak{g}, {}^{\backprime}\mathfrak{a}) \\ \alpha(X_1) > -w\cdot\alpha(X_2)}} \big({}^{\backprime}\mathfrak{g}_\alpha \times \{0\}\big) \oplus \bigoplus_{\substack{\alpha \in \Sigma({}^{\backprime}\mathfrak{g}, {}^{\backprime}\mathfrak{a}) \\ \alpha(X_1) < -w\cdot\alpha(X_2)}} \big(\{0\} \times {}^{\backprime}\mathfrak{g}_{w\cdot\alpha}\big).$$

From this formula it follows that every P-orbit in G/H is of maximal rank.

If $w = e$, so that $\mathcal{O}(w)$ is the open P orbit in G/H, then

$$\mathfrak{h}_{z,X} = \mathrm{diag}({}^{\backprime}\mathfrak{m} \oplus {}^{\backprime}\mathfrak{a}) \oplus \big({}^{\backprime}\overline{\mathfrak{n}} \times \{0\}\big) \oplus \big(\{0\} \times {}^{\backprime}\mathfrak{n}\big) =: \mathfrak{h}_\emptyset$$

is independent of the choice of X. For other orbits $\mathfrak{h}_{z,X}$ does depend on X. We illustrate this by considering the most extreme case: the closed P-orbit in G/H. Let $w \in \mathcal{R}$ represent the longest Weyl group element, so that $\mathcal{O}(w)$ is the closed orbit. Every choice of X_1 and X_2 corresponds to a unique positive system ${}^{\backprime}\Sigma^+$ of $\Sigma({}^{\backprime}\mathfrak{g}, {}^{\backprime}\mathfrak{a})$ satisfying.

$$\alpha(X_1) > -w \cdot \alpha(X_2) \qquad (\alpha \in {}^{\backprime}\Sigma^+). \tag{3.4}$$

Vice versa, given a positive system ${}^{\backprime}\Sigma^+$ of $\Sigma({}^{\backprime}\mathfrak{g}, {}^{\backprime}\mathfrak{a})$, we may choose X_1 and X_2 so that (3.4) holds. If (3.4) is satisfied, then

$$\mathfrak{h}_{z,X} = \{(Y, \mathrm{Ad}(w)Y) : Y \in {}^{\backprime}\mathfrak{m} \oplus {}^{\backprime}\mathfrak{a}\} \oplus \bigoplus_{\alpha \in {}^{\backprime}\Sigma^+} \big({}^{\backprime}\mathfrak{g}_\alpha \times \{0\}\big) \oplus \bigoplus_{\alpha \in {}^{\backprime}\Sigma^+} \big(\{0\} \times {}^{\backprime}\mathfrak{g}_\alpha\big).$$

In this case there exists a $v \in N_{{}^{\backprime}G}({}^{\backprime}\mathfrak{a})$ so that

$$\mathfrak{h}_{z,X} = \mathrm{Ad}(v, wv)\mathfrak{h}_\emptyset.$$

Note that v is a representative for the element of the Weyl group mapping $\Sigma({}^{\backprime}\overline{P})$ to ${}^{\backprime}\Sigma^+$.

For the proof of Proposition 3.1 we need a slight strengthening of [39, Lemma 10.8].

Lemma 3.4. *Let $z \in Z$ and let $X \in \mathfrak{a}$ be order regular. Then $\dim(\mathfrak{a}_\mathfrak{h}) \geq \dim(\mathfrak{h}_{z,X} \cap \mathfrak{a})$ if and only if there exists a $w \in N_G(\mathfrak{a})$ so that $\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$. In that case*

$$\mathfrak{a} \cap \mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{a}_\mathfrak{h}.$$

The proof for the lemma is essentially the same as the proof of [39, Lemma 10.8]; in the proof the equality $\dim(\mathfrak{a}_\mathfrak{h}) = \dim(\mathfrak{h}_{z,X} \cap \mathfrak{a})$ can straightforwardly be replaced by the inequality $\dim(\mathfrak{a}_\mathfrak{h}) \geq \dim(\mathfrak{h}_{z,X} \cap \mathfrak{a})$.

Proof of Proposition 3.1. Let $z \in \mathcal{O}$ and $X \in \mathfrak{a}^-$. If $\operatorname{rank}(\mathcal{O}) \geq \operatorname{rank}(Z)$, then

$$\dim(\mathfrak{h}_{z,X} \cap \mathfrak{a}) = \dim(\mathfrak{a}_{\mathcal{O}}) \leq \dim(\mathfrak{a}_{\mathfrak{h}}).$$

By Lemma 3.4 there exists a $w \in N_G(\mathfrak{a})$ so that $\mathfrak{h}_{z,X} = \operatorname{Ad}(w)\mathfrak{h}_\emptyset$. Moreover, if $\mathfrak{h}_{z,X} = \operatorname{Ad}(w)\mathfrak{h}_\emptyset$ for some $w \in N_G(\mathfrak{a})$, then

$$\mathfrak{a}_{\mathcal{O}} = \mathfrak{h}_{z,X} \cap \mathfrak{a} = \operatorname{Ad}(w)\mathfrak{h}_\emptyset \cap \mathfrak{a} = \operatorname{Ad}(w)\mathfrak{a}_{\mathfrak{h}},$$

and hence $\operatorname{rank}(\mathcal{O}) = \operatorname{rank}(Z)$.

It remains to prove the existence of an open P-orbit $\mathcal{O}'$ in Z so that $w \cdot \mathcal{O} \subset \mathcal{O}'$. We first prove that $w^{-1} \cdot z$ lies in an open P-orbit. As $\mathfrak{g} = \mathfrak{p} + \mathfrak{h}_\emptyset$ we have

$$\mathfrak{h}_{z,X} + \operatorname{Ad}(w)\mathfrak{p} = \operatorname{Ad}(w)\big(\mathfrak{h}_\emptyset + \mathfrak{p}\big) = \mathfrak{g}.$$

It follows that for sufficiently large $t > 0$ we have

$$\operatorname{Ad}\big(\exp(tX)\big)\mathfrak{h}_z + \operatorname{Ad}(w)\mathfrak{p} = \mathfrak{g}.$$

As $\operatorname{Ad}(w)\mathfrak{p}$ and $\mathfrak{g}$ are both A-stable, it follows that

$$\mathfrak{h}_z + \operatorname{Ad}(w)\mathfrak{p} = \mathfrak{g}.$$

Therefore, $wPw^{-1} \cdot z$ is open in Z, and hence $Pw^{-1} \cdot z$ is open in Z.

Now set $\mathcal{O}' = Pw^{-1} \cdot z$. Let $n \in N_P$. In view of Proposition 3.1 (iv) we have

$$\mathfrak{h}_{n\cdot z,X} = \operatorname{Ad}(w)\mathfrak{h}_\emptyset.$$

By the argument above, the P-orbit $Pw^{-1}n \cdot z$ is open. It follows that $w^{-1}N_P \cdot z$ is contained in the union of all open P-orbits in Z. As $w^{-1}N \cdot z$ is connected, intersects with $\mathcal{O}'$ and the boundary of $\mathcal{O}'$ only contains non-open P-orbits, it follows that $w^{-1}N \cdot z$ is contained in $\mathcal{O}'$. Moreover, since MA is a normal subgroup of $N_G(\mathfrak{a})$ we have

$$Pw^{-1}man \cdot z = Pw^{-1}n \cdot z = \mathcal{O}'$$

for all $m \in M$, $a \in A$ and $n \in N_P$. This proves the last assertion. □

3.7 Weakly adapted points

Let $X \in \mathfrak{a}$. If $\mathcal{O}$ is a P-orbit of maximal rank, then we say that X is $\mathcal{O}$-regular if $X \in \mathfrak{a} \cap \mathfrak{a}_{\mathcal{O}}^\perp$ and $\alpha(X) \neq 0$ for all roots $\alpha \in \Sigma$ that do not vanish on $\mathfrak{a} \cap \mathfrak{a}_{\mathcal{O}}^\perp$. We say that a point $z \in Z$ is weakly adapted (to the Langlands decomposition $P = MAN$) if the following two conditions are satisfied.

(i) The P-orbit $\mathcal{O} = P \cdot z$ is of maximal rank.

(ii) There exists an $\mathcal{O}$-regular element in $\mathfrak{a} \cap \mathfrak{h}_z^\perp$.

Note that an adapted point $z \in Z$ is also weakly adapted.

The weakly adapted points in a given maximal rank P-orbit admit a similar parametrization as the adapted points in Proposition 3.2.

Proposition 3.1. *Let $\mathcal{O} \in P\backslash Z$ be of maximal rank. The following hold.*

(i) For every $\mathcal{O}$-regular element $X \in \mathfrak{a}$ there exists a weakly adapted point $z \in \mathcal{O}$ so that $X \in \mathfrak{h}_z^\perp$. Moreover, if $z' \in \mathcal{O}$ is another adapted point so that $X \in \mathfrak{h}_{z'}^\perp$, then there exist $m \in M$ and $a \in A$ so that $z' = ma \cdot z$.

(ii) Let $z \in \mathcal{O}$ be weakly adapted and $w \in N_G(\mathfrak{a})$. If there exists an $X \in \mathfrak{a}^-$ so that $\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$, then $w^{-1} \cdot z$ is adapted.

For the proof of the proposition we need the following lemma. We write $p_\mathfrak{a} : \mathfrak{g} \to \mathfrak{a}$ for the projection onto $\mathfrak{a}$ along the root space decomposition.

Lemma 3.2. *Let $\mathcal{O}$ be a P-orbit of maximal rank and let $z \in \mathcal{O}$. Let $w \in N_G(\mathfrak{a})$ be so that $\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$ for some order-regular element $X \in \mathfrak{a}^-$. Then*

$$p_\mathfrak{a}\big((\mathfrak{p} + \mathrm{Ad}(w)\mathfrak{q}) \cap \mathfrak{h}_z\big) = \mathfrak{a}_\mathcal{O}.$$

Proof. It follows from (3.1) that

$$p_\mathfrak{a}(\mathfrak{p} \cap \mathfrak{h}_z) = \mathfrak{a} \cap \mathfrak{h}_{z,X} = \mathfrak{a}_\mathcal{O},$$

where X is any element in $\mathfrak{a}^-$. Therefore,

$$\mathfrak{a}_\mathcal{O} \subseteq p_\mathfrak{a}\big((\mathfrak{p} + \mathrm{Ad}(w)\mathfrak{q}) \cap \mathfrak{h}_z\big).$$

We move on to the other inclusion. Let $Y \in \mathfrak{p} + \mathrm{Ad}(w)\mathfrak{q}$ and assume that $Y \in \mathfrak{h}_z$. We will prove that $p_\mathfrak{a}(Y) \in \mathfrak{a}_\mathcal{O}$. We decompose Y as

$$Y = Y_\mathfrak{p} + Y_0 + Y_-,$$

where $Y_\mathfrak{p} \in \mathfrak{p}$, $Y_0 \in \mathrm{Ad}(w)\mathfrak{l}_{Q,\mathrm{nc}}$ and $Y_- \in \mathrm{Ad}(w)\mathfrak{n}_Q \cap \overline{\mathfrak{n}}_P$.

In view of Proposition 3.1 the P-orbit $Pw^{-1} \cdot z$ is open. Therefore, there exists a $n \in N_Q$ so that the connected subgroup $L_{Q,\mathrm{nc}}$ with Lie algebra $\mathfrak{l}_{Q,\mathrm{nc}}$ is contained in $H_{nw^{-1}\cdot z}$. It follows that $\mathrm{Ad}(wn^{-1})\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathfrak{h}_z$. Note that $\mathrm{Ad}(w)\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathrm{Ad}(wn^{-1})\mathfrak{l}_{Q,\mathrm{nc}} + \mathrm{Ad}(w)\mathfrak{n}_Q$. Therefore, there exists a $Y' \in \mathrm{Ad}(w)\mathfrak{n}_Q$ so that $Y_0 + Y' \in \mathrm{Ad}(wn^{-1})\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathfrak{h}_z$. Note that $p_\mathfrak{a}(Y_0 + Y') \in \mathrm{Ad}(w)\mathfrak{a}_\mathfrak{h} = \mathfrak{a}_\mathcal{O}$. By subtracting $Y_0 + Y'$ from Y we may thus without loss of generality assume that $Y_0 = 0$, i.e.,

$$Y = Y_\mathfrak{p} + Y_-.$$

Let $X \in \mathfrak{a}^-$ be order-regular and satisfy $\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$. The line $(\mathbb{R}Y)_X$ is $\mathfrak{a}$-stable and contained in $\mathrm{Ad}(w)\mathfrak{h}_\emptyset$. Note that Y_- is a linear combination of eigenvectors of $\mathrm{ad}(X)$ with strictly positive eigenvalues, whereas $Y_\mathfrak{p}$ is a linear combination of eigenvectors with non-positive eigenvalues. It follows that $Y_- = 0$ as $(\mathbb{R}Y)_X$ would otherwise be a line in $\mathrm{Ad}(w)\mathfrak{n}_Q \cap \overline{\mathfrak{n}}_P$, which would be in contradiction with the fact that $(\mathbb{R}Y)_X$ is contained in $\mathrm{Ad}(w)\mathfrak{h}_\emptyset$. Now $Y \in \mathfrak{p} \cap \mathfrak{h}_z$ and hence $p_\mathfrak{a}(Y) \in p_\mathfrak{a}(\mathfrak{p} \cap \mathfrak{h}_z) = \mathfrak{a}_\mathcal{O}$. □

Proof of Proposition 3.1. Let $z_0 \in \mathcal{O}$ and let $X \in \mathfrak{a}$ be $\mathcal{O}$-regular. Let $X' \in \mathfrak{a}^-$ be order-regular. By Proposition 3.1 there exists a $w \in N_G(\mathfrak{a})$ so that $\mathfrak{h}_{z_0,X'} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$. It follows from Lemma 3.2 that

$$X \in \big((\mathfrak{p} + \mathrm{Ad}(w)\mathfrak{q}) \cap \mathfrak{h}_{z_0}\big)^\perp = (\mathfrak{n}_P \cap \mathrm{Ad}(w)\mathfrak{n}_Q) + \mathfrak{h}_{z_0}^\perp.$$

In particular, there exists a $Y \in \mathfrak{n}_P \cap \mathrm{Ad}(w)\mathfrak{n}_Q$ so that $X + Y \in \mathfrak{h}_{z_0}^\perp$. Since X is $\mathcal{O}$-regular, it follows from (3.2) that $\alpha(X) \neq 0$ for every $\alpha \in \Sigma$ so that $\alpha|_{\mathfrak{a}\cap\mathrm{Ad}(w)\mathfrak{a}_\mathfrak{h}^\perp} \neq 0$. As the roots of $\mathfrak{a}$ in $\mathfrak{n}_P \cap \mathrm{Ad}(w)\mathfrak{n}_Q$ do not vanish on $\mathfrak{a} \cap \mathrm{Ad}(w)\mathfrak{a}_\mathfrak{h}^\perp$, this implies that there exists a $n \in N_P \cap wN_Qw^{-1}$ so that $\mathrm{Ad}(n)X = X + Y$. Set $z = n^{-1} \cdot z_0$. Then

$$X \in \mathrm{Ad}(n^{-1})\mathfrak{h}_{z_0}^\perp = \mathfrak{h}_z^\perp.$$

This proves the first assertion in (i).

We move on to the second assertion in (i). Let $z' \in \mathcal{O}$ be another point so that $X \in \mathfrak{h}_{z'}^\perp$. By Proposition 3.1 the points $w^{-1} \cdot z$ and $w^{-1} \cdot z'$ lie in the same open P-orbit. Moreover,

$$\mathrm{Ad}(w^{-1})X \in \mathfrak{h}_{w^{-1}\cdot z}^\perp \cap \mathfrak{h}_{w^{-1}\cdot z'}^\perp.$$

By Proposition 3.2

$$w^{-1} \cdot z' \in MAw^{-1} \cdot z.$$

As MA is a normal subgroup of $N_G(\mathfrak{a})$, it follows that

$$z' \in wMAw^{-1} \cdot z = MA \cdot z.$$

This concludes the proof of (i).

It remains to prove (ii). Assume that z is weakly adapted and there exists an $X \in \mathfrak{a}^-$ so that $\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$. By Proposition 3.1 the P-orbit through $w^{-1} \cdot z$ is open. Moreover, as $\mathfrak{a} \cap \mathfrak{h}_z^\perp$ contains $\mathcal{O}$-regular elements, the set

$$\mathfrak{a} \cap \mathfrak{h}_{w^{-1}\cdot z}^\perp = \mathrm{Ad}(w^{-1})\big(\mathfrak{a} \cap \mathfrak{h}_z^\perp\big)$$

contains elements X so that $\alpha(X) \neq 0$ for all $\Sigma(Q)$. It follows that $w^{-1}\cdot z$ is adapted. $\square$

3.8 Structure of orbits of maximal rank

In this section we show that the P-orbits of maximal rank admit a structure theorem that is similar to the local structure theorem for open P-orbits. We begin with a decomposition of N_P.

Proposition 3.1. *Let $\mathcal{O} \in P\backslash Z$ be of maximal rank and $w \in N_G(\mathfrak{a})$. Assume that there exist a point $z \in \mathcal{O}$ and an order-regular element $X \in \mathfrak{a}^-$ so that*

$$\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset.$$

Then for every $y \in \mathcal{O}$ the multiplication map

$$(N_P \cap wN_Qw^{-1}) \times (N_P \cap H_y) \to N_P; \quad (n, n_H) \mapsto nn_H \tag{3.1}$$

is a diffeomorphism.

Proof. We prove the assertion first for $y = z$. It follows from (3.1) in Proposition 3.1 that

$$(\mathfrak{n}_P \cap \mathfrak{h}_z)_X = \mathfrak{n}_P \cap \mathfrak{h}_{z,X} = \mathfrak{n}_P \cap \mathrm{Ad}(w)\mathfrak{h}_\emptyset.$$

Since

$$\mathfrak{g} = \mathrm{Ad}(w)\mathfrak{n}_Q \oplus (\mathrm{Ad}(w)\mathfrak{h}_\emptyset + \mathfrak{m} + \mathfrak{a}),$$

and this decomposition is compatible with the root space decomposition of $\mathfrak{g}$, it follows that

$$\mathfrak{n}_P = (\mathfrak{n}_P \cap \mathrm{Ad}(w)\mathfrak{n}_Q) \oplus (\mathfrak{n}_P \cap \mathfrak{h}_z)_X.$$

Hence, for sufficiently large $t > 0$

$$\mathfrak{n}_P = (\mathfrak{n}_P \cap \mathrm{Ad}(w)\mathfrak{n}_Q) \oplus \mathrm{Ad}\left(\exp(tX)\right)(\mathfrak{n}_P \cap \mathfrak{h}_z)$$

As $\mathfrak{n}_P$ and $\mathfrak{n}_P \cap \mathrm{Ad}(w)\mathfrak{n}_Q$ are both $\mathfrak{a}$-stable, it follows, that

$$\mathfrak{n}_P = (\mathfrak{n}_P \cap \mathrm{Ad}(w)\mathfrak{n}_Q) \oplus (\mathfrak{n}_P \cap \mathfrak{h}_z)$$

and thus (3.1) is a local diffeomorphism onto an open neighborhood of e in N_P. It remains to show that (3.1) is a bijection.

The intersection $(N_P \cap wN_Qw^{-1}) \cap (N_P \cap H_z)$ is an algebraic subgroup of N_P of dimension 0. The only such subgroup is the trivial one. Therefore, (3.1) is injective.

By [44, Theorem 2] both $N_P \cdot z$ and $(N_P \cap wN_Qw^{-1}) \cdot z$ are closed submanifolds of Z. Since the image of (3.1) is open in N_P, the set $(N_P \cap wN_Qw^{-1}) \cdot z$ is a relatively open subset of $N_P \cdot z$. Hence, $(N_P \cap wN_Qw^{-1}) \cdot z$ is open and closed in $N_P \cdot z$. As $N_P \cdot z$ is connected, it follows that

$$(N_P \cap wN_Qw^{-1}) \cdot z = N_P \cdot z. \tag{3.2}$$

From this we conclude that (3.1) is surjective and this concludes the proof of the proposition for $z = y$.

Let now $y \in \mathcal{O}$ and let $m \in M$, $a \in A$ and $n \in N_P$ be such that $y = man \cdot z$. The identity (3.2) shows that we may choose $n \in N_P \cap wN_Qw^{-1}$. Since the groups $(N_P \cap wN_Qw^{-1})$ and N_P are normalized by $MA(N_P \cap wN_Qw^{-1})$, the assertion in the proposition now follows from the case $z = y$. □

Theorem 3.2. *Let $\mathcal{O} \in P\backslash Z$ be of maximal rank and $z \in \mathcal{O}$ weakly adapted. Let $w \in N_G(\mathfrak{a})$ be so that*

$$\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$$

for some order-regular $X \in \mathfrak{a}^-$. Then $\mathfrak{a}_\mathcal{O} = \mathrm{Ad}(w)\mathfrak{a}_\mathfrak{h} \subseteq \mathfrak{h}_z$. Moreover, $Pw^{-1} \cdot z$ is open and the maps

$$N_Q \times M/(M \cap H_{w^{-1}\cdot z}) \times A/\exp(\mathfrak{a}_\mathfrak{h}) \to Pw^{-1} \cdot z; \quad (n, m, a) \mapsto nmaw^{-1} \cdot z \tag{3.3}$$

$$(N_P \cap wN_Qw^{-1}) \times M/(M \cap H_z) \times A/\exp(\mathfrak{a}_\mathcal{O}) \to \mathcal{O}; \quad (n, m, a) \mapsto nma \cdot z \tag{3.4}$$

$$(\overline{N}_P \cap wN_Qw^{-1}) \times \mathcal{O} \to wPw^{-1} \cdot z; \quad (n, x) \mapsto n \cdot x \tag{3.5}$$

are diffeomorphisms.

Remark 3.3. The diffeomorphism (3.4) may be viewed as a structure theorem for a P-orbit of maximal rank. For complex spherical spaces this structure theorem was first proven by Brion in [11, Proposition 6 & Theorem 3]. For our purposes the diffeomorphism (3.5) will be of particular importance for the construction of distributions in Section 6.

Proof of Theorem 3.2. In view of Proposition 3.1 (ii) the P-orbit through $w^{-1}\cdot z$ is open and $w^{-1}\cdot z$ is adapted. The map (3.3) is a diffeomorphism by Proposition 3.1.

It follows from Proposition 3.1 that

$$w^{-1}\cdot\mathcal{O}\subseteq Pw^{-1}\cdot z.$$

In view of Proposition 3.1

$$\mathcal{O}=(N_P\cap wN_Qw^{-1})MA\cdot z,$$

and hence

$$w^{-1}\cdot\mathcal{O}=(w^{-1}N_Pw\cap N_Q)MAw^{-1}\cdot z$$

Since (3.3) is a diffeomorphism, the map

$$(w^{-1}N_Pw\cap N_Q)\times M/(M\cap H_{w^{-1}\cdot z})\times A/\exp(\mathfrak{a}_\mathfrak{h})\to w^{-1}\mathcal{O};\quad (n,ma)\mapsto nmaw^{-1}\cdot z$$

is a diffeomorphism. As $\mathfrak{a}_\mathcal{O}=\mathrm{Ad}(w)\mathfrak{a}_\mathfrak{h}$, this implies that (3.4) is a diffeomorphism.

Finally, the map (3.5) is a diffeomorphism since (3.3), (3.4) and the product map

$$(N_P\cap wN_Qw^{-1})\times(\overline{N}_P\cap wN_Qw^{-1})\to wN_Qw^{-1}$$

are diffeomorphisms. □

3.9 Admissible points and the little Weyl group

Following [39, Definition 10.1] we call a point $z\in Z$ admissible if it is adapted and if for every order-regular element $X\in\mathfrak{a}$ there exists a $w\in N_G(\mathfrak{a})$ so that $\mathfrak{h}_{z,X}=\mathrm{Ad}(w)\mathfrak{h}_\emptyset$. By [39, Proposition 10.4] the set of admissible points is open and dense in the set of adapted points in Z (with respect to the subspace topology). In particular, every open P-orbit in Z contains an admissible point. We may and will assume that the point $eH\in G/H\in Z$ is admissible.

We define the groups

$$\mathcal{N}:=N_G(\mathfrak{a})\cap N_G(\mathfrak{a}_\mathfrak{h})\tag{3.1}$$

and

$$\mathcal{Z}:=\{w\in N_G(\mathfrak{a}):\mathrm{Ad}(w)\mathfrak{h}_\emptyset=\mathrm{Ad}(m)\mathfrak{h}_\emptyset\text{ for some }m\in M\}=N_{L_Q}(\mathfrak{a})\tag{3.2}$$

Note that $\mathcal{Z}$ is a normal subgroup of $\mathcal{N}$. For an admissible point $z\in Z$ we set

$$\mathcal{W}:=\{w\in N_G(\mathfrak{a}):\mathfrak{h}_{z,X}=\mathrm{Ad}(wm)\mathfrak{h}_\emptyset\text{ for some }X\in\mathfrak{a}\text{ and }m\in M\}.\tag{3.3}$$

By [39, Proposition 10.4] the set $\mathcal{W}$ does not depend on the choice of the admissible point. Furthermore, by [39, Theorem 11.1] it is a subgroup of $\mathcal{N}$. The quotient group

$$W_Z := \mathcal{W}/\mathcal{Z}$$

is equal to the little Weyl group of Z as defined in [28]. The little Weyl group acts on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$ as a finite reflection group. The set $\overline{\mathcal{C}}/\mathfrak{a}_\mathfrak{h}$ is a fundamental domain for this action.

Let $\mathfrak{a}_E$ be the edge of $\overline{\mathcal{C}}$, i.e.,

$$\mathfrak{a}_E := \overline{\mathcal{C}} \cap -\overline{\mathcal{C}}.$$

The little Weyl group acts trivially on $\mathfrak{a}_E/\mathfrak{a}_\mathfrak{h}$. See [39, Lemma 12.1]. Moreover, by [28, Proposition 10.3] and [39, Theorem 12.2] the little Weyl group is the Weyl group of a root system in $(\mathfrak{a}/\mathfrak{a}_E)^*$. This root system is called the spherical root system. We will indicate it by Σ_Z.

For our purposes the following characterization of $\mathcal{W}$ is important.

Proposition 3.1. *Let $z \in Z$ be admissible and let $w \in N_G(\mathfrak{a})$. Then $Pw^{-1} \cdot z$ is open if and only if $w \in \mathcal{W}$. In that case $w^{-1} \cdot z$ is admissible.*

Proof. The assertion follows from [39, Proposition 7.2] and the equivariance of limits of subalgebras

$$\mathfrak{h}_{w^{-1}\cdot z,X} = \mathrm{Ad}(w^{-1})\mathfrak{h}_{z,\mathrm{Ad}(w)X} \qquad (X \in \mathfrak{a}).$$

□

If Z is wavefront, then the $\mathcal{W} = \mathcal{N}$ and the little Weyl group is equal to $W_Z = \mathcal{W}/\mathcal{Z}$. See Proposition A.1 in Appendix A.

3.10 Weakly admissible points

We call a point $z \in Z$ weakly admissible if it is weakly adapted and for every order-regular element $X \in \mathfrak{a}$ there exists a $w \in N_G(\mathfrak{a})$ so that $\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$. Note that every admissible point is weakly admissible.

Proposition 3.1. *Let $z \in Z$. If z is weakly admissible, then $w \cdot z$ is weakly admissible for every $w \in N_G(\mathfrak{a})$.*

Remark 3.2. As the set of admissible points is (relatively) open and dense in the set of adapted points, it follows from the proposition and Proposition 3.1 (ii) that the set of weakly admissible points is open and dense (with respect to the subspace topology) in the set of weakly adapted points.

Proof of Proposition 3.1. Assume that z is weakly admissible and $w \in N_G(\mathfrak{a})$. If $X \in \mathfrak{a}$ is order-regular then $\mathrm{Ad}(w)X$ is also order-regular. Therefore, there exists a $w' \in N_G(\mathfrak{a})$ so that $\mathfrak{h}_{z,\mathrm{Ad}(w^{-1})X} = \mathrm{Ad}(w')\mathfrak{h}_\emptyset$, and hence

$$\mathfrak{h}_{w\cdot z,X} = \mathrm{Ad}(w)\mathfrak{h}_{z,\mathrm{Ad}(w^{-1})X} = \mathrm{Ad}(ww')\mathfrak{h}_\emptyset.$$

By Proposition 3.1 $\mathcal{O} := Pw \cdot z$ has maximal rank. It remains to prove that $w \cdot z$ is weakly adapted.

Let $\mathcal{O}' = P \cdot z$. Since z is weakly adapted, we have $\mathfrak{a}_{\mathcal{O}'} \subseteq \mathfrak{h}_z$ by Theorem 3.2, and hence $\mathrm{Ad}(w)\mathfrak{a}_{\mathcal{O}'} \subseteq \mathfrak{h}_{w \cdot z}$. Let $X \in \mathfrak{a}^-$. Then

$$\mathrm{Ad}(w)\mathfrak{a}_{\mathcal{O}'} \subseteq \mathfrak{a} \cap \mathfrak{h}_{w \cdot z, X} = \mathfrak{a}_{\mathcal{O}}.$$

As both $\mathfrak{a}_{\mathcal{O}'}$ and $\mathfrak{a}_{\mathcal{O}}$ are conjugate to $\mathfrak{a}_{\mathfrak{h}}$, these two spaces are of equal dimension. Therefore, $\mathrm{Ad}(w)\mathfrak{a}_{\mathcal{O}'} = \mathfrak{a}_{\mathcal{O}}$. Since z is weakly adapted, there exists $\mathcal{O}'$-regular elements in $\mathfrak{a} \cap \mathfrak{h}_z^\perp$. It follows that there exist $\mathcal{O}$-regular elements in $\mathfrak{a} \cap \mathfrak{h}_{w \cdot z}^\perp = \mathrm{Ad}(w)\big(\mathfrak{a} \cap \mathfrak{h}_z^\perp\big)$. This proves that $w \cdot z$ is weakly adapted. □

Proposition 3.3. *Let $z \in Z$ be weakly admissible, let $X \in \mathfrak{a}$ be order-regular and let $w \in N_G(\mathfrak{a})$. Then $\mathfrak{h}_{z,X} = \mathrm{Ad}(wm)\mathfrak{h}_\emptyset$ for some $m \in M$ if and only if $Pw^{-1} \cdot z$ is open and $X \in \mathrm{Ad}(w)\mathcal{C}$.*

Proof. We have

$$\mathrm{Ad}(w)^{-1}\mathfrak{h}_{z,X} = \mathfrak{h}_{w^{-1} \cdot z, \mathrm{Ad}(w^{-1})X}.$$

By Proposition 3.1 the point $w^{-1} \cdot z$ is weakly admissible. In view of Proposition 3.1 the limit subalgebra $\mathfrak{h}_{w^{-1} \cdot z, \mathrm{Ad}(w^{-1})X}$ is equal to $\mathrm{Ad}(m)\mathfrak{h}_\emptyset$ for some $m \in M$ if and only if $w^{-1} \cdot z$ is open and $\mathrm{Ad}(w^{-1})X \in \mathcal{C}$. □

3.11 An action of the Weyl group

We write $(P\backslash Z)_{\max}$ for the subset of $P\backslash Z$ consisting of all P-orbits in Z of maximal rank and $(P\backslash Z)_{\text{open}}$ for set of all open P-orbits in Z.

Proposition 3.1. *Let $\mathcal{O}_1, \mathcal{O}_2 \in (P\backslash Z)_{\max}$, and let $z_1 \in \mathcal{O}_1$ and $z_2 \in \mathcal{O}_2$ be weakly admissible. Let further $X_1, X_2 \in \mathfrak{a}$ be order-regular, and let $m \in M$. If $\mathfrak{h}_{z_1,X_1} = \mathrm{Ad}(m)\mathfrak{h}_{z_2,X_1}$, then $\mathfrak{h}_{z_1,X_2} = \mathrm{Ad}(m)\mathfrak{h}_{z_2,X_2}$.*

Proof. Let $w \in N_G(\mathfrak{a})$ be so that $\mathfrak{h}_{z_1,X_1} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$. By Proposition 3.1 (ii) the point $w^{-1} \cdot z_1$ is adapted. In view of Proposition 3.1 it is also weakly admissible, and hence $w^{-1} \cdot z_1$ is admissible. Moreover,

$$\mathfrak{h}_{w^{-1} \cdot z_1, \mathrm{Ad}(w^{-1})X_1} = \mathrm{Ad}(w^{-1})\mathfrak{h}_{z_1,X_1} = \mathfrak{h}_\emptyset,$$

and hence $\mathrm{Ad}(w^{-1})X_1 \in \mathcal{C}$. Let $v \in \mathcal{W}$ be so that $\mathrm{Ad}(w^{-1})X_2 \in \mathrm{Ad}(v)\mathcal{C}$. Then $v^{-1}w^{-1} \cdot z_1$ is admissible by Proposition 3.1. As $\mathrm{Ad}(v^{-1}w^{-1})X_2 \in \mathcal{C}$, it follows that there exists an $m_1 \in M$ so that

$$\mathfrak{h}_{z_1,X_2} = \mathrm{Ad}(wv)\mathfrak{h}_{v^{-1}w^{-1} \cdot z, \mathrm{Ad}(v^{-1}w^{-1})X_2} = \mathrm{Ad}(m_1 wv)\mathfrak{h}_\emptyset.$$

In the same way we find

$$\mathfrak{h}_{z_2,X_2} = \mathrm{Ad}(m_2 wv)\mathfrak{h}_\emptyset$$

for some $m_2 \in M$. Now

$$\mathfrak{h}_{z_1,X_2} = \mathrm{Ad}(m_1 m_2^{-1})\mathfrak{h}_{z_2,X_2}.$$

The latter is equal to $\mathrm{Ad}(m)\mathfrak{h}_{z_2,X_2}$ if and only if

$$\mathfrak{m}\cap\mathfrak{h}_{z_1,X_2}=\mathrm{Ad}(m)\big(\mathfrak{m}\cap\mathfrak{h}_{z_2,X_2}\big). \tag{3.1}$$

It thus suffices to prove the latter.

We claim that

$$\mathfrak{m}\cap\mathfrak{h}_{z,X}=\mathfrak{m}\cap\mathfrak{h}_z \tag{3.2}$$

for all weakly adapted points $z\in Z$ and all order-regular elements $X\in\mathfrak{a}$. To prove the claim we first consider an adapted point $z\in Z$. Then (3.2) follows from Proposition 3.1 if $X\in\mathfrak{a}^-$. Since the limit $\mathfrak{h}_{z,X}$ is the same for all $X\in\mathcal{C}$, (3.2) also holds for $X\in\mathcal{C}$. If $X\in\mathfrak{a}$ is any order-regular element, then there exists a $u\in\mathcal{W}$ so that $\mathrm{Ad}(u)X\in\mathcal{C}$. Then

$$\mathfrak{m}\cap\mathfrak{h}_{z,X}=\mathfrak{m}\cap\mathrm{Ad}(u^{-1})\mathfrak{h}_{u\cdot z,\mathrm{Ad}(u)X}=\mathrm{Ad}(u^{-1})\big(\mathfrak{m}\cap\mathfrak{h}_{u\cdot z,\mathrm{Ad}(u)X}\big).$$

By Proposition 3.1 the point $u\cdot z$ is adapted. Therefore,

$$\mathrm{Ad}(u^{-1})\big(\mathfrak{m}\cap\mathfrak{h}_{u\cdot z,\mathrm{Ad}(u)X}\big)=\mathrm{Ad}(u^{-1})\big(\mathfrak{m}\cap\mathfrak{h}_{u\cdot z}\big)=\mathfrak{m}\cap\mathfrak{h}_z.$$

This proves (3.2) in case $z\in Z$ is adapted. Let now $z\in Z$ be weakly adapted. Then there exists a $u\in N_G(\mathfrak{a})$ so that $u\cdot z$ is adapted. In that case

$$\mathfrak{m}\cap\mathfrak{h}_{z,X}=\mathrm{Ad}(u^{-1})\big(\mathfrak{m}\cap\mathfrak{h}_{u\cdot z,\mathrm{Ad}(u)X}\big)=\mathrm{Ad}(u^{-1})\big(\mathfrak{m}\cap\mathfrak{h}_{u\cdot z}\big)=\mathfrak{m}\cap\mathfrak{h}_z.$$

This proves the claim (3.2).

The required identity (3.1) follows from (3.2) as $\mathfrak{h}_{z_1,X_1}=\mathrm{Ad}(m)\mathfrak{h}_{z_2,X_1}$ and hence

$$\mathfrak{m}\cap\mathfrak{h}_{z_1,X_2}=\mathfrak{m}\cap\mathfrak{h}_{z_1,X_1}=\mathrm{Ad}(m)\big(\mathfrak{m}\cap\mathfrak{h}_{z_2,X_1}\big)=\mathrm{Ad}(m)\big(\mathfrak{m}\cap\mathfrak{h}_{z_2,X_2}\big).$$

□

If $\mathcal{O}\in P\backslash Z$ and $X\in\mathfrak{a}^-$, then up to M-conjugacy the limits $\mathfrak{h}_{z,X}$ do not depend on the point $z\in\mathcal{O}$, see Remark 3.2 (b). In view of Proposition 3.1 we may thus define an equivalence relation $\sim$ on $(P\backslash Z)_{\max}$ by requiring that

$$\mathcal{O}_1\sim\mathcal{O}_2$$

if and only if for a given order-regular element $X\in\mathfrak{a}$ there exists weakly admissible points $z_1\in\mathcal{O}_1$ and $z_2\in\mathcal{O}_2$ so that

$$\mathfrak{h}_{z_1,X}=\mathfrak{h}_{z_2,X}.$$

The equivalence relation does not depend on the choice of the order-regular element $X\in\mathfrak{a}$.

Remark 3.2.

(a) If $\mathcal{O}\in P\backslash Z$, then by Proposition 3.1 the limit subalgebra $\mathfrak{h}_{z,X}$ for a given order-regular element $X\in\mathfrak{a}^-$ does not depend on $z\in\mathcal{O}$ up to M-conjugacy. Moreover, $\mathfrak{h}_{m\cdot z,X}=\mathrm{Ad}(m)\mathfrak{h}_{z,X}$ for every $m\in M$. Therefore, two P-orbits $\mathcal{O}_1,\mathcal{O}_2$ of maximal rank are equivalent if and only if there exists an order-regular elements $X\in\mathfrak{a}^-$, points $z_1\in\mathcal{O}_1$ and $z_2\in\mathcal{O}_2$, and an $m\in M$ so that $\mathfrak{h}_{z_1,X}=\mathrm{Ad}(m)\mathfrak{h}_{z_2,X}$.

(b) Let $X \in \mathfrak{a}^-$ be order-regular. If $z \in Z$, then by Proposition 3.1 the limit subalgebra $\mathfrak{h}_{z,X}$ is an M-conjugate of $\mathfrak{h}_\emptyset$ if and only if $P \cdot z$ is open. Therefore, if $\mathcal{O}_1 \in P\backslash Z$ is open and $\mathcal{O}_2 \in (P\backslash Z)_{\max}$, then $\mathcal{O}_1 \sim \mathcal{O}_2$ if and only if $\mathcal{O}_2$ is open. In particular, the set $(P\backslash Z)_{\mathrm{open}}$ of all open P-orbits in Z forms an equivalence class.

We denote the equivalence classes of $\sim$ by $[\cdot]$ and recall the subgroup $\mathcal{W}$ of $N_G(\mathfrak{a})$ from (3.3).

Theorem 3.3. *For any $v \in N_G(\mathfrak{a})$ and any $\mathcal{O} \in (P\backslash Z)_{\max}$, the equivalence class $[Pv \cdot z]$ is independent of the choice of the weakly admissible point $z \in \mathcal{O}$. For $w \in W$ and $\mathcal{O} \in (P\backslash Z)_{\max}$ we may thus set*

$$w \cdot [\mathcal{O}] = [Pv \cdot z],$$

where $v \in N_G(\mathfrak{a})$ is any representative of w and $z \in \mathcal{O}$ is any weakly admissible point. The map

$$W \times (P\backslash Z)_{\max}/_{\sim} \to (P\backslash Z)_{\max}/_{\sim}$$

thus obtained defines an action of W on $(P\backslash Z)_{\max}/_{\sim}$. This action has the following properties.

(i) W acts transitively on $(P\backslash Z)_{\max}/_{\sim}$.

(ii) The stabilizer of the equivalence class $(P\backslash Z)_{\mathrm{open}}$ is equal to $\mathcal{W}/MA$.

(iii) Let $w \in W$ and let $v \in N_G(\mathfrak{a})$ be a representative for w. If $X \in \mathfrak{a} \cap \mathfrak{a}_{\mathfrak{h}}^{\perp}$ satisfies $Z_{\mathfrak{g}}(X) = \mathfrak{l}_Q$, and $z_1, \dots, z_n$ is a set of admissible points representing the open P-orbits in Z with

$$X \in \mathfrak{a} \cap \mathfrak{h}_{z_i}^{\perp} \qquad (1 \leq i \leq n),$$

then

$$Pw \cdot z_i \neq Pw \cdot z_j \qquad (1 \leq i < j \leq n)$$

and

$$w \cdot (P\backslash Z)_{\mathrm{open}} = \{Pw \cdot z_i : 1 \leq i \leq n\}.$$

In particular, the cardinalities of the equivalence classes are all equal, i.e, for every $\mathcal{O} \in (P\backslash Z)_{\max}$ the cardinality of $[\mathcal{O}]$ is equal to the number of open P-orbits in Z.

(iv) If $w \in W$ and $\mathcal{O} \in (P\backslash Z)_{\max}$, then $\mathfrak{a}_{\mathcal{O}'} = \mathrm{Ad}(w)\mathfrak{a}_{\mathcal{O}}$ for every $\mathcal{O}' \in w \cdot [\mathcal{O}]$.

The action of W on $(P\backslash Z)_{\max}$ lifts to an action of $N_G(\mathfrak{a})$. In later sections we will use interchangeably the actions of W and $N_G(\mathfrak{a})$ on $(P\backslash Z)_{\mathrm{open}}$ and use the same notation without further indication.

Remark 3.4.

(a) Let $\underline{P} \subseteq \underline{G}$ be a minimal parabolic subgroup defined over $\mathbb{R}$ and let $\underline{Z}$ be an algebraic $\underline{G}$-variety. Assume that $\underline{Z}$ is real spherical, i.e., that $\underline{P}$ admits an open orbit in $\underline{Z}$. In [32] Knop and Zhgoon constructed an action of W on the set of $\underline{P}$-orbits $\underline{\mathcal{O}}$ in $\underline{Z}$ with the property that $\underline{\mathcal{O}}(\mathbb{R}) \neq \emptyset$. If Z is an open G-orbit in $\underline{Z}(\mathbb{R})$, then each equivalence class in $(\underline{P}(\mathbb{R})\backslash Z)_{\max}$ corresponds to one $\underline{P}$-orbit $\underline{\mathcal{O}}$ of maximal rank in $\underline{Z}$ with the property that $\underline{\mathcal{O}}(\mathbb{R}) \neq \emptyset$. The construction of Knop and Zhgoon then coincides with the W-action on $(P\backslash Z)_{\max}/\sim$ from Theorem 3.3.

(b) If P admits only one open orbit in Z, then each equivalence class of $\sim$ consists of precisely one P-orbit in Z. The above action then yields a transitive action of W on $(P\backslash Z)_{\max}$. This is in particular the case if G and H are both complex groups; the latter action then coincides with Knop's W-action on $P\backslash Z$ from [27] restricted to the set of maximal rank orbits.

(c) Assume that in each open P-orbit in Z the set of adapted points is precisely equal to one MA-orbit, equivalently if there exists a $z \in Z$ so that $P \cdot z$ is open and $\mathfrak{a} \cap \mathfrak{a}_{\mathfrak{h}}^{\perp} \subseteq \mathfrak{h}_z^{\perp}$. Then every adapted point is admissible. If $\mathcal{O}$ is an open orbit in Z, $z \in \mathcal{O}$ is adapted, and $w = vMA \in W$, then the orbit

$$w \cdot \mathcal{O} := Pv \cdot z \tag{3.3}$$

does not depend on the choice of z. In view of Proposition 3.1 the map

$$W \times (P\backslash Z)_{\max} \to (P\backslash Z)_{\max}; \quad (w, \mathcal{O}) \mapsto w \cdot \mathcal{O} \tag{3.4}$$

defines an action of W on $(P\backslash Z)_{\max}$. This action is a refinement of the action of W on $(P\backslash Z)_{\max}/\sim$ defined in Theorem 3.3. The condition that in each open P-orbit in Z the set of adapted points forms one MA-orbit is in particular satisfied in case Z is symmetric, i.e., in case H is an open subgroup of the fixed point subgroup of an involutive automorphism of G.

Proof of Theorem 3.3. Recall from Proposition 3.1 that the set of weakly admissible points is stable under $N_G(\mathfrak{a})$. In particular, if $\mathcal{O} \in (P\backslash Z)_{\max}$ and $z \in \mathcal{O}$ is weakly admissible, then $Pw \cdot z \in (P\backslash Z)_{\max}$ for all $w \in N_G(\mathfrak{a})$.

Let $\mathcal{O} \in (P\backslash Z)_{\max}$ and let $z_1, z_2 \in \mathcal{O}$ be weakly admissible. Let further $w \in N_G(\mathfrak{a})$. We claim that

$$Pw \cdot z_1 \sim Pw \cdot z_2. \tag{3.5}$$

By Remark 3.2 (b) there exists an $m \in M$ so that for all $X \in \mathfrak{a}^-$ we have $\mathfrak{h}_{z_1,X} = \mathrm{Ad}(m)\mathfrak{h}_{z_2,X}$. After replacing z_2 by $m \cdot z_2$ we may assume that $\mathfrak{h}_{z_1,X} = \mathfrak{h}_{z_2,X}$ for all $X \in \mathfrak{a}^-$. By Proposition 3.1 the limits $\mathfrak{h}_{z_1,X}$ and $\mathfrak{h}_{z_2,X}$ are equal for all order-regular $X \in \mathfrak{a}$. Fix now an order-regular $X \in \mathfrak{a}$ and let $w \in N_G(\mathfrak{a})$. Then $\mathrm{Ad}(w^{-1})X$ is order-regular, and hence

$$\mathfrak{h}_{w\cdot z_1,X} = Ad(w)\mathfrak{h}_{z_1,\mathrm{Ad}(w^{-1})X} = Ad(w)\mathfrak{h}_{z_2,\mathrm{Ad}(w^{-1})X} = \mathfrak{h}_{w\cdot z_2,X}.$$

This proves the claim (3.5).

From the claim it follows that for a given weakly adapted point z_0 and $v \in N_G(\mathfrak{a})$ we have $[Pv \cdot z] = [Pv \cdot z_0]$ for all weakly adapted point $z \in P \cdot z_0$. This proves the first assertion in the theorem and we thus obtain an action of W on $(P\backslash Z)_{\max}$.

We move on to prove the listed properties of this action. It follows from Proposition 3.1 that the action is transitive, and from Proposition 3.1 that the stabilizer of the equivalence class of open P-orbits is equal to $\mathcal{W}/MA$.

We move on to prove (iii). Let $w \in W$, let $v \in N_G(\mathfrak{a})$ be a representative for w and let $X \in \mathfrak{a} \cap \mathfrak{a}_{\mathfrak{h}}^{\perp}$ satisfy $Z_{\mathfrak{g}}(X) = \mathfrak{l}_Q$. Every open P-orbit admits by Proposition 3.2 an admissible point z with $X \in \mathfrak{a} \cap \mathfrak{h}_z^\perp$. This point is unique up to translation by an element in MA. Let $z_1, \dots, z_n$ be a set of adapted point representing the open P-orbits in Z and assume that

$$X \in \mathfrak{a} \cap \mathfrak{h}_{z_i}^\perp \qquad (1 \leq i \leq n).$$

Then $Pv \cdot z_i \in w \cdot (P\backslash Z)_{\mathrm{open}}$ for all $1 \leq i \leq n$. Moreover, the points $v \cdot z_1, \dots, v \cdot z_n$ are weakly admissible and

$$\mathrm{Ad}(w)X \in \mathfrak{a} \cap \mathfrak{h}_{v \cdot z_i}^\perp \qquad (1 \leq i \leq n).$$

If $Pv \cdot z_i = Pv \cdot z_j$ for some $1 \leq i < j \leq n$, then it follows from Proposition 3.1 (i) that $z_i \in MA \cdot z_j$, which leads to a contradiction. We conclude that the orbits $Pv \cdot z_1, \dots Pv \cdot z_n$ are pairwise distinct. It now suffices to show that the number of orbits in $w \cdot (P\backslash Z)_{\mathrm{open}}$ does not exceed the number of open orbits.

Let $z \in Z$ be any point so that $[P \cdot z] = w \cdot (P\backslash Z)$ and let $Y \in \mathfrak{a}^-$. There exists a $u \in N_G(\mathfrak{a})$ so that $\mathfrak{h}_{z,Y} = \mathrm{Ad}(u)\mathfrak{h}_\emptyset$. By Proposition 3.1 the assignment

$$\mathcal{O} \mapsto Pu^{-1} \cdot \mathcal{O} \tag{3.6}$$

maps $w \cdot (P\backslash Z)_{\mathrm{open}}$ to $(P\backslash Z)_{\mathrm{open}}$. We claim that this map is injective.

Let $\mathcal{O}, \mathcal{O}' \in w \cdot (P\backslash Z)_{\mathrm{open}}$ be so that $Pu^{-1} \cdot \mathcal{O} = Pu^{-1} \cdot \mathcal{O}'$. By Proposition 3.1 (i) there exist weakly adapted points $z \in \mathcal{O}$ and $z' \in \mathcal{O}'$ so that $\mathrm{Ad}(u)X \in \mathfrak{h}_z^\perp$ and $\mathrm{Ad}(u)X \in \mathfrak{h}_{z'}^\perp$. Now $u^{-1} \cdot z$ and $u^{-1} \cdot z'$ are adapted points in the same open orbit. It follows from Proposition 3.1 (i) that $u^{-1} \cdot z \in MAu^{-1} \cdot z'$. This implies that $z \in MA \cdot z'$, and hence

$$\mathcal{O} = P \cdot z = P \cdot z' = \mathcal{O}'.$$

This proves the injectivity of (3.6) and hence (iii).

Finally, we prove (iv). Let $\mathcal{O} \in (P\backslash Z)_{\max}$ and $w \in W$. If $z \in \mathcal{O}$ is weakly admissible, then for every order-regular element $X \in \mathfrak{a}$ we have $\mathfrak{a}_\mathcal{O} = \mathfrak{a} \cap \mathfrak{h}_{z,X}$. Since

$$\mathfrak{a} \cap \mathfrak{h}_{w \cdot z, X} = \mathrm{Ad}(w)\big(\mathfrak{a} \cap \mathfrak{h}_{z, \mathrm{Ad}(w^{-1}X)}\big) = \mathrm{Ad}(w)\mathfrak{a}_\mathcal{O} \qquad (X \in \mathfrak{a} \text{ order-regular})$$

it follows that $\mathfrak{a}_{Pw \cdot z} = \mathrm{Ad}(w)\mathfrak{a}_\mathcal{O}$. This proves (iv). $\square$

4 Distribution vectors of principal series representations

4.1 Basic Definitions

For a parabolic subgroup S of G with Langlands decomposition $S = M_S A_S N_S$ and a representation ξ of M_S on a Hilbert space V_ξ and $\lambda \in \mathfrak{a}_{S,\mathbb{C}}^*$, we define $C^\infty(S : \xi : \lambda)$ to be the space smooth vectors in the principal series representation induced from S with induction data $\xi \otimes \lambda \otimes 1$, i.e., the space of smooth V_ξ-valued functions f on G with the property that

$$f(manx) = a^{\lambda + \rho_S}\xi(m)f(x) \qquad (m \in M_S, a \in A_S, n \in N_S, x \in G).$$

Recall that K is a maximal compact subgroup of G. The pairing

$$C^\infty(S:\xi^\vee:-\lambda)\times C^\infty(S:\xi:\lambda)\to\mathbb{C};\quad (\chi,f)\mapsto\int_K\big(\chi(k),f(k)\big)\,dk$$

is non-degenerate and G-equivariant. We thus obtain a G-equivariant inclusion

$$C^\infty(S:\xi^\vee:-\lambda)\hookrightarrow C^\infty(S:\xi:\lambda)'.$$

For a smooth manifold $\mathcal{M}$ and a Hilbert space V we define $\mathcal{E}(\mathcal{M},V)$ to be the vector space of all smooth functions $\mathcal{M}\to V$, and $\mathcal{D}(\mathcal{M},V)$ to be the subspace of $\mathcal{E}(\mathcal{M},V)$ consisting of all functions with compact support. We write $\mathcal{D}'(\mathcal{M},V)$ for the continuous dual of $\mathcal{D}(\mathcal{M},V)$. Note that in case $\mathcal{M}$ is an open subset of G, there is a natural injection $\mathcal{E}(\mathcal{M},V^*)\hookrightarrow\mathcal{D}'(\mathcal{M},V)$ (using the Haar measure on G to identify densities with functions).

Let $L^\vee$ and $R^\vee$ be the contragredients of the left-regular representation L and the right-regular representation R, respectively. We define $\mathcal{D}'(S:\xi:\lambda)$ to be the subspace of $\mathcal{D}'(G,V_\xi)$ consisting of all distributions μ such that

$$L^\vee(man)\mu=a^{\lambda-\rho_S}\xi^\vee(m^{-1})\mu\qquad(m\in M_S,a\in A_S,n\in N_S).\tag{4.1}$$

Let V be a Hilbert space. We write $\mathcal{D}'(G,V)^H$ for the subspace of $\mathcal{D}'(G,V)$ of distributions that are invariant under the right-regular representation of H on $\mathcal{D}'(G,V)$, i.e.,

$$\mathcal{D}'(G,V)^H=\{\mu\in\mathcal{D}'(G,V):R^\vee(h)\mu=\mu\text{ for all }h\in H\}.$$

If $\phi\in\mathcal{D}(G,V)$, then in view of the identification $Z=G/H$ the function

$$gH\mapsto\int_H\phi(gh)\,dh$$

defines an element of $\mathcal{D}(Z,V)$. The map $\mathcal{D}(G,V)\to\mathcal{D}(Z,V)$ thus obtained is continuous. Moreover, the induced map

$$\mathcal{D}'(Z,V)\to\mathcal{D}'(G,V)^H;\quad \mu\mapsto\Big(\phi\mapsto\mu\Big(\int_H\phi(\,\cdot\,h)\,dh\Big)\Big)\tag{4.2}$$

is a topological isomorphism. We will use this isomorphism to identify $\mathcal{D}'(Z,V)$ with $\mathcal{D}'(G,V)^H$. Finally, we define the space

$$\mathcal{D}'(Z,S:\xi:\lambda)=\mathcal{D}'(Z,V_\xi)\cap\mathcal{D}'(S:\xi:\lambda).$$

4.2 Distribution vectors versus functionals

Let $S=M_SA_SN_S$, (ξ,V_ξ) and $\lambda\in\mathfrak{a}^*_{S,\mathbb{C}}$ be as before. In this section we compare the spaces $C^\infty(S:\xi:\lambda)'$ and $\mathcal{D}'(S:\xi:\lambda)$. We follow for this the analysis in [14, Section 2.3].

Let $\psi_0 \in \mathcal{D}(G)$ satisfy

$$\int_{M_S}\int_{A_S}\int_{N_S} a^{2\rho_S}\psi_0(manx)\,dn\,da\,dm = 1 \qquad (x \in G).$$

One may for instance take $\psi_0 \in \mathcal{D}(G)$ to be right K-invariant and satisfying

$$\int_G \psi_0(x)a_S^{2\rho_S}(x)\,dx = 1,$$

where $a_S : G \to A_S$ is the map given by

$$x \in N_S a_S(x) M_S K \qquad (x \in G).$$

For $\mu \in \mathcal{D}'(S:\xi:\lambda)$, let $\omega^S_{\xi,\lambda}\mu \in C^\infty(S:\xi:\lambda)'$ be given by

$$\big(\omega^S_{\xi,\lambda}\mu\big)(f) = \mu(\psi_0 f) \qquad \big(f \in C^\infty(S:\xi:\lambda)\big).$$

The map

$$\omega^S_{\xi,\lambda} : \mathcal{D}'(S:\xi:\lambda) \to C^\infty(S:\xi:\lambda)' \tag{4.1}$$

we thus obtain is a topological isomorphism; it is easily seen that the map

$$\theta^S_{\xi,\lambda} : C^\infty(S:\xi:\lambda)' \to \mathcal{D}'(S:\xi:\lambda),$$

which for $\eta \in C^\infty(S:\xi:\lambda)'$ and $\phi \in \mathcal{D}(G,V_\xi)$ is given by

$$\big(\theta^S_{\xi,\lambda}\eta\big)(\phi) = \eta\Big(x \mapsto \int_{M_S}\int_{A_S}\int_{N_S} a^{-\lambda+\rho_S}\xi(m^{-1})\phi(manx)\,dn\,da\,dm\Big), \tag{4.2}$$

is the inverse of $\omega^S_{\xi,\lambda}$. In particular it follows that $\omega^S_{\xi,\lambda}$ does not depend on the choice of the function ψ_0. Note that $\theta^S_{\xi,\lambda}$ intertwines the representation $\pi^\vee_{S:\xi:\lambda}$ on $C^\infty(S:\xi:\lambda)'$ with $R^\vee$ on $\mathcal{D}'(S:\xi:\lambda)$. The restriction of $\omega^S_{\xi,\lambda}$ to $\mathcal{D}'(S:\xi:\lambda)^H$ is a G-equivariant isomorphism to the space of H-fixed functionals on $C^\infty(S:\xi:\lambda)$.

4.3 Intertwining operators

Let $S = M_SA_SN_S$, (ξ, V_ξ) and $\lambda \in \mathfrak{a}^*_{S,\mathbb{C}}$ be as before. For $u \in \mathcal{U}(\mathfrak{g})$ we set

$$p_{S,\xi,\lambda,u} : \mathcal{E}(G,V_\xi) \to [0,\infty]; \quad \phi \mapsto \int_G \|a_S(x)^{-\lambda+\rho_S}\, R(u)\phi(x)\|_\xi\,dx$$

and endow the space

$$\mathcal{V}_{S,\xi,\lambda} := \big\{\phi \in \mathcal{E}(G,V_\xi) : p_{S,\xi,\lambda,u}(\phi) < \infty \text{ for every } u \in \mathcal{U}(\mathfrak{g})\big\},$$

with the Fréchet topology induced by the seminorms $p_{S,\xi,\lambda,u}$. Note that $\mathcal{D}(G,V_\xi) \subseteq \mathcal{V}_{S,\xi,\lambda}$. Further, for two parabolic subgroups S_1 and S_2 of G with $A_{S_1} = A_{S_2} = A_S$, we write

$$A(S_2:S_1:\xi:\lambda) : C^\infty(S_1:\xi:\lambda) \to C^\infty(S_2:\xi:\lambda)$$

for the standard Knapp-Stein intertwining operators and define

$$\mathcal{A}(S_2:S_1:\xi:\lambda) := \theta^{S_2}_{\xi,\lambda} \circ A(S_1:S_2:\xi:\lambda)^* \circ \omega^{S_1}_{\xi,\lambda}.$$

We assume that $A_S \subseteq A$ and identify $\mathfrak{a}^*_{S,\mathbb{C}}$ with the annihilator of $\mathfrak{m}_S \cap \mathfrak{a}$ in $\mathfrak{a}^*_{\mathbb{C}}$.

Proposition 4.1. *Let S_1, S_2, ξ and λ be as above. The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{D}'(S_1 : \xi : \lambda) \arrow[r, "\mathcal{A}(S_2:S_1:\xi:\lambda)"] \arrow[d, shift right, "\omega^{S_1}_{\xi,\lambda}"'] & \mathcal{D}'(S_2 : \xi : \lambda) \arrow[d, shift right, "\omega^{S_2}_{\xi,\lambda}"'] \\
C^\infty(S_1 : \xi : \lambda)' \arrow[r, "A(S_1:S_2:\xi:\lambda)^*"] \arrow[u, shift right, "\theta^{S_1}_{\xi,\lambda}"'] & C^\infty(S_2 : \xi : \lambda)' \arrow[u, shift right, "\theta^{S_2}_{\xi,\lambda}"']
\end{tikzcd}$$

Assume that $\lambda \in \mathfrak{a}^*_{S,\mathbb{C}}$ satisfies

$$\langle \operatorname{Re}\lambda, \alpha\rangle > 0 \qquad \big(\alpha \in \Sigma(\mathfrak{a} : S_2) \cap -\Sigma(\mathfrak{a} : S_1)\big),$$

Then for every $\phi \in \mathcal{V}_{S_2,\xi,\lambda}$ and every $x \in G$ the integral

$$\int_{N_{S_2}\cap \overline{N}_{S_1}} \phi(nx)\, dx$$

is absolutely convergent and the function $\int_{N_{S_2}\cap \overline{N}_{S_1}} \phi(n\,\cdot\,)\, dx$ thus obtained is an element of $\mathcal{V}_{S_1,\xi,\lambda}$. Moreover, the map

$$\mathcal{V}_{S_2,\xi,\lambda} \to \mathcal{V}_{S_1,\xi,\lambda}; \quad \phi \mapsto \int_{N_{S_2}\cap \overline{N}_{S_1}} \phi(n\,\cdot\,)\, dn$$

is continuous. Finally, if $\mu \in \mathcal{D}'(S_1 : \xi : \lambda)$, then μ extends to a continuous linear functional on $\mathcal{V}_{S_1,\xi,\lambda}$, and the distribution $\mathcal{A}(S_2 : S_1 : \xi : \lambda)\mu \in \mathcal{D}'(S_2 : \xi : \lambda)$ is given by

$$\big[\mathcal{A}(S_2 : S_1 : \xi : \lambda)\mu\big](\phi) = \mu\Big(\int_{N_{S_2}\cap \overline{N}_{S_1}} \phi(n\,\cdot\,)\, dn\Big) \qquad \big(\phi \in \mathcal{V}_{S_2,\xi,\lambda}\big). \tag{4.1}$$

For the proof of the proposition we refer to Appendix B.

We define an inner product $\langle\cdot,\cdot\rangle_{S,\xi,\lambda}$ on $C^\infty(S : \xi : \lambda)$ by

$$\langle \phi, \psi\rangle_{S,\xi,\lambda} = \int_K \langle \phi(k), \psi(k)\rangle_\xi\, dk \qquad \big(\phi, \psi \in C^\infty(S : \xi : \lambda)\big), \tag{4.2}$$

where $\langle\cdot,\cdot\rangle_\xi$ is the inner product on V_ξ. We consider parabolic subgroup S_1 and S_2 with $A_{S_1} = A_{S_2} = A_S \subseteq A$ as before. The adjoint of $A(S_2 : S_1 : \xi : \lambda)$ with respect to (4.2) is given by

$$A(S_2 : S_1 : \xi : \lambda)^\dagger = A(S_1 : S_2 : \xi : -\overline{\lambda})$$

The composition $A(S_2 : S_1 : \xi : \lambda) \circ A(S_1 : S_2 : \xi : \lambda)$ is an intertwining operator from $C^\infty(S_1 : \xi : \lambda)$ to itself. It is therefore given by multiplication by a scalar. As in [24, §14.6] we choose a meromorphic function on $\mathfrak{a}^*_{S,\mathbb{C}}$

$$\lambda \mapsto \gamma(S_1 : S_2 : \xi : \lambda)$$

that is real and non-negative on $\mathfrak{a}_S^*$ and satisfies the identity of meromorphic operators

$$A(S_2 : S_1 : \xi : \lambda) \circ A(S_1 : S_2 : \xi : \lambda) = \gamma(S_2 : S_1 : \xi : \lambda)\gamma(S_1 : S_2 : \xi : \lambda)\mathrm{Id}.$$

We may choose these functions so that for all $\lambda \in \mathfrak{a}_{S,\mathbb{C}}^*$

$$\gamma(S_1 : S_2 : \xi : \lambda) = \overline{\gamma(S_2 : S_1 : \xi : -\overline{\lambda})},$$

$$\gamma(S_1 : S_2 : \xi : \lambda) = \gamma(S_1 : S_2 : \xi' : \lambda) \qquad (\xi' \simeq \xi, \lambda \in \mathfrak{a}_{S,\mathbb{C}}^*)$$

for every ξ' equivalent to ξ, and

$$\gamma\big(vS_1v^{-1} : vS_2v^{-1} : v \cdot \xi : \mathrm{Ad}^*(v)\lambda\big) = \gamma(S_1 : S_2 : \xi : \lambda)$$

for every $v \in N_G(\mathfrak{a}_S)$. Here $v \cdot \xi$ is the representation of M_S with representation space V_ξ given by

$$(v \cdot \xi)(m) = \xi(v^{-1}mv) \qquad (m \in M_S).$$

If we normalize the intertwining operators with these γ-functions as

$$A^\circ(S_1 : S_2 : \xi : \lambda) := \frac{1}{\gamma(S_1 : S_2 : \xi : \lambda)} A(S_1 : S_2 : \xi : \lambda),$$

then we obtain the identities

$$A^\circ(S_3 : S_1 : \xi : \lambda) = A^\circ(S_3 : S_2 : \xi : \lambda) \circ A^\circ(S_2 : S_1 : \xi : \lambda)$$

for all parabolic subgroups S_1, S_2, S_3 with $A_{S_1} = A_{S_2} = A_{S_3} = A_S \subseteq A$, $\lambda \in (\mathfrak{a}_S)_\mathbb{C}^*$ and unitary representations ξ of $M_{S_1} = M_{S_2} = M_{S_3} = M_S$. In particular,

$$A^\circ(S_1 : S_2 : \xi : \lambda)^{-1} = A^\circ(S_2 : S_1 : \xi : \lambda),$$

and hence the operator $A^\circ(S_1 : S_2 : \xi : \lambda)$ is unitary if $\lambda \in i\mathfrak{a}_S^*$.

For $v \in N_G(\mathfrak{a}_S)$ we define the intertwining operator

$$\mathcal{I}_v(S : \xi : \lambda) : \mathcal{D}'(S : \xi : \lambda) \to \mathcal{D}'(S : v \cdot \xi : \mathrm{Ad}^*(v)\lambda)$$

by

$$\mathcal{I}_v(S : \xi : \lambda) := L^\vee(v) \circ \mathcal{A}(v^{-1}Sv : S : \xi : \lambda)$$

and the corresponding normalized intertwining operator

$$\mathcal{I}_v^\circ(S : \xi : \lambda) : \mathcal{D}'(S : \xi : \lambda) \to \mathcal{D}'(S : v \cdot \xi : \mathrm{Ad}^*(v)\lambda)$$

by

$$\mathcal{I}_v^\circ(S : \xi : \lambda) := \frac{1}{\gamma(v^{-1}Sv : S : \xi : \lambda)} \mathcal{I}_v(S : \xi : \lambda).$$

We note that

$$\mathcal{I}_v^\circ(S : w \cdot \xi : w \cdot \lambda) \circ \mathcal{I}_w^\circ(S : \xi : \lambda) = \mathcal{I}_{vw}^\circ(S : \xi : \lambda) \qquad \big(v, w \in N_G(\mathfrak{a}_S)\big).$$

The family of operators $\lambda \mapsto \mathcal{I}_v^\circ(S : \xi : \lambda)$ is meromorphic. There exists a locally finite union $\mathcal{H}$ of complex affine hyperplanes in $\mathfrak{a}_{S,\mathbb{C}}^*$ of the form $\{\lambda \in \mathfrak{a}_{S,\mathbb{C}}^* : \lambda(\alpha^\vee) = c\}$ for some $\alpha \in \Sigma(S)$ and $c \in \mathbb{R}$, so that for all unitary representations ξ of M_S the poles of the families $\lambda \mapsto \mathcal{I}_v(S : \xi : \lambda)$ and $\lambda \mapsto \mathcal{I}_v^\circ(S : \xi : \lambda)$ lie on $\mathcal{H}$.

4.4 Comparison between induction from different parabolic subgroups

Let S and T be parabolic subgroups of G and assume that $S \subseteq T$. Let $S = M_S A_S N_S$ $T = M_T A_T N_T$ be Langlands decompositions of S and T, respectively, and assume that $A_T \subseteq A_S \subseteq A$. Observe that $S \cap M_T$ is a parabolic subgroup of M_T. Moreover, $\mathfrak{a} = \mathfrak{a}_T \oplus (\mathfrak{m}_T \cap \mathfrak{a})$. We identify $\mathfrak{a}_T^*$ as a subspace $\mathfrak{a}^*$ by extending the functionals by 0 on $\mathfrak{m}_T \cap \mathfrak{a}$. Note that

$$\rho_{S \cap M_T} = \rho_S - \rho_T.$$

Let (ξ, V_ξ) be a representation of M_T on a Hilbert space and assume that

$$M_T \cap N_S \subseteq \ker(\xi).$$

Let further $\lambda \in \mathfrak{a}_{T,\mathbb{C}}^*$. There is a natural M_T-equivariant embedding

$$i : \xi \hookrightarrow \mathrm{Ind}_{M_T \cap S}^{M_T}(\xi|_{M_S} \otimes \rho_T - \rho_S \otimes 1),$$

see [1, Lemma 4.4]. Concretely, the map i from V_ξ into $C^\infty(M_T \cap S : \xi|_{M_S} : \rho_T - \rho_S)$ of smooth vectors for the principal series representation on the right-hand side is given by

$$i(v)(m_T) = \xi(m_T)v, \qquad (v \in V_\xi,\ m_T \in M_T).$$

Let $\lambda \in \mathfrak{a}_{T,\mathbb{C}}^*$. Using induction by stages, we obtain a G-equivariant embedding

$$\mathrm{Ind}_T^G(\xi \otimes \lambda \otimes 1) \hookrightarrow \mathrm{Ind}_S^G(\xi|_{M_S} \otimes (\lambda + \rho_T - \rho_S) \otimes 1).$$

On the level of smooth vectors this results in a G-equivariant embedding

$$i_{\xi,\lambda}^{\#} : C^\infty(T : \xi : \lambda) \hookrightarrow C^\infty(S : \xi|_{M_S} : \lambda + \rho_T - \rho_S),$$

which is the natural inclusion map. Note that $i_{\xi^\vee,-\lambda}^{\#}$ extends to a continuous inclusion

$$\mathcal{D}'(T : \xi : \lambda) \hookrightarrow \mathcal{D}'(S : \xi|_{M_S} : \lambda - \rho_T + \rho_S).$$

Using the isomorphisms from Section 4.2, we now arrive at the following result.

Proposition 4.1. *Let $\lambda \in \mathfrak{a}_{T,\mathbb{C}}^*$ and let (ξ, V_ξ) be a representation of M_T on a Hilbert space and assume that $M_T \cap N_S \subseteq \ker(\xi)$. There exists a G-equivariant injective map*

$$C^\infty(T : \xi : \lambda)' \hookrightarrow C^\infty(S : \xi|_{M_S} : \lambda - \rho_T + \rho_S)'$$

so that

$$\begin{tikzcd}
\mathcal{D}'(T : \xi : \lambda) \arrow[r, hook] \arrow[d, shift right, "\omega^T_{\xi,\lambda}"'] & \mathcal{D}'(S : \xi|_{M_S} : \lambda - \rho_T + \rho_S) \arrow[d, shift right, "\omega^S_{\xi|_{M_S},\lambda-\rho_T+\rho_S}"'] \\
C^\infty(T : \xi : \lambda)' \arrow[r, hook] \arrow[u, shift right, "\theta^T_{\xi,\lambda}"'] & C^\infty(S : \xi|_{M_S} : \lambda - \rho_T + \rho_S)' \arrow[u, shift right, "\theta^S_{\xi|_{M_S},\lambda-\rho_T+\rho_S}"']
\end{tikzcd}$$

is a commuting diagram.

4.5 $L_{Q,\mathrm{nc}}$-spherical representations of M_Q

Let $Q = M_Q A_Q N_Q$ be the Langlands decomposition of Q with $A_Q \subseteq A$. Then $L_Q = M_Q A_Q$. We recall that $\mathfrak{l}_{Q,\mathrm{nc}}$ is the sum of the simple ideals of non-compact type in $\mathfrak{l}_Q$. We write $L_{Q,\mathrm{nc}}$ for the connected subgroup of L_Q with Lie algebra $\mathfrak{l}_{Q,\mathrm{nc}}$. Note that $L_{Q,\mathrm{nc}} \subseteq M_Q$.

We first look at a few properties of M_Q and $L_{Q,\mathrm{nc}}$ which will be needed in this and later sections.

Lemma 4.1.

(i) $L_{Q,\mathrm{nc}}$ *is a closed normal subgroup of* M_Q.

(ii) $M_Q = ML_{Q,\mathrm{nc}} \simeq M \times_{M\cap L_{Q,\mathrm{nc}}} L_{Q,\mathrm{nc}}$.

(iii) The group $M_{L,\mathrm{nc}}$ *acts trivially on* $M_Q/M_{L,\mathrm{nc}}$.

(iv) Let $z \in Z$ *be a weakly adapted point and* $w \in N_G(A)$ *so that* $\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$ *for some order-regular element* $X \in \mathfrak{a}^-$. *Then* $wL_{Q,\mathrm{nc}}w^{-1} \subseteq M_Q \cap H_z$. *If* w *normalizes* $\mathfrak{a}_\mathfrak{h}$, *or equivalently if* $\mathfrak{a}_{P\cdot z} = \mathfrak{a}_\mathfrak{h}$, *then* $L_{Q,\mathrm{nc}} \subseteq M_Q \cap H_z$.

Proof. Since $\mathfrak{m}_Q$ is reductive, there exists an ideal $\mathfrak{m}_{Q,c}$ complementary to $\mathfrak{l}_{Q,\mathrm{nc}}$. The group $L_{Q,\mathrm{nc}}$ is equal to the connected component of $Z_{M_Q}(\mathfrak{m}_{Q,c})$ and therefore $L_{Q,\mathrm{nc}}$ is closed. As $\mathfrak{m}_Q = \mathfrak{m} + \mathfrak{l}_{Q,\mathrm{nc}}$ the set $ML_{Q,\mathrm{nc}}$ is open. Moreover, since M is compact and $L_{Q,\mathrm{nc}}$ is closed, $ML_{Q,\mathrm{nc}}$ is also closed. From the fact that M intersects with every connected component of M_Q assertion (ii) now follows.

The subalgebra $\mathfrak{l}_{Q,\mathrm{nc}}$ is an M-stable ideal of $\mathfrak{m}_Q$. Assertion (i) therefore follows from (ii).

Since $L_{Q,\mathrm{nc}}$ is normal in M_Q, it acts trivially on the quotient $M_Q/L_{Q,\mathrm{nc}}$, and hence (iii) follows.

Finally we prove (iv). Let $\mathcal{O}$ be a P-orbit in Z of maximal rank and let $z \in \mathcal{O}$ be weakly admissible. We select a regular element $X \in \mathfrak{a}^-$. Then there exists a $w \in N_G(\mathfrak{a})$ so that $\mathfrak{h}_{z,X} = \mathrm{Ad}(w)\mathfrak{h}_\emptyset$. By Proposition 3.1 (ii) the point $w^{-1}\cdot z$ is adapted. Therefore, $\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathfrak{h}_{w^{-1}\cdot z} = \mathrm{Ad}(w^{-1})\mathfrak{h}_z$, and hence $\mathrm{Ad}(w)\mathfrak{l}_{Q,\mathrm{nc}} \subseteq \mathfrak{h}_z$. The assertion now follows as $L_{Q,\mathrm{nc}}$ is connected. By Remark 3.2 the roots of $\mathfrak{a}$ in $\mathfrak{l}_{Q,\mathrm{nc}}$ are precisely those roots that vanish on $\mathfrak{a} \cap \mathfrak{a}_\mathfrak{h}^\perp$. If w normalizes $\mathfrak{a}_\mathfrak{h}$, then it follows that w normalizes $\mathfrak{l}_{Q,\mathrm{nc}}$ and hence $L_{Q,\mathrm{nc}}$. $\square$

Given a continuous representation of M_Q in a Fréchet space V, we denote its space of smooth vectors by V^∞ and equip it with the structure of a continuous Fréchet M_Q-module in the usual way. The continuous linear dual we denote by $V^{\infty\prime}$.

Corollary 4.2. *Let* (ξ, V_ξ) *be an irreducible continuous representation of* M_Q *in a Fréchet space* V *such that*

$$(V_\xi^{\infty\prime})^{L_{Q,\mathrm{nc}}} \neq 0.$$

Then $\xi|_{L_{Q,\mathrm{nc}}}$ *is trivial and* $\xi|_M$ *is irreducible. In particular,* ξ *is finite dimensional and unitarizable. In particular this is the case if*

$$(V_\xi^{\infty\prime})^{H_z} \neq \{0\}.$$

for some weakly adapted point z in a P-orbit $\mathcal{O}$ in Z with $\mathfrak{a}_{\mathcal{O}} = \mathfrak{a}_{\mathfrak{h}}$.

Proof. The proof is the same as the one for [4, Corollary 4.4]. For convenience we give it here. Let $\eta \in (V_\xi^{\infty\prime})^{L_{Q,\mathrm{nc}}}$. If $\eta \neq 0$, then there is a unique injective continuous linear M_Q-equivariant map $j : V^\infty \to \mathcal{E}(M_Q/L_{Q,\mathrm{nc}})$ such that $j^*\delta = \eta$, with δ denoting the Dirac measure of $M_Q/L_{Q,\mathrm{nc}}$ at $eL_{Q,\mathrm{nc}}$. It follows from Lemma 4.1 (iii) that $L_{Q,\mathrm{nc}}$ acts trivially $\mathcal{E}(M_Q/L_{Q,\mathrm{nc}})$ and hence on V^∞. We conclude that $L_{Q,\mathrm{nc}} \subseteq \ker(\xi)$. By application of Lemma 4.1 (ii) it follows that $\xi|_M$ is irreducible. The final assertion follows from Lemma 4.1 (iv). $\square$

Let $\widehat{M}_{Q,\mathrm{fu}}$ be the set of equivalence classes of finite dimensional irreducible unitary representations of M_Q.

Corollary 4.3. *Every representation in $\widehat{M}_{Q,\mathrm{fu}}$ restricts to the trivial representation on $L_{Q,\mathrm{nc}}$. The restriction map $\xi \mapsto \xi|_{M_Q} := \xi|_M$ induces an injection*

$$\widehat{M}_{Q,\mathrm{fu}} \hookrightarrow \widehat{M}.$$

The image of this injection equals

$$\left\{[\xi] \in \widehat{M} : \xi\big|_{M\cap L_{Q,\mathrm{nc}}} \text{ is trivial}\right\}.$$

Proof. Since the $L_{Q,\mathrm{nc}}$ is connected semisimple of the non-compact type, the restriction of a representation from $\widehat{M}_{Q,\mathrm{fu}}$ to $L_{Q,\mathrm{nc}}$ is trivial. The remaining assertions follow from Lemma 4.1. $\square$

4.6 Comparison between induction from P and Q

The following proposition follows directly from Corollary 4.2 and the comparison of induction from different parabolic subgroups in Section 4.4.

Proposition 4.1. *Let ξ be a representation of M_Q on a Hilbert space V_ξ and $\lambda \in \mathfrak{a}^*_{Q,\mathbb{C}}$. Assume that*

$$(V_\xi^{\infty\prime})^{L_{Q,\mathrm{nc}}} \neq \{0\}.$$

Then ξ is finite dimensional, $\xi|_M$ is irreducible and

$$\mathcal{D}'(Q : \xi : \lambda) \subseteq \mathcal{D}'(P : \xi|_M : \lambda + \rho_P - \rho_Q).$$

Moreover, there exists a natural inclusion

$$C^\infty(Q : \xi : \lambda)' \hookrightarrow C^\infty(P : \xi|_M : \lambda + \rho_P - \rho_Q)'$$

so that

$$\begin{tikzcd}
\mathcal{D}'(Q : \xi : \lambda) \arrow[r, hook] \arrow[d, shift right, "\omega^Q_{\xi,\lambda}"'] & \mathcal{D}'(P : \xi|_M : \lambda + \rho_P - \rho_Q) \arrow[d, shift right, "\omega^P_{\xi|_M,\lambda+\rho_P-\rho_Q}"'] \\
C^\infty(Q : \xi : \lambda)' \arrow[r, hook] \arrow[u, shift right, "\theta^Q_{\xi,\lambda}"'] & C^\infty(P : \xi|_M : \lambda + \rho_P - \rho_Q)' \arrow[u, shift right, "\theta^P_{\xi|_M,\lambda+\rho_P-\rho_Q}"']
\end{tikzcd}$$

is a commuting diagram. In particular this is the case if

$$(V_\xi^{\infty\prime})^{H_z} \neq \{0\}.$$

for some weakly adapted point z contained in a P-orbit $\mathcal{O}$ of maximal rank in Z with $\mathfrak{a}_{\mathcal{O}} = \mathfrak{a}_{\mathfrak{h}}$.

The following describes $\mathcal{D}'(Q : \xi : \lambda)$ as a subspace of $\mathcal{D}'(P : \xi|_M : \lambda + \rho_P - \rho_Q)$.

Lemma 4.2. *Let $\sigma \in \widehat{M}$ be so that $\sigma|_{M\cap L_{Q,\mathrm{nc}}}$ is trivial, and let $\lambda \in \mathfrak{a}^*_{Q,\mathbb{C}}$. Let ξ be the representation of M_Q so that $\xi|_{L_{Q,\mathrm{nc}}}$ is trivial and $\xi|_M = \sigma$. If $\mu \in \mathcal{D}'(P : \sigma : \lambda+\rho_P-\rho_Q)$ satisfies*

$$L^\vee(\overline{n})\mu = \mu \qquad (n \in M_Q \cap \overline{N}_P),$$

then $\mu \in \mathcal{D}'(Q : \xi : \lambda)$.

Proof. Let G_μ be the closed subgroup of G consisting of elements $g \in G$ so that

$$L^\vee(g)\mu = \mu.$$

Since $M_Q = ML_{Q,\mathrm{nc}}$, see Lemma 4.1, it suffices to prove that $L_{Q,\mathrm{nc}} \subseteq G_\mu$. The latter follows from the assumptions as $L_{Q,\mathrm{nc}}$ is the smallest closed subgroup of G containing $M_Q \cap N_P$ and $M_Q \cap \overline{N}_P$.

□

5 Support and transversal degree

Throughout this section we fix $\sigma \in \widehat{M}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. In this section we study the support and transversal derivatives of distributions in $\mathcal{D}'(Z, P : \sigma : \lambda)$.

5.1 Transversal degree

Let $\mathcal{M}$ be a smooth submanifold of G and let U be an open subset of G.

We fix a set of smooth vector fields $v_1, \dots, v_n$ on U so that at every point $y \in \mathcal{M} \cap U$

$$T_yG = \mathbb{R}v_1(y) \oplus \cdots \oplus \mathbb{R}v_n(y) \oplus T_y\mathcal{M}.$$

For a multi-index β in n-variables, let ∂^β be the differential operator

$$C^\infty(U : V) \to C^\infty(U : V)$$

given by

$$\partial^\beta\phi = \underbrace{v_1 \cdots v_1}_{\beta_1 \text{ times}} \cdots \underbrace{v_n \cdots v_n}_{\beta_n \text{ times}}(\phi) \qquad \big(\phi \in C^\infty(U : V)\big).$$

Let $\mu \in \mathcal{D}'(G, V)$ and assume that $\operatorname{supp}\mu = \mathcal{M} \cap U$. It follows from [48, p. 102] that for every multi-index β there exists a distribution $\mu_\beta \in \mathcal{D}'(\mathcal{M} \cap U, V)$ such that for all $\phi \in \mathcal{D}(U, V)$

$$\mu(\phi) = \sum_\beta \mu_\beta\big(\partial^\beta\phi\big).$$

This decomposition of μ is unique. Let $k_U = \max\{|\beta| : \mu_\beta \neq 0\}$. The *transversal degree of μ at a point $y \in \mathcal{M}$*, is defined to be the minimum of the numbers k_U, where U runs over all neighborhoods of y in G. The transversal degree is independent of the choice of the vector fields v_i.

For a distribution $\mu \in \mathcal{D}'(Z, P : \sigma : \lambda)$ let $(P\backslash Z)_\mu$ be the set of $\mathcal{O} \in P\backslash Z$ with the property that there exists an open neighborhood U of $\mathcal{O}$ in G such that

$$\operatorname{supp}\mu \cap U = \mathcal{O}.$$

The proof for the following proposition can be found in Remark 5.2 in [35].

Proposition 5.1. *Let $\mu \in \mathcal{D}'(Z, P : \sigma : \lambda)$. Then*

$$\operatorname{supp}\mu = \overline{\bigcup_{\mathcal{O}\in(P\backslash Z)_\mu} \mathcal{O}}.$$

Let $\mu \in \mathcal{D}'(Z, P : \sigma : \lambda)$ and let $\mathcal{O} \in (P\backslash Z)_\mu$. Then the transversal degree of μ at $z \in \mathcal{O}$ does not depend on $z \in \mathcal{O}$, see [35, Lemma 5.5]. Therefore, we may define the *transversal degree of μ at the orbit $\mathcal{O}$* to be the transversal degree of μ at any point $z \in \mathcal{O}$. We write $\operatorname{trdeg}_{\mathcal{O}}(\mu)$ for the transversal degree of μ at $\mathcal{O}$.

5.2 Principal asymptotics

In this section we introduce are main tool, principal asymptotics from [35], to analyse the support and transversal degree of distributions in $\mathcal{D}'(Z, P : \sigma : \lambda)$ and use to obtain some first restrictions on the support and transversal derivatives for given λ and σ.

Let $X \in \mathfrak{a}^-$ be order-regular and let $z \in Z$. We define the $\mathfrak{a}$-stable subalgebra

$$\overline{\mathfrak{n}}_{z,X} := \mathfrak{h}_{z,X} \cap \overline{\mathfrak{n}}_P$$

and write $\overline{N}_{z,X}$ for the connected subgroup of G with Lie algebra equal to $\overline{\mathfrak{n}}_{z,X}$. Let $\overline{\mathfrak{n}}_X^z$ be an $\mathfrak{a}$-stable complementary subspace to $\overline{\mathfrak{n}}_{z,X}$ in $\overline{\mathfrak{n}}_P$, so that

$$\overline{\mathfrak{n}}_P = \overline{\mathfrak{n}}_{z,X} \oplus \overline{\mathfrak{n}}_X^z.$$

We define $\Sigma(\overline{\mathfrak{n}}_X^z; \mathfrak{a})$ to be the set of roots of $\mathfrak{a}$ in $\overline{\mathfrak{n}}_X^z$ and $\rho_{\mathcal{O},X} \in \mathfrak{a}^*$ by setting

$$\rho_{\mathcal{O},X}(Y) = \frac{1}{2}\operatorname{tr}\left(\operatorname{ad}(Y)\big|_{\overline{\mathfrak{n}}_{z,X}}\right) \qquad (Y \in \mathfrak{a}).$$

Note that for $m \in M$, $a \in A$ and $n \in N_P$ we have

$$\overline{\mathfrak{n}}_{man\cdot z,X} = \operatorname{Ad}(m)\overline{\mathfrak{n}}_{z,X},$$

and thus $\Sigma(\overline{\mathfrak{n}}_X^z; \mathfrak{a})$ and $\rho_{\mathcal{O},X}$ only depend on $\mathcal{O}$, not on the choice for $z \in \mathcal{O}$.

Let $e_1, \dots, e_n$ be a basis of $\overline{\mathfrak{n}}_X^z$ consisting of joint eigenvectors for the action of $\operatorname{ad}(\mathfrak{a})$ on $\overline{\mathfrak{n}}_X^z$. For a multi-index β, let $\kappa_\beta \in \mathbb{N}_0\Sigma(\overline{\mathfrak{n}}_X^z; \mathfrak{a})$ be the $\mathfrak{a}$-weight of $e_1^{\beta_1} \cdots e_n^{\beta_n} \in \mathcal{U}(\overline{\mathfrak{n}}_P)$, where $\mathcal{U}(\overline{\mathfrak{n}}_P)$ denotes the universal enveloping algebra of $\overline{\mathfrak{n}}_P$. We write ∂^β for the differential operator on $P\overline{N}$ that for $\phi \in \mathcal{E}(P\overline{N}, V)$ is given by

$$\big(\partial^\beta \phi\big)(p\overline{n}) := \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} \phi\big(p \exp(\sum_{i=1}^n x_i e_i)\overline{n}\big)\Big|_{x_i=0} \qquad (p \in P, \overline{n} \in \overline{N}).$$

The following theorem was proven in [35]; see Theorem 5.1 and its proof and Corollary 5.3. We formulate the results here using distributions instead of functionals, for which we use the identifications in Section 4.

Proposition 5.1. *Let $\mu \in \mathcal{D}'(Z, P : \sigma : \lambda)$ and let $\mathcal{O} \in (P\backslash Z)_\mu$. We fix a point $z \in \mathcal{O}$ and identify μ with an H_z-invariant distribution in $\mathcal{D}'(P : \sigma : \lambda)$ as in section 4, for which we, with abuse of notation, also write μ. Let $X \in \mathfrak{a}^-$ be order regular and satisfy*

$$\kappa_\beta(X) \neq \kappa_\gamma(X) \tag{5.1}$$

for any two multi-indices β, γ with $|\beta|, |\gamma| \leq \operatorname{trdeg}_{\mathcal{O}}(\mu)$ and $\kappa_\beta \neq \kappa_\gamma$. Then there exist a left-P-invariant open neighborhood Ω of e in G, a $\kappa \in \mathbb{N}_0\Sigma(\overline{\mathfrak{n}}_X^z; \mathfrak{a})$, and a unique non-zero distribution $\mu_{z,X} \in \mathcal{D}'(\Omega, V_\sigma)$, so that

$$\lim_{t\to\infty} e^{t\big(\lambda+\rho_P+2\rho_{\mathcal{O},X}-\kappa\big)(X)} R^\vee\big(\exp(tX)\big)\mu = \mu_{z,X}\,.$$

Here the limit is with respect to weak-$*$ topology on $\mathcal{D}'(\Omega, V_\sigma)$. The distribution $\mu_{z,X}$ is given by the following. For every multi-index β with $\kappa_\beta = \kappa$ there exists a $c_\beta \in V_\sigma^*$ such that for all $\phi \in \mathcal{D}(\Omega, V_\sigma)$

$$\mu_{z,X}(\phi) = \sum_{\substack{\beta\\ \kappa_\beta=\kappa}} \int_M \int_A \int_{N_P} \int_{\overline{N}_{z,X}} a^{-\lambda+\rho_P}\Big(\sigma^\vee(m)c_\beta, \partial^\beta\phi(man\overline{n})\Big)\, d\overline{n}\, dn\, da\, dm.$$

Finally, $\mu_{z,X}$ has the following properties.

(i) $L^\vee(man)\mu_{z,X} = a^{\lambda-\rho_P}\sigma^\vee(m^{-1})\mu_{z,X}$ *for every $m \in M$, $a \in A$ and $n \in N_P$.*

(ii) $R^\vee(Y)\mu_{z,X} = \big(-\lambda-\rho_P-2\rho_{\mathcal{O},X}+\kappa\big)(Y)\mu_{z,X}$ *for every $Y \in \mathfrak{a}$.*

(iii) $R^\vee(Y)\mu_{z,X} = 0$ *for every $Y \in \mathfrak{h}_{z,X}$.*

(iv) The following are equivalent:

(a) $\operatorname{trdeg}_{\mathcal{O}}(\mu) \neq 0$

(b) the transversal degree of $\mu_{z,X}$ (w.r.t. the submanifold $P\overline{N}_{z,X} \cap \Omega$ of G) at any point in $\operatorname{supp}\mu_{z,X}$ is non-zero.

(c) $\kappa \neq 0$.

Corollary 5.2. *Let $\mu \in \mathcal{D}'(Z, P : \sigma : \lambda)$ and let $\mathcal{O} \in (P\backslash Z)_\mu$. Let $X \in \mathfrak{a}^-$ satisfy (5.1). Then*

$$\lambda\big|_{\mathfrak{a}_\mathcal{O}} \in \big(-\rho_P - 2\rho_{\mathcal{O},X} + \mathbb{N}_0\Sigma(\overline{\mathfrak{n}}_X^z; \mathfrak{a})\big)\big|_{\mathfrak{a}_\mathcal{O}}. \tag{5.2}$$

Moreover, $\operatorname{trdeg}_\mathcal{O}(\mu) \neq 0$ *if and only if there exists a non-zero element* $\kappa \in \mathbb{N}_0\Sigma(\overline{\mathfrak{n}}_X^z; \mathfrak{a})$ *so that*

$$\lambda\big|_{\mathfrak{a}_\mathcal{O}} \in \big(-\rho_P - 2\rho_{\mathcal{O},X} + \kappa\big)\big|_{\mathfrak{a}_\mathcal{O}}.$$

Remark 5.3. If $\mathcal{O}$ is of maximal rank, then $\rho_{\mathcal{O},X}$ can be explicitly determined. Let $z \in \mathcal{O}$ and $X \in \mathfrak{a}^-$. By Proposition 3.1 there exists a $w \in N_G(\mathfrak{a})$ so that $\mathfrak{h}_{z,X} = \operatorname{Ad}(w)\mathfrak{h}_\emptyset$. In view of Remark 3.2 (a) we may choose w so that (3.3) is satisfied. The latter guarantees that

$$\operatorname{Ad}(w)(\overline{\mathfrak{n}}_P \cap \mathfrak{l}_Q) \subseteq \overline{\mathfrak{n}}_P,$$

and hence

$$\overline{\mathfrak{n}}_{z,X} = \operatorname{Ad}(w)\overline{\mathfrak{n}}_P \cap \overline{\mathfrak{n}}_P, \quad \overline{\mathfrak{n}}_X^z = \operatorname{Ad}(w)\mathfrak{n}_P \cap \overline{\mathfrak{n}}_P = \operatorname{Ad}(w)\mathfrak{n}_Q \cap \overline{\mathfrak{n}}_P.$$

It follows that

$$\rho_{\mathcal{O},X} = -\frac{1}{2}\rho_{wPw^{-1}} - \frac{1}{2}\rho_P.$$

In particular (5.2) can be rewritten as

$$\lambda\big|_{\mathfrak{a}_\mathcal{O}} \in \Big(\operatorname{Ad}^*(w)\rho_P + \mathbb{N}_0\Big(-\Sigma(P) \cap \operatorname{Ad}^*(w)\Sigma(Q)\Big)\Big)\Big|_{\mathfrak{a}_\mathcal{O}}.$$

Proof of Corollary 5.2. The functional $\rho_{\mathcal{O},X}$ only depends on the connected component of the set of order-regular elements in $\mathfrak{a}$ in which X is chosen. Every connected component of the set of order-regular elements in $\mathfrak{a}$ contains elements X that satisfy $\kappa_\beta(X) \neq \kappa_\gamma(X)$ for any two multi-indices β, γ with $|\beta|, |\gamma| \leq \operatorname{trdeg}_\mathcal{O}(\mu)$ and $\kappa_\beta \neq \kappa_\gamma$. The claim therefore follows from (ii) and (iii) in Theorem 5.1. □

For $\mathcal{O} \in P\backslash Z$ we set

$$\mathcal{H}_\mathcal{O} := \big\{\lambda \in \mathfrak{a}_\mathbb{C}^* : \lambda\big|_{\mathfrak{a}_\mathcal{O}} \in \frac{1}{2}\mathbb{Z}\Sigma\big|_{\mathfrak{a}_\mathcal{O}}\big\}$$

and we define

$$\mathcal{H}_{\mathrm{nm}} := \bigcup_{\substack{\mathcal{O} \in P\backslash Z \\ \operatorname{rank}(\mathcal{O}) < \operatorname{rank}(Z)}} \mathcal{H}_\mathcal{O}.$$

Here nm stands for not maximal. Then $\mathcal{H}_{\mathrm{nm}}$ is a locally finite set of complex affine subspaces in $\mathfrak{a}_\mathbb{C}^*$ of codimension at least 1. Note that if $\lambda \in \mathcal{H}_\mathcal{O}$, then $\operatorname{Im}(\lambda) \in (\mathfrak{a}/\mathfrak{a}_\mathcal{O})^*$.

Theorem 5.4. *Let $\sigma \in \widehat{M}$ and $\lambda \in \mathfrak{a}_\mathbb{C}^*$. The following assertions hold true.*

(i) Let $\mu \in \mathcal{D}'(Z, P : \sigma : \lambda)$. If $\mathcal{O} \in (P\backslash Z)_\mu$, then $\lambda \in \mathcal{H}_\mathcal{O}$.

(ii) If $\lambda \notin \mathcal{H}_{\mathrm{nm}}$, then for all $\mu \in \mathcal{D}'(Z, P : \sigma : \lambda)$

$$(P\backslash Z)_\mu \subseteq (P\backslash Z)_{\max}.$$

Proof. The assertions follow directly from Corollary 5.2. □

5.3 P-Orbits of maximal rank and transversal degree

We now focus on P-orbits of maximal rank in $(P\backslash Z)_\mu$ and use the principal asymptotics for these orbits to obtain further restrictions on the support and the transversal derivatives.

Let $\mathcal{Z}(\mathfrak{g})$ be the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ and let $\mathfrak{t}$ be a maximal abelian subalgebra in $\mathfrak{m}$. We fix a positive system $\Sigma_\mathfrak{m}^+$ of the root system of $i\mathfrak{t}$ in $\mathfrak{m}$ and write $\rho_\mathfrak{m}$ for the corresponding half-sum of positive roots. We further write $W_\mathbb{C}$ for the Weyl group of the root system $\Sigma_\mathbb{C}$ of $(\mathfrak{a}+i\mathfrak{t})_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$. Let γ be the Harish-Chandra homomorphism $\gamma:\mathcal{Z}(\mathfrak{g})\to\mathrm{Sym}\big((\mathfrak{a}\oplus i\mathfrak{t})_\mathbb{C}\big)^{W_\mathbb{C}}\simeq\mathbb{C}\big[(\mathfrak{a}\oplus i\mathfrak{t})^*\big]^{W_\mathbb{C}}$.

Proposition 5.1. *Let $\lambda\in\mathfrak{a}_\mathbb{C}^*$ and $\sigma\in\widehat{M}$. Further, let $\mu\in\mathcal{D}'(Z,P:\sigma:\lambda)$ and let $\mathcal{O}\in(P\backslash Z)_\mu$. Assume that $\mathcal{O}$ is of maximal rank. Let $\Lambda_\sigma\in i\mathfrak{t}^*$ be the highest-weight of σ. If $\mathrm{trdeg}_\mathcal{O}(\mu)\neq 0$, then there exists a non-zero element $\nu\in\mathbb{N}_0\Sigma(P)$ and a dominant $\Sigma_\mathfrak{m}$-integral weight $\Lambda\in i\mathfrak{t}^*$ so that*

$$W_\mathbb{C}\cdot(\lambda+\Lambda_\sigma+\rho_\mathfrak{m})=W_\mathbb{C}\cdot\big(\lambda+\Lambda+\rho_\mathfrak{m}+\nu\big).$$

Moreover, if $X_\nu\in\mathfrak{a}$ satisfies $B(X_\nu,\,\cdot\,)=\nu$, then $X_\nu\notin\mathfrak{a}_\mathcal{O}$.

Proof. Let $z\in\mathcal{O}$ and $X\in\mathfrak{a}^-$ be as in Proposition 5.1. By Proposition 3.1 there exists a $w\in N_G(\mathfrak{a})$ so that $\mathfrak{h}_{z,X}=\mathrm{Ad}(w)\mathfrak{h}_\emptyset$. In view of Remark 3.2 (a) we may choose w so that (3.3) is satisfied. Then

$$\rho_{\mathcal{O},X}=-\frac{1}{2}\rho_{wPw^{-1}}-\frac{1}{2}\rho_P,$$

see Remark 5.3. In view of Proposition 5.1 there exists a left-P-invariant open neighborhood Ω of e in G, a $\kappa\in\Sigma(\mathrm{Ad}(w)\mathfrak{n}_P\cap\overline{\mathfrak{n}}_P;\mathfrak{a})$ and non-zero distribution $\mu_{z,X}\in\mathcal{D}'(\Omega,V_\sigma)$ so that (i) – (iv) in Proposition 5.1 hold. In particular,

$$R^\vee(Y)\mu_{z,X}=0\qquad(Y\in\mathrm{Ad}(w)\overline{\mathfrak{n}}_P).\tag{5.1}$$

and

$$R^\vee(Y)\mu_{z,X}=\big(-\lambda+\rho_{wPw^{-1}}+\kappa\big)(Y)\mu_{z,X}\qquad(Y\in\mathfrak{a}).\tag{5.2}$$

Moreover, for every multi-index β with $\kappa_\beta=\kappa$ there exists a unique $c_\beta\in V_\sigma^*$ such that for all $\phi\in\mathcal{D}(\Omega,V_\sigma)$

$$\mu_{z,X}(\phi)=\sum_{\substack{\beta\\ \kappa_\beta=\kappa}}\int_M\int_A\int_{N_P}\int_{w\overline{N}_Pw^{-1}\cap\overline{N}_P}a^{-\lambda+\rho_P}\Big(\sigma^\vee(m)c_\beta,\partial^\beta\phi(man\overline{n})\Big)\,d\overline{n}\,dn\,da\,dm.\tag{5.3}$$

The representation $\mathcal{D}'(P:\xi:\lambda)$ admits an infinitesimal character, which via the Harish-Chandra isomorphism is identified with $-(\lambda+\Lambda_\sigma+\rho_\mathfrak{m})$. Therefore,

$$R^\vee(u)\mu=\gamma(u)\big(-(\lambda+\Lambda_\sigma+\rho_\mathfrak{m})\big)\mu\qquad\big(u\in\mathcal{Z}(\mathfrak{g})\big).$$

Since the elements of $\mathcal{Z}(\mathfrak{g})$ commutate with the adjoint action of G, we find for all $u\in\mathcal{Z}(\mathfrak{g})$

$$\begin{aligned}\gamma(u)\big(-(\lambda+\Lambda_\sigma+\rho_\mathfrak{m})\big)\mu_{z,X}&=\lim_{t\to\infty}e^{t\big(\lambda+\rho_P+2\rho_{\mathcal{O},X}-\kappa\big)(X)}R^\vee\big(\exp(tX)\big)R^\vee(u)\mu\\&=R^\vee(u)\mu_{z,X}\,.\end{aligned}\tag{5.4}$$

We will prove the proposition by computing $R^\vee(u)\mu_{z,X}$ using the formula (5.3) for $\mu_{z,X}$. For this we first look at the action of M on $\mu_{z,X}$.

Let $m_0 \in M$. By (5.3) we have for every $\phi \in \mathcal{D}(\Omega, V_\sigma)$

$$\begin{aligned} &R^\vee(m_0)\mu_{z,X}(\phi) \\ &= \sum_{\substack{\beta \\ \kappa_\beta=\kappa}} \int_M \int_A \int_{N_P} \int_{w\overline{N}_P w^{-1}\cap \overline{N}_P} a^{-\lambda+\rho_P} \Big(\sigma^\vee(m)c_\beta, \partial^\beta\big(R(m_0^{-1})\phi\big)(man\overline{n})\Big)\, d\overline{n}\, dn\, da\, dm. \end{aligned}$$

The Haar-measure on each of the groups $w\overline{N}_P w^{-1} \cap \overline{N}_P$, N_P and A is invariant under conjugation by m_0, and hence the right-hand side is equal to

$$\begin{aligned} &\sum_{\substack{\beta \\ \kappa_\beta=\kappa}} \int_M \int_A \int_{N_P} \int_{w\overline{N}_P w^{-1}\cap \overline{N}_P} a^{-\lambda+\rho_P} \\ &\quad \times \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} \bigg(\sigma^\vee(mm_0)c_\beta, \phi\Big(man \exp\big(\sum_{i=1}^n x_i \operatorname{Ad}(m_0)e_i\big)\overline{n}\Big)\bigg)\Big|_{x_i=0} d\overline{n}\, dn\, da\, dm. \end{aligned}$$

For multi-indices β and β', let $\chi^\beta_{\beta'} : M \to \mathbb{C}$ be determined by

$$\partial^\beta\big(\psi \circ C_m\big) = \sum_{\beta'} \chi^\beta_{\beta'}(m)\partial^{\beta'}\psi \qquad \big(\psi \in C^\infty(wN_P w^{-1} \cap \overline{N}_P), m \in M\big).$$

Here C_m denotes conjugation by m. If we denote $\operatorname{trdeg}_{\mathcal{O}}(\mu)$ by k and write S for the set of multi-indices of length at most k, then the representation χ of M on $\mathbb{C}^S$ that for all multi-indices $\beta \in S$ is given by

$$\big(\chi(m)v\big)_\beta = \sum_{\beta'} \chi^\beta_{\beta'}(m)v_{\beta'} \qquad \big(m \in M, v = (v_{\beta'})_{\beta' \in S} \in \mathbb{C}^S\big)$$

is isomorphic to the adjoint representation of M on $\bigoplus_{l=0}^k \big(\operatorname{Ad}(w)\mathfrak{n}_P \cap \overline{\mathfrak{n}}_P\big)^{\otimes l}$.

Let $c \in \mathbb{C}^S \otimes V_\sigma^* \simeq (V_\sigma^*)^S$ be the element of which the β'th component is equal to c_β for each $\beta \in S$. Then for all $\phi \in \mathcal{D}(\Omega, V_\sigma)$

$$\begin{aligned} R^\vee(m_0)\mu_{z,X}(\phi) = &\sum_{\substack{\beta \\ \kappa_\beta=\kappa}} \int_M \int_A \int_{N_P} \int_{w\overline{N}_P w^{-1}\cap \overline{N}_P} a^{-\lambda+\rho_P} \\ &\times \bigg(\sigma^\vee(m)\Big(\big(\chi \otimes \sigma^\vee\big)(m_0)c\Big)_\beta, \partial^\beta\phi(man\overline{n})\bigg)\, d\overline{n}\, dn\, da\, dm. \end{aligned} \tag{5.5}$$

The center $\mathcal{Z}(\mathfrak{g})$ is contained in

$$\mathcal{Z}(\mathfrak{m} \oplus \mathfrak{a}) \oplus \mathcal{U}(\mathfrak{g}) \operatorname{Ad}(w)\overline{\mathfrak{n}}_P = \big(\mathcal{Z}(\mathfrak{m}) \otimes \operatorname{Sym}(\mathfrak{a})\big) \oplus \mathcal{U}(\mathfrak{g}) \operatorname{Ad}(w)\overline{\mathfrak{n}}_P.$$

Let $u \in \mathcal{Z}(\mathfrak{g})$. There exist $v_{\mathfrak{m},1}, \dots, v_{\mathfrak{m},k} \in \mathcal{Z}(\mathfrak{m})$ and $v_{\mathfrak{a},1}, \dots, v_{\mathfrak{a},k} \in \operatorname{Sym}(\mathfrak{a}) \simeq \mathbb{C}[\mathfrak{a}^*]$ so that

$$u - \sum_{j=1}^k v_{\mathfrak{m},j} \otimes v_{\mathfrak{a},j} \in \mathcal{U}(\mathfrak{g}) \operatorname{Ad}(w)\overline{\mathfrak{n}}_P.$$

We may assume that $\delta_j := v_{\mathfrak{a},j}\big(-\lambda+\rho_{wPw^{-1}}+\kappa\big)$ is for every $1\leq j\leq k$ either equal to 0 or to 1. In view of (5.1), (5.2) and (5.5) we have for all $\phi\in\mathcal{D}(\Omega,V_\sigma)$

$$
\begin{aligned}
R^\vee(u)\mu_{z,X}(\phi)=\sum_{j=1}^k\delta_j\sum_{\substack{\beta\\ \kappa_\beta=\kappa}}\int_M\int_A\int_{N_P}\int_{w\overline{N}_Pw^{-1}\cap\overline{N}_P} a^{-\lambda+\rho_P}&\\
\times\Big(\sigma^\vee(m)\Big(\big(\chi\otimes\sigma^\vee\big)(v_{\mathfrak{m},j})c\Big)_\beta,\partial^\beta\phi(man\overline{n})\Big)\,d\overline{n}\,dn\,da\,dm.&
\end{aligned}
$$

Let $\Xi_{\chi\otimes\sigma^\vee}\subseteq(i\mathfrak{t})^*$ be the set of lowest weights of $\chi\otimes\sigma^\vee$ and let $\gamma_\mathfrak{m}:\mathcal{Z}(\mathfrak{m})\to\mathrm{Sym}(\mathfrak{t}_\mathbb{C})\simeq\mathbb{C}[\mathfrak{t}^*]$ be the Harish-Chandra homomorphism for $\mathfrak{m}$. Then $\big(\chi\otimes\sigma^\vee\big)(v_{\mathfrak{m},j})$ acts diagonalizably on $\mathbb{C}^S\otimes V_\sigma^*$ with eigenvalues $\gamma_\mathfrak{m}(v_{\mathfrak{m},j})(\eta-\rho_\mathfrak{m})$ for $\eta\in\Xi_{\chi\otimes\sigma^\vee}$. From (5.4) it follows that $R^\vee(u)\mu_{z,X}$ is a multiple of $\mu_{z,X}$. It follows from the uniqueness of the element c that there exists an $\eta\in\Xi_{\chi\otimes\sigma^\vee}$ so that

$$
\delta_j\big(\chi\otimes\sigma^\vee\big)(v_{\mathfrak{m},j})c=\delta_j\gamma_\mathfrak{m}(v_{\mathfrak{m},j})(\eta-\rho_\mathfrak{m})c\qquad(1\leq j\leq k).
$$

Therefore,

$$
\begin{aligned}
R^\vee(u)\mu_{z,X}&=\sum_{j=1}^k\delta_j\gamma_\mathfrak{m}(v_{\mathfrak{m},j})(\eta-\rho_\mathfrak{m})\mu_{z,X}\\
&=\sum_{j=1}^k\Big(v_{\mathfrak{a},j}\big(-\lambda+\rho_{wPw^{-1}}+\kappa\big)\Big)\Big(\gamma_\mathfrak{m}(v_{\mathfrak{m},j})(\eta-\rho_\mathfrak{m})\Big)\mu_{z,X}\\
&=\gamma(u)\big(-\lambda+\kappa+\eta-\rho_\mathfrak{m}\big)\mu_{z,X},
\end{aligned}
$$

and hence by (5.4)

$$
\gamma(u)\big(-\lambda-\Lambda_\sigma-\rho_\mathfrak{m}\big)\mu_{z,X}=\gamma(u)\big(-\lambda+\kappa+\eta-\rho_\mathfrak{m}\big)\mu_{z,X}.
$$

As $\mu_{z,X}\neq 0$ and this identity holds for all $u\in\mathcal{Z}(\mathfrak{g})$, the first assertion now follows with $\nu=-\kappa$ and $\Lambda=-\eta$. The second assertion follows from the Remarks 3.2 and 5.3. $\square$

Theorem 5.2. *Let $\lambda\in\mathfrak{a}_\mathbb{C}^*$, $\sigma\in\widehat{M}$ and $\mu\in\mathcal{D}'(Z,P:\sigma:\lambda)$. Let $\mathcal{O}\in(P\backslash Z)_\mu$ be of maximal rank. The following assertions hold true.*

(i) $\mathrm{Im}\,\lambda\in(\mathfrak{a}/\mathfrak{a}_\mathcal{O})^*$. *Assume that* $\mathrm{Im}\,\lambda$ *is regular in the sense that if $w\in W_\mathbb{C}$ stabilizes* $\mathrm{Im}\,\lambda$*, then w normalizes $\mathfrak{a}_\mathcal{O}+i\mathfrak{t}$ and acts trivially on $(\mathfrak{a}+i\mathfrak{t})/(\mathfrak{a}_\mathcal{O}+i\mathfrak{t})$. Then*

$$
\mathrm{trdeg}_\mathcal{O}(\mu)=0.
$$

(ii) If $\mathrm{trdeg}_\mathcal{O}(\mu)=0$ *and $v\in N_G(\mathfrak{a})$ satisfies (3.3) and $\mathfrak{h}_{z,X}=\mathrm{Ad}(v)\mathfrak{h}_\emptyset$ for some $z\in\mathcal{O}$ and order-regular element $X\in\mathfrak{a}^-$, then*

$$
\mathrm{Re}\,\lambda\in\rho_{vPv^{-1}}+(\mathfrak{a}/\mathfrak{a}_\mathcal{O})^*=\mathrm{Ad}^*(v)\big(\rho_P+(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\big)
$$

Proof. Let $\mathcal{O} \in (P\backslash Z)_\mu$ be of maximal rank and assume that $\operatorname{trdeg}_{\mathcal{O}}(\mu) \neq 0$. Let $\Lambda_\sigma \in i\mathfrak{t}^*$ be the highest-weight of σ. By Proposition 5.1 there exists a non-zero $\nu \in \mathbb{N}_0\Sigma(P)$, a dominant $\Sigma_{\mathfrak{m}}$-integral element $\Lambda \in i\mathfrak{t}^*$, and a $w \in W_{\mathbb{C}}$ so that

$$w \cdot \big(\lambda + \Lambda_\sigma + \rho_{\mathfrak{m}}\big) = \lambda + \Lambda + \rho_{\mathfrak{m}} + \nu.$$

Moreover, the element $X_\nu \in \mathfrak{a}$ so that $B(X_\nu, \cdot\,) = \nu$ is not contained in $\mathfrak{a}_{\mathcal{O}}$.

Note that $W_{\mathbb{C}}$ stabilizes the real subspace $(\mathfrak{a} \oplus i\mathfrak{t})^*$ of $(\mathfrak{a} \oplus i\mathfrak{t})^*_{\mathbb{C}}$. Therefore,

$$w \cdot \big(\operatorname{Re}\lambda + \Lambda_\sigma + \rho_{\mathfrak{m}}\big) = \operatorname{Re}\lambda + \Lambda + \rho_{\mathfrak{m}} + \nu \tag{5.6}$$

and

$$w \cdot \operatorname{Im}\lambda = \operatorname{Im}\lambda. \tag{5.7}$$

Furthermore, it follows from Corollary 5.2 that $\operatorname{Im}\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathcal{O}})^*$. Now assume that $\operatorname{Im}\lambda$ satisfies the regularity condition stated in (i). In view of (5.7) the element w normalizes $\mathfrak{a}_{\mathcal{O}} + i\mathfrak{t}$ and acts trivially on $\big((\mathfrak{a}+i\mathfrak{t})/(\mathfrak{a}_{\mathcal{O}}+i\mathfrak{t})\big)^*$. Let $\mathfrak{a}_{\mathcal{O}}^\perp$ be the Killing orthocomplement of $\mathfrak{a}_{\mathcal{O}}$ in $\mathfrak{a}$. Then $(\mathfrak{a} + i\mathfrak{t})/(\mathfrak{a}_{\mathcal{O}} + i\mathfrak{t})$ is identified with $\mathfrak{a}_{\mathcal{O}}^\perp$ via the Killing form and hence w acts trivially on $\mathfrak{a}_{\mathcal{O}}^\perp$. It follows from (5.6) that

$$(\operatorname{Re}\lambda + \nu)\big|_{\mathfrak{a}_{\mathcal{O}}^\perp} = w \cdot (\operatorname{Re}\lambda + \Lambda_\sigma + \rho_{\mathfrak{m}})\big|_{\mathfrak{a}_{\mathcal{O}}^\perp} = \operatorname{Re}\lambda\big|_{\mathfrak{a}_{\mathcal{O}}^\perp}$$

and thus $\nu|_{\mathfrak{a}_{\mathcal{O}}^\perp} = 0$. This is in contradiction with $X_\nu \notin \mathfrak{a}_{\mathcal{O}}$. We thus conclude that $\operatorname{trdeg}_{\mathcal{O}}(\mu) = 0$. This proves (i).

Assertion (ii) follows from Corollary 5.2 and Remark 5.3. □

6 Construction and properties of H-fixed distribution vectors

In this section we construct meromorphic families of distributions in $\mathcal{D}'(Z, P : \sigma : \lambda)$ and study some of their properties.

6.1 H-spherical finite dimensional representations

We write $Z_{\mathbb{C}}$ for the complexification $\underline{G}(\mathbb{C})/\underline{H}(\mathbb{C})$ of Z. Note that Z naturally embeds into $Z_{\mathbb{C}}$. We further write $\mathbb{C}[Z]^{(P)}$ for the multiplicative monoid of functions $f : Z \to \mathbb{C}$ so that

(a) there exists a non-zero regular function ϕ on $Z_{\mathbb{C}}$ so that $f = \phi\big|_Z$,

(b) there exists a $\nu \in \mathfrak{a}^*$ so that

$$f(man \cdot z) = a^\nu f(z) \qquad (m \in M, a \in A, n \in N).$$

It follows from [29, Lemma 5.6] that $\mathbb{C}[Z]^{(P)}$ is finitely generated.

For every function $f \in \mathbb{C}[Z]^{(P)}$ there exists a finite dimensional representation (π, V), an H-fixed vector $v_H \in V$ and a MN-fixed vector $v^* \in V^*$ for the contragredient representation $\pi^\vee$ of π such that f is the matrix-coefficient of v_H and v^*. If π has lowest weight $\nu \in \mathfrak{a}^*$, then v^* is a highest weight vector of $\pi^\vee$ with weight $-\nu$. Note that for $m \in M$, $a \in A$, $n \in N_P$, $g \in G$ and $h \in H$

$$f(mangh) = v^*\big(\pi(mangh)v_H\big) = a^\nu v^*\big(\pi(g)v_H\big) = a^\nu f(g). \tag{6.1}$$

We define Λ to be the monoid of $\mathfrak{a}$-weights ν that occur in $\mathbb{C}[Z]^{(P)}$, i.e., Λ is the monoid of lowest $\mathfrak{a}$-weights of finite dimensional representations π with $V^H \neq \{0\}$ and $(V^*)^{MN} \neq \{0\}$. It follows from (6.1) that

$$\Lambda \subseteq (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*.$$

The rank of the lattice generated by Λ is equal to $\operatorname{rank}(Z)$, see the proof of [29, Proposition 3.13].

We define the submonoid $\mathbb{C}[Z]^{(P)}_+$ of $\mathbb{C}[Z]^{(P)}$ by

$$\mathbb{C}[Z]^{(P)}_+ := \Big\{ f \in \mathbb{C}[Z]^{(P)} : f^{-1}(\mathbb{C} \setminus \{0\}) = \bigcup_{\substack{\mathcal{O} \in P\backslash Z \\ \mathcal{O} \text{ open}}} \mathcal{O} \Big\}.$$

By [29, Lemma 3.6] the set $\mathbb{C}[Z]^{(P)}_+$ is non-empty. Furthermore, we write Λ_+ for the submonoid of Λ corresponding to $\mathbb{C}[Z]^{(P)}_+$. We note that $\mathbb{C}[Z]^{(P)}_+\mathbb{C}[Z]^{(P)} = \mathbb{C}[Z]^{(P)}_+$, and hence $\Lambda_+ + \Lambda = \Lambda_+$. As $\mathbb{C}[Z]^{(P)}_+$ is non-empty, the submonoid Λ_+ has full rank.

Lemma 6.1. *Let $z \in Z$ be adapted, let $\sigma \in \widehat{M}$ and let $\eta \in (V_\sigma^*)^{M \cap H_z}$. Then there exists a $\nu \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ and a regular function $f_\eta : Z \to V_\sigma^*$ so that $f_\eta(z) = \eta$ and*

$$f_\eta(man \cdot z') = a^\nu \sigma^\vee(m) f_\eta(z') \qquad (m \in M, a \in A, n \in N_P, z' \in Z).$$

Proof. We may assume that $(V_\sigma^*)^{M \cap H_z} \neq \{0\}$. It suffices to prove the existence of a regular function $\phi : Z_\mathbb{C} \to \mathbb{C}$ so that ϕ is $N_{P,\mathbb{C}}$-invariant and $\sigma^\vee$ occurs as a direct summand in the representation of M generated by ϕ. Let $z \in Z$ be adapted. We define the algebras

$$\mathcal{A} := \big\{ \phi : Z_\mathbb{C} \to \mathbb{C} : \phi \text{ is regular and } \phi(n \cdot z') = \phi(z') \text{ for all } n \in N_{P_\mathbb{C}}, z' \in Z_\mathbb{C} \big\}$$

and

$$\mathcal{A}_M := \big\{ M/(M \cap H_z) \ni m \mapsto \phi(m \cdot z) : \phi \in \mathcal{A} \big\}$$

In order to prove the existence of a function ϕ with the properties mentioned above, it suffices to prove that $\mathcal{A}_M$ is dense in $C\big(M/(M \cap H_z)\big)$. For this we use the Stone-Weierstrass theorem.

Note that $\mathcal{A}_M$ is a subalgebra of $C\big(M/(M \cap H_z)\big)$, is closed under complex conjugation and contains the unit, i.e., the constant function 1. By the Stone-Weierstrass theorem $\mathcal{A}_M$ is dense in $C\big(M/(M \cap H_z)\big)$ if $\mathcal{A}_M$ separates points in $M/(M \cap H_z)$. For the latter it suffices to prove that $\mathcal{A}$ separates points in $M \cdot z \subseteq Z$.

Let $m_1, m_2 \in M$ and assume that $m_1 \cdot z \neq m_2 \cdot z$. By [44, Theorem 2] the N_P-orbits $N_P m_1 \cdot z$ and $N_P m_2 \cdot z$ are closed in Z. The space $N_P \backslash Z$ is isomorphic to the quasi-affine space $G/N_P \times_{\mathrm{diag}(G)} Z$ and

$$\mathcal{A} \simeq \mathbb{C}[G/N_P \times Z]^{\mathrm{diag}(G)}.$$

Therefore, also $\mathcal{D}_1 := \mathrm{diag}(G) \cdot \big(eN_P \times m_1 \cdot z\big)$ and $\mathcal{D}_2 := \mathrm{diag}(G) \cdot \big(eN_P \times m_2 \cdot z\big)$ are closed. It is an straightforward corollary of the main result in [10] that then $\mathcal{D}_1$ and $\mathcal{D}_2$ are also Zariski closed. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be the ideals of $\mathbb{C}[G/N_P \times Z]$ of functions vanishing on $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. As $m_1 \cdot z \neq m_2 \cdot z$ and z is adapted, it follows from the local structure theorem, Proposition 3.1 that $\mathcal{D}_1$ and $\mathcal{D}_2$ are disjoint. Together with the fact that $\mathcal{D}_1$ and $\mathcal{D}_2$ are Zariski-closed this implies that

$$\mathbb{C}[G/N_P \times Z] = \mathcal{I}_1 + \mathcal{I}_2.$$

Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are $\mathrm{diag}(G)$-orbits, the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are $\mathrm{diag}(G)$-stable. As $\mathrm{diag}(G)$ is reductive, it follows that

$$\mathbb{C}[G/N_P \times Z]^{\mathrm{diag}(G)} = \mathcal{I}_1^{\mathrm{diag}(G)} + \mathcal{I}_2^{\mathrm{diag}(G)}.$$

In particular, there exist a $\phi_1 \in \mathcal{I}_1^{\mathrm{diag}(G)}$ and a $\phi_2 \in \mathcal{I}_2^{\mathrm{diag}(G)}$ so that $\phi_1 + \phi_2$ is the constant function 1. Now $\phi_2(m_1 \cdot z) = 1$ and $\phi_2(m_2 \cdot z) = 0$. We thus conclude that $\mathcal{A}$ separates points in $M \cdot z \subseteq Z$. □

6.2 Construction on open P-orbits

In this section we construct meromorphic families of distributions in $\mathcal{D}'(Z, P : \sigma : \lambda)$ with support equal to the closure of an open P-orbit. For an adapted point $z \in Z$, $\lambda \in \rho_P + (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$, a finite dimensional representation (σ, V_σ) of M, and $\eta \in (V_\sigma^*)^{M \cap H_z}$ we define the function

$$\epsilon_z(P : \sigma : \lambda : \eta) : Z \to V_\sigma^*$$

by

$$\begin{cases} \epsilon_z(P : \sigma : \lambda : \eta)(nma \cdot z) = a^{-\lambda + \rho_P} \sigma^\vee(m)\eta, & (n \in N_Q, a \in A, m \in M); \\ \epsilon_z(P : \sigma : \lambda : \eta)(y) = 0, & (y \notin P \cdot z). \end{cases}$$

We note that in view of Proposition 3.1 this function is well defined.

Let Γ be the cone in $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ generated by Λ, i.e.,

$$\Gamma = \sum_{\lambda \in \Lambda} \mathbb{R}_{\geq 0} \lambda.$$

Since Λ has full rank, the interior of Γ is non-empty.

Proposition 6.1. *Let $z \in Z$ be adapted. Let (σ, V_σ) be a finite dimensional unitary representation of M. Assume that η is a non-zero $M \cap H_z$-fixed vector in V_σ^*. For $\lambda \in \rho_P - \Gamma + i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ the function $\epsilon_z(P : \sigma : \lambda : \eta)$ is measurable and bounded on every compact subset of G.*

Proof. Let $\mathcal{O} = P \cdot z$. Note that $\mathcal{O}$ is open. The function $\epsilon_z(P : \sigma : \lambda : \eta)$ is continuous outside of the set $\partial(\mathcal{O})$, which has measure 0 in G. Therefore, $\epsilon_z(P : \sigma : \lambda : \eta)$ is measurable. Now let $\lambda \in \rho_P - \Gamma + i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. Let $f_1, \dots, f_r$ be a set of generators of $\mathbb{C}[Z]^{(P)}$ with $f_i(z) = 1$, and let $\lambda_1, \dots, \lambda_r$ be the corresponding set of generators of Λ. Let $\nu_1, \dots, \nu_r \in \mathbb{C}$ with $\operatorname{Re} \nu_i \geq 0$ be such that

$$\rho_P - \lambda = \sum_{i=1}^{r} \nu_i \lambda_i.$$

Then

$$\epsilon_z(P : \sigma : \lambda : \eta) = \Big(\prod_{j=1}^{r} f_j^{\nu_j}\Big)\epsilon_z(P : \sigma : \rho_P : \eta).$$

Therefore,

$$\|\epsilon_z(P : \sigma : \lambda : \eta)(x)\|_\sigma = \Big(\prod_{j=1}^{r} |f_j(x)|^{\operatorname{Re}\nu_j}\Big)\|\epsilon_z(P : \sigma : \rho_P, \eta)(x)\|_\sigma.$$

As

$$\|\epsilon_z(P : \sigma : \rho_P, \eta)(x)\|_\sigma = \begin{cases} \|\eta\|_\sigma, & (x \in \mathcal{O}) \\ 0, & (x \notin \mathcal{O}), \end{cases}$$

the function $\epsilon_z(P : \sigma : \rho_P, \eta)$ is bounded. Since the functions f_j are continuous, it follows that $\epsilon_z(P : \sigma : \lambda : \eta)$ is bounded on every compact subset of G. $\square$

For every adapted point $z \in Z$, finite dimensional unitary representation (σ, V_σ) of M, non-zero $M \cap H_z$-fixed vector η in V_σ^*, and $\lambda \in \rho_P - \Gamma + i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$, the function $\epsilon_z(P : \sigma : \lambda : \eta)$ defines in view of Proposition 6.1 a distribution $\mu_z(P : \sigma : \lambda : \eta)$ in $\mathcal{D}'(Z, P : \sigma : \lambda)$ given by

$$\mu_z(P : \sigma : \lambda : \eta) : \mathcal{D}(Z, V_\sigma) \to \mathbb{C}; \quad \phi \mapsto \int_Z \Big(\epsilon_z(P : \sigma : \lambda : \eta)(x), \phi(x)\Big)\, dx. \tag{6.1}$$

It follows from Proposition 3.1, that for all $\phi \in \mathcal{D}(Z, V_\sigma)$

$$\begin{aligned} &\mu_z(P : \sigma : \lambda : \eta)(\phi) \\ &\quad = \int_{N_Q} \int_{M/M\cap H_z} \int_{A/A\cap H_z} a^{-\lambda+\rho_P-2\rho_Q}\Big(\sigma^\vee(m)\eta, \phi(nma \cdot z)\Big)\, dm\, dn\, da. \end{aligned} \tag{6.2}$$

It is easily seen that for a given adapted point $z \in Z$, finite dimensional representation σ of M, and $M \cap H_z$-fixed vector η in V_σ^*, the family

$$\rho_P - \Gamma + i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \ni \lambda \mapsto \mu_z(P : \sigma : \lambda : \eta)$$

is a holomorphic family of distributions in $\mathcal{D}'(Z, V_\sigma)$. We will show that this family extends meromorphically to all of $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$. To do so we use the theorem of Bernstein and Sato. Our proof is similar to that of [42, Theorem 5.1].

Proposition 6.2. *Let $z \in Z$ be adapted. Let (σ, V_σ) be a finite dimensional unitary representation of M and let η be a non-zero $M \cap H_z$-fixed vector in V_σ^*. The family*

$$\rho_P - \Gamma + i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \ni \lambda \mapsto \mu_z(P : \sigma : \lambda : \eta)$$

of distributions in $\mathcal{D}'(Z, V_\sigma)$ defined in (6.1) is holomorphic and extends to a meromorphic family on $\rho_P + (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$. There exists a locally finite union $\mathcal{H}$ of complex affine hyperplanes in $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ of the form

$$\{\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C} : \lambda(X) = a\} \quad \textit{for some } X \in \mathfrak{a} \textit{ and } a \in \mathbb{R}, \tag{6.3}$$

so that the poles of the family $\lambda \mapsto \mu_z(P : \sigma : \lambda : \eta)$ lie on $\rho_P + \mathcal{H}$. For $\lambda \in \rho_P + (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ outside of the set $\rho_P + \mathcal{H}$ the distribution $\mu_z(P : \sigma : \lambda : \eta)$ thus obtained is contained in $\mathcal{D}'(Z, P : \sigma : \lambda)$.

Proof. Let $\nu_1, \ldots, \nu_r \in \Lambda$ be a basis of $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ and let $f_1, \ldots, f_r \in \mathbb{C}[Z]^{(P)}$ be so that

$$f_j(man \cdot z) = a^{\nu_j} \qquad (1 \leq j \leq r, m \in M, a \in A, n \in N_P).$$

Note that each f_j is real valued and thus f_j^2 is non-negative. For

$$\lambda \in \sum_{j=1}^r \mathbb{R}_{\geq 0}\nu_j + i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*,$$

we define

$$\varphi^\lambda := \prod_{j=1}^r \left(f_j^2\right)^{u_j} : Z \to \mathbb{C},$$

where $u_j \in \mathbb{C}$ is determined by

$$\lambda = 2\sum_{j=1}^r u_j \nu_j.$$

By the theorem of Bernstein (see [13, Appendice A]) there exists for every $1 \leq j \leq r$ a polynomial function b_j on $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ and a differential operator D_j on Z with coefficients in $\mathbb{C}[Z][\lambda]$, so that for all $\lambda \in \sum_{j=1}^r \mathbb{R}_{\geq 1}\nu_j + i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$

$$D_j \varphi^\lambda = b_j(\lambda)\varphi^{\lambda - \nu_j}. \tag{6.4}$$

Furthermore, there exists a locally finite union $\mathcal{H}'$ of complex affine hyperplanes of the form (6.3) in $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ so that the zero's of the polynomials b_j are contained in $\mathcal{H}'$.

We now write $\mathcal{O}$ for the open P-orbit $P \cdot z$ and $\mathbf{1}_\mathcal{O}$ for its characteristic function. Let n be the maximum of the degrees of the differential operators D_j. Let further $f_0 \in \mathbb{C}[Z]^{(P)}_+$ and let $\gamma \in \Lambda_+$ be its weight. Then for all $\lambda \in (n+1)\gamma + \sum_{j=1}^r \mathbb{R}_{\geq 0}\nu_j$ we have

$$\varphi^\lambda = f_0^{n+1}\varphi^{\lambda - (n+1)\gamma}.$$

Since f_0 vanishes on $\partial\mathcal{O}$ and $\varphi^{\lambda-(n+1)\gamma}$ is continuous, it follows that $\varphi^\lambda \mathbf{1}_\mathcal{O}$ is at least n times continuously differentiable. From (6.4) it then follows that for all

$$\lambda \in (n+1)\gamma + \sum_{j=1}^{r} \mathbb{R}_{\geq 1}\nu_j$$

and every $1 \leq j \leq r$

$$D_j\Big(\varphi^\lambda \mathbf{1}_\mathcal{O}\Big) = \Big(D_j\varphi^\lambda\Big)\mathbf{1}_\mathcal{O} = b_j(\lambda)\varphi^{\lambda-\nu_j}\mathbf{1}_\mathcal{O}.$$

By means of this functional equation the family

$$\lambda \mapsto \varphi^{\mathcal{O},\lambda} := \varphi^\lambda \mathbf{1}_\mathcal{O}$$

can be extended to a meromorphic family of distributions on Z. The poles of this family lie on $\mathcal{H}'$.

Let $\sigma \in \widehat{M}$ and $\eta \in (V_\sigma^*)^{M\cap H_z}$. By Lemma 6.1 there exists a regular function $f_\eta : Z \to V_\sigma^*$ and a $\nu \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ so that

$$f_\eta(man \cdot z) = a^\nu \sigma^\vee(m)\eta \qquad (m \in M, a \in A, n \in N_P).$$

Now for $\lambda \in \rho_P - \nu - \sum_{j=1}^r \mathbb{R}_{\geq 0}\nu_j + i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$

$$\epsilon_z(P : \sigma : \lambda : \eta) = \varphi^{\mathcal{O},\rho_P-\lambda-\nu} f_\eta.$$

It follows that for these λ the distribution $\mu_z(P : \sigma : \lambda : \eta)$ is given by

$$\mu_z(P : \sigma : \lambda : \eta)(\phi) = \varphi^{\mathcal{O},\rho_P-\lambda-\nu}\Big(\big(f_\eta, \phi\big)\Big) \qquad \big(\phi \in \mathcal{D}(Z, V_\sigma)\big),$$

where $\big(f_\eta, \phi\big)$ is short-hand notation for the function $Z \to \mathbb{C}$, $z' \mapsto \big(f_\eta(z'), \phi(z')\big)$. As $\lambda \mapsto \varphi^{\mathcal{O},\rho_P-\lambda-\nu}$ is a meromorphic family of distributions, it follows that $\mu_z(P : \sigma : \lambda : \eta)$ extends to a meromorphic family of distributions in $\mathcal{D}'(Z, V_\sigma)$. The poles of this family lie on $\rho_Q - \nu - \mathcal{H}'$. Moreover, $\mu_z(P : \sigma : \lambda : \eta)$ is contained in $\mathcal{D}'(Z, P : \sigma : \lambda)$ for all $\lambda \in \rho_P + (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ outside of the poles of the family as this is true for all λ in the open subset $\rho_P - \Gamma + i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ of $\rho_P + (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$. $\square$

6.3 Construction on P-orbits of maximal rank

In this section we apply standard intertwining operators to the construction of meromorphic families on open P-orbits from the previous section to obtain meromorphic families whose support equals the closure of a P-orbit of maximal rank.

Lemma 6.1. *Let $w \in N_G(\mathfrak{a})$ be so that (3.3) holds. Let $\sigma \in \widehat{M}$. Then for λ in $\mathrm{Ad}(w)^*\big(\rho_P + (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}\big)$ outside of a locally finite set of complex affine hyperplanes of the form*

$$\{\lambda \in \mathrm{Ad}(w)^*\big(\rho_P + (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}\big) : \lambda(X) = c\} \quad \textit{for some } X \in \mathfrak{a} \textit{ and } a \in \mathbb{R} \qquad (6.1)$$

the intertwining operator

$$\mathcal{A}(wPw^{-1} : P : \sigma : \lambda) : \mathcal{D}'(P : \sigma : \lambda) \to \mathcal{D}'(wPw^{-1} : \sigma : \lambda)$$

is an isomorphism.

Proof. Let l be the length of w and let $P = P_0, \cdots, P_l = wPw^{-1}$ be a sequence of minimal parabolic subgroups so that $A \subseteq P_j$ and P_j and P_{j+1} are adjacent for every j. For $0 \leq j < l$ let $\alpha_j \in \Sigma(\mathfrak{a})$ be the reduced root such that

$$\Sigma(P_{j+1}, \mathfrak{a}) \cap \Sigma(\overline{P_j}, \mathfrak{a}) \subseteq \{\alpha_j, 2\alpha_j\}.$$

The rank one standard intertwining operators $A(P_j : P_{j+1} : \sigma : \lambda)$ are isomorphisms for λ in $\rho_P + (\mathfrak{a}/\mathfrak{a}_{\mathcal{O}})^*_{\mathbb{C}}$ outside a locally finite union $\mathcal{H}_j$ of complex affine hyperplanes of the form $\{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : \lambda(\alpha_j^\vee) = c\}$ with $c \in \mathbb{Q}$. See for example [35, Proposition B.1]. From [47, Théorème 1] it now follows that $A(P : wPw^{-1} : \sigma : \lambda)$ is an isomorphism for

$$\lambda \in \Big(\rho_P + (\mathfrak{a}/\mathfrak{a}_{\mathcal{O}})^*_{\mathbb{C}}\Big) \setminus \bigcup_{j=0}^{l-1} \mathcal{H}_j.$$

The same then holds for $A(P : wPw^{-1} : \sigma : \lambda)^*$ and $\mathcal{A}(wPw^{-1} : P : \sigma : \lambda)$.

Since (3.3) is assumed to hold, we have

$$\{\alpha_j : 0 \leq j < l\} \subseteq \Sigma(wPw^{-1}, \mathfrak{a}) \cap \Sigma(\overline{P}, \mathfrak{a}) = \Sigma(wQw^{-1}, \mathfrak{a}) \cap \Sigma(\overline{P}, \mathfrak{a}).$$

In view of Remark 3.2 $\alpha_j^\vee \notin \mathrm{Ad}(w)\mathfrak{a}_{\mathfrak{h}}$ for all j. Therefore, the intersection of $\mathcal{H}_j$ with $\mathrm{Ad}(w)^*\big(\rho_P + (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}\big)$ is for every $1 \leq j \leq l-1$ a locally finite union of affine hyperplanes of the form (6.1). $\square$

We now come to construction of distributions on maximal rank orbits.

Proposition 6.2. *Let $w \in N_G(\mathfrak{a})$ be so that (3.3) holds. Let further $\mathcal{O} \in w \cdot (P\backslash Z)_{\mathrm{open}}$ and let $z \in \mathcal{O}$ be weakly adapted. Now the wPw^{-1}-orbit $wPw^{-1} \cdot z$ is open in Z and z is adapted to $wPw^{-1} = MAwN_Pw^{-1}$. Let (σ, V_σ) be a finite dimensional unitary representation of M and let η be a $M \cap H_z$-fixed vector in V_σ^*. The assignment*

$$\lambda \mapsto \mu_z(P : \sigma : \lambda : \eta) := \mathcal{A}(wPw^{-1} : P : \sigma : \lambda)^{-1}\mu_z(wPw^{-1} : \sigma : \lambda : \eta) \tag{6.2}$$

defines a meromorphic family on $\rho_{wPw^{-1}} + (\mathfrak{a}/\mathfrak{a}_{\mathcal{O}})^*_{\mathbb{C}} = \mathrm{Ad}(w)^*\big(\rho_P + (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}\big)$ of distributions in $\mathcal{D}'(Z, P : \sigma : \lambda)$. These distributions have the following properties.

(i) If

$$\lambda \in \rho_{wPw^{-1}} - \mathrm{Ad}^*(w)\Gamma + i(\mathfrak{a}/\mathfrak{a}_{\mathcal{O}})^* = \mathrm{Ad}(w)^*\big(\rho_P - \Gamma + i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big),$$

then the distribution $\mu_z(P : \sigma : \lambda : \eta)$ is for $\phi \in \mathcal{D}(G, V_\sigma)$ given by the absolutely convergent integral

$$\begin{aligned}\mu_z(P : \sigma : \lambda : \eta)(\phi)& \\ = \int_{N_P \cap wN_Qw^{-1}} \int_{M/M\cap H_z} \int_{A/A\cap H_z} &a^{-\lambda + \mathrm{Ad}^*(w)\rho_P - 2\,\mathrm{Ad}^*(w)\rho_Q} \\ &\times \Big(\sigma^\vee(m)\eta, \phi(nma \cdot z)\Big)\, da\, dm\, dn.\end{aligned} \tag{6.3}$$

(ii) *There exists a locally finite union $\mathcal{H}$ of complex affine hyperplanes of the form*

$$\{\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathcal{O}})^*_{\mathbb{C}} : \lambda(Y) = a\} \quad \textit{for some } Y \in \mathfrak{a} \textit{ and } a \in \mathbb{R} \tag{6.4}$$

in $(\mathfrak{a}/\mathfrak{a}_{\mathcal{O}})^*_{\mathbb{C}}$, so that the poles of the family $\lambda \mapsto \mu_z(P : \sigma : \lambda : \eta)$ lie on $\rho_{wPw^{-1}} + \mathcal{H}$.

(iii) *For every $\lambda \in \rho_{wPw^{-1}} + \Big((\mathfrak{a}/\mathfrak{a}_{\mathcal{O}})^*_{\mathbb{C}} \setminus \mathcal{H}\Big)$ and $\eta \in (V_\sigma^*)^{M\cap H_z} \setminus \{0\}$ we have*

$$\operatorname{supp} \mu_z(P : \sigma : \lambda : \eta) = \overline{\mathcal{O}}.$$

(iv) *Up to scaling the distributions $\mu_z(P : \sigma : \lambda : \eta)$ do not depend on the choice of w, i.e., if $w' \in N_G(\mathfrak{a})$ satisfies (3.3) and $w \cdot (P\backslash Z)_{\mathrm{open}} = w' \cdot (P\backslash Z)_{\mathrm{open}}$, then there exists a $c > 0$ so that*

$$\mu_z(P : \sigma : \lambda : \eta) = c\mathcal{A}(w'Pw'^{-1} : P : \sigma : \lambda)^{-1}\mu_z(w'Pw'^{-1} : \sigma : \lambda : \eta)$$

as a meromorphic identity on $\rho_{wPw^{-1}} + (\mathfrak{a}/\mathfrak{a}_{\mathcal{O}})^*_{\mathbb{C}}$.

Remark 6.3. For reductive symmetric spaces the distributions $\mu_z(Q : \xi : \lambda : \eta)$ from Proposition 6.2 were constructed in [4]. The proof uses the same crucial point: the geometric decomposition (3.5) in Theorem 3.2 translates to a decomposition of a distribution $\mu_z(P : \sigma : \lambda : \eta)$ for an adapted point z, as constructed in Section 6.2, into an intertwining operator and a distribution that transforms under a conjugate minimal parabolic subgroup wPw^{-1} and is supported on the closure of a non-open wPw^{-1}-orbit. In [4] the objects under consideration are H-invariant functionals on $C^\infty(P : \sigma : \lambda)$; we consider here distributions in $\mathcal{D}'(Z, P : \sigma : \lambda)$. The resulting analysis is formally the same. However, we choose here to look at distributions rather than functionals since in this way we can avoid working with densities.

Proof. In view of Proposition 6.2 the family (6.2) is a meromorphic family of distributions in $\mathcal{D}'(Z, P : \sigma : \lambda)$. It follows from Proposition 6.2 and Lemma 6.1 that the poles of this family lie on a locally finite union of complex affine hyperplanes of the form (6.4). This proves (ii).

We move on to (i). Let $\lambda \in \mathrm{Ad}(w)^*\big(\rho_P - \Gamma + i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big)$ and $\phi \in \mathcal{D}(Z, V_\sigma)$. By Proposition 3.1 the wPw^{-1}-orbit $wPw^{-1} \cdot z$ is open. Moreover, if $P = MAN_P$ is replaced by $wPw^{-1} = MA(wN_Pw^{-1})$, then the point z is adapted. Therefore, it follows from Proposition 6.1 that the integral

$$\int_{wN_Qw^{-1}} \int_{M/M\cap H_z} \int_{A/A\cap H_z} \int_K \big|\big(\eta, \phi(knma \cdot z)\big)\big|\, dk\, a^{-\mathrm{Re}\,\lambda + \mathrm{Ad}^*(w)\rho_P - 2\,\mathrm{Ad}^*(w)\rho_Q}\, da\, dm\, dn$$

is absolutely convergent. The product map

$$(wN_Qw^{-1} \cap \overline{N}_P) \times (wN_Qw^{-1} \cap N_P) \to wN_Qw^{-1}; \quad (\overline{n}, n) \mapsto \overline{n}n$$

is a diffeomorphism with Jacobean equal to the constant function 1. Therefore, we may replace the integral over wN_Qw^{-1} by a repeated integral, the first over $wN_Qw^{-1} \cap N_P$

and the second over $wN_Qw^{-1} \cap \overline{N}_P$. For $\phi \in \mathcal{D}(Z, V_\sigma)$ we set

$$\chi_z(P : \sigma : \lambda : \eta)(\phi) \\ := \int_{N_P \cap wN_Q w^{-1}} \int_{M/M\cap H_z} \int_{A/A\cap H_z} a^{-\lambda + \mathrm{Ad}^*(w)\rho_P - 2\,\mathrm{Ad}^*(w)\rho_Q} \\ \times \Big(\sigma^\vee(m)\eta, \phi(nma \cdot z)\Big)\, da\, dm\, dn.$$

It follows from Fubini's theorem that the integral $\chi_z(P : \sigma : \lambda : \eta)(L_g\phi)$ is absolutely convergent for almost every $g \in G$, and the resulting function

$$I(\phi) : g \mapsto \chi_z(P : \sigma : \lambda : \eta)(L_g\phi)$$

is locally integrable on G. We claim that the integral is absolutely convergent for every $g \in G$ and that $I(\phi)$ is smooth. Indeed, in view of [19, Théorème 3.1] we may write ϕ is a finite sum of convolutions $\psi * \chi$ with $\psi \in \mathcal{D}(G)$ and $\chi \in \mathcal{D}(Z, V_\sigma)$. It follows from the above analysis that the integral

$$\int_G \psi(y) I(\chi)(y^{-1}g)\, dy$$

is absolutely convergent for every $g \in G$. Moreover, it depends smoothly on g and by Fubini's theorem it is equal to $I(\psi * \chi)(g)$. This proves the claim. It is now easily seen that $\mathrm{Ad}(w)^*\big(\rho_P - \Gamma + i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\big) \ni \lambda \mapsto \chi_z(P : \sigma : \lambda : \eta)$ defines a holomorphic family of distributions in $\mathcal{D}'(Z, V_\sigma)$.

We claim that $\chi_z(P : \sigma : \lambda : \eta) \in \mathcal{D}'(Z, P : \sigma : \lambda)$. To prove the claim, we first note that

$$L^\vee(ma)\chi_z(P : \sigma : \lambda : \eta) = a^{\lambda - \rho_P}\sigma^\vee(m^{-1})\chi_z(P : \sigma : \lambda : \eta)$$

for every $m \in M$ and $a \in A$. To prove the claim it thus suffices to show that

$$L^\vee(n)\chi_z(P : \sigma : \lambda : \eta) = \chi_z(P : \sigma : \lambda : \eta) \qquad (n \in N_P). \tag{6.5}$$

Let M_0 be a submanifold of M so that

$$M_0 \to M/(M \cap H_z); \quad m_0 \mapsto m_0(M \cap H_z)$$

is a diffeomorphism onto an open and dense subset of $M/(M \cap H_z)$ and let $d\mu$ be the pull back of the invariant measure on $M/(M \cap H_z)$ along this map. Let further A_0 be a closed subgroup of A so that

$$A_0 \to A/(A \cap wH_zw^{-1}); \quad a_0 \mapsto a_0(A \cap wH_zw^{-1})$$

is a diffeomorphism. For every $p \in P$ the map

$$N_P \cap wN_Qw^{-1} \to N_P/(N_P \cap H_{p\cdot z}); \quad n \mapsto n(N_P \cap H_{p\cdot z}) \tag{6.6}$$

is a diffeomorphism by Proposition 3.1. We normalize the N_P-invariant measure $d_{p\cdot z}\nu$ on $N_P/N_P\cap H_{p\cdot z}$ so that its pull-back along (6.6) is the Haar measure on $N_P\cap wN_Qw^{-1}$. After changing the order of integration we get for all $\phi\in\mathcal{D}(G,V_\sigma)$

$$\begin{aligned}&\chi_z(P:\sigma:\lambda:\eta)(\phi)\\&\quad=\int_{M_0}\int_{A_0}\int_{N_P/(N_P\cap H_{ma\cdot z})}a^{-\lambda+\operatorname{Ad}^*(w)\rho_P-2\operatorname{Ad}^*(w)\rho_Q}\\&\qquad\qquad\times\Big(\sigma^\vee(m)\eta,\phi(nma\cdot z)\Big)\,d\nu_{ma\cdot z}(n)\,da\,d\mu(m).\end{aligned}\tag{6.7}$$

The identity (6.5) follows from the invariance of the measures on the homogeneous spaces $N_P/N_P\cap H_{p\cdot z}$. We have thus proven the claim that $\chi_z(P:\sigma:\lambda:\eta)\in\mathcal{D}'(Z,P:\sigma:\lambda)$.

We move on to show that $\chi_z(P:\sigma:\lambda:\eta)=\mu_z(P:\sigma:\lambda:\eta)$. By (3.3) and (4.1) we have for all $\phi\in\mathcal{D}(G,V_\sigma)$

$$\begin{aligned}&\big[\mathcal{A}(wPw^{-1}:P:\sigma:\lambda)\chi_z(P:\sigma:\lambda:\eta)\big](\phi)\\&\quad=\int_{\overline{N}_P\cap wN_Qw^{-1}}\int_{N_P\cap wN_Qw^{-1}}\int_{M/M\cap H_z}\int_{A/A\cap H_z}a^{-\lambda+\operatorname{Ad}^*(w)\rho_P-2\operatorname{Ad}^*(w)\rho_Q}\\&\qquad\qquad\times\Big(\sigma^\vee(m)\eta,\phi(\overline{n}nma\cdot z)\Big)\,da\,dm\,dn\,d\overline{n}\\&\quad=\int_{wN_Qw^{-1}}\int_{M/M\cap H_z}\int_{A/A\cap H_z}a^{-\lambda+\operatorname{Ad}^*(w)\rho_P-2\operatorname{Ad}^*(w)\rho_Q}\\&\qquad\qquad\times\Big(\sigma^\vee(m)\eta,\phi(nma\cdot z)\Big)\,da\,dm\,dn.\end{aligned}$$

The right-hand side is equal to $\mu_z(wPw^{-1}:\sigma:\lambda:\eta)(\phi)$, and hence

$$\mathcal{A}(wPw^{-1}:P:\sigma:\lambda)\chi_z(P:\sigma:\lambda:\eta)=\mathcal{A}(wPw^{-1}:P:\sigma:\lambda)\mu_z(P:\sigma:\lambda:\eta).$$

It follows that (6.3) holds for $\lambda\in\operatorname{Ad}(w)^*\big(\rho_P-\Gamma+i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\big)$ for which the intertwining operator $\mathcal{A}(wPw^{-1}:P:\sigma:\lambda)$ is an isomorphism. In view of Lemma 6.1 this is the case for λ outside of a locally finite union of hyperplanes. Since $\chi_z(P:\sigma:\lambda:\eta)$ depends holomorphically and $\mu_z(P:\sigma:\lambda:\eta)$ meromorphically on λ, the identity (6.3) holds in fact for all $\lambda\in\operatorname{Ad}(w)^*\big(\rho_P-\Gamma+i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\big)$.

We move on to prove (iii). Assume that $\eta\neq0$. From (6.3) it follows that

$$\operatorname{supp}\mu_z(P:\sigma:\lambda:\eta)\subseteq\overline{\mathcal{O}}.$$

Since the support of $\mu_z(P:\sigma:\lambda:\eta)$ is a union of P-orbits in Z, it suffices to prove that the restriction of $\mu_z(P:\sigma:\lambda:\eta)$ to the open subset $Z\setminus\partial\mathcal{O}$ is non-zero. The right-hand side of (6.3) defines for every $\lambda\in\operatorname{Ad}(w)^*\big(\rho_P-\Gamma+i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\big)$ a non-zero distribution on $Z\setminus\partial\mathcal{O}$. Moreover, the dependence on λ is holomorphic, and hence the right-hand side of (6.3) defines a holomorphic family of distributions on $Z\setminus\partial\mathcal{O}$ with family parameter $\lambda\in\operatorname{Ad}(w)^*\big(\rho_P-\Gamma+i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\big)$. As this family coincides on a non-empty open subset of $\operatorname{Ad}(w)^*\big(\rho_P-\Gamma+i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\big)$ with the meromorphic family

$$\lambda\mapsto\mu_z(P:\sigma:\lambda:\eta)\big|_{Z\setminus\partial\mathcal{O}},$$

it follows that $\mu_z(P:\sigma:\lambda:\eta)\big|_{Z\setminus\partial\mathcal{O}} \neq 0$ for all λ for which $\mu_z(P:\sigma:\lambda:\eta)$ is defined. This proves (iii).

Finally we prove (iv). Let $w' \in N_G(\mathfrak{a})$ satisfy (3.3) and $w\cdot(P\backslash Z)_{\text{open}} = w'\cdot(P\backslash Z)_{\text{open}}$. Because of meromorphical continuation, it suffices to prove the uniqueness for λ in the open subset $\mathrm{Ad}^*(w)\big(\rho_P-\Gamma+i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\big)$ of $\mathrm{Ad}^*(w)\big(\rho_P+(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}\big)$. For these λ the distribution $\mu_z(P:\sigma:\lambda:\eta)$ is given by the right-hand side of (6.7). It follows from Proposition 3.1 that there exists a $\gamma: M\times A\to\mathbb{R}_{>0}$ so that for every $\psi\in\mathcal{D}(Z)$, $m\in M$ and $a\in A$

$$\int_{N_P/(N_P\cap H_{ma\cdot z})}\psi(nma\cdot z)\,d\nu_{ma\cdot z}(n) = \gamma(m,a)\int_{N_P\cap w'N_Qw'^{-1}}\psi(n'ma\cdot z)\,dn'.$$

The function

$$M\times A\to\mathbb{R}_{>0};\quad (m,a)\mapsto\frac{\gamma(m,a)}{\gamma(e,e)}$$

is a character of $M\times A$. Therefore, there exists a $c>0$ and a $\nu\in\mathfrak{a}^*$ so that

$$\gamma(m,a)=ca^\nu\qquad(m\in M, a\in A).$$

It follows that,

$$\begin{aligned}&\mu_z(P:\sigma:\lambda:\eta)(\phi)\\&\quad= c\int_{N_P\cap w'N_Qw'^{-1}}\int_{M/M\cap H_z}\int_{A/A\cap H_z} a^{-\lambda+\mathrm{Ad}^*(w)\rho_P-2\,\mathrm{Ad}^*(w)\rho_Q+\nu}\\&\qquad\qquad\times\Big(\sigma^\vee(m)\eta,\phi(nma\cdot z)\Big)\,da\,dm\,dn.\end{aligned}$$

Since w' satisfies (3.3), we have $w'N_Pw'^{-1}\cap\overline{N}_P = w'N_Qw'^{-1}\cap\overline{N}_P$. Now for every $\phi\in\mathcal{D}'(G,V_\sigma)$

$$\begin{aligned}&\big[\mathcal{A}(w'Pw'^{-1}:P:\sigma:\lambda)\mu_z(P:\sigma:\lambda:\eta)\big](\phi)\\&\quad= c\int_{w'N_Qw'^{-1}\cap\overline{N}_P}\int_{N_P\cap w'N_Qw'^{-1}}\int_{M/M\cap H_z}\int_{A/A\cap H_z} a^{-\lambda+\mathrm{Ad}^*(w)\rho_P-2\,\mathrm{Ad}^*(w)\rho_Q+\nu}\\&\qquad\qquad\times\Big(\sigma^\vee(m)\eta,\phi(nma\cdot z)\Big)\,da\,dm\,dn\\&\quad= c\int_{w'N_Qw'^{-1}}\int_{M/M\cap H_z}\int_{A/A\cap H_z} a^{-\lambda+\mathrm{Ad}^*(w)\rho_P-2\,\mathrm{Ad}^*(w)\rho_Q+\nu}\\&\qquad\qquad\times\Big(\sigma^\vee(m)\eta,\phi(nma\cdot z)\Big)\,da\,dm\,dn.\end{aligned}$$

Since $\mathcal{A}(w'Pw'^{-1}:P:\sigma:\lambda)\mu_z(P:\sigma:\lambda:\eta)$ is a distribution in $\mathcal{D}'(w'Pw'^{-1}:\sigma:\lambda)$, ν must satisfy

$$-\lambda+\mathrm{Ad}^*(w)\rho_P-2\,\mathrm{Ad}^*(w)\rho_Q+\nu = -\lambda+\mathrm{Ad}^*(w')\rho_P-2\,\mathrm{Ad}^*(w')\rho_Q.$$

Thus, in view of (6.2), we have

$$\mathcal{A}(w'Pw'^{-1}:P:\sigma:\lambda)\mu_z(P:\sigma:\lambda:\eta) = c\mu_z(w'Pw'^{-1}:\sigma:\lambda:\eta).$$

This concludes the proof of (iv). $\square$

Let $\mathcal{O} \in P\backslash Z$ be of maximal rank and let $z \in \mathcal{O}$ be weakly adapted. Assume that $\mathfrak{a}_{\mathcal{O}} = \mathfrak{a}_{\mathfrak{h}}$. Let $\xi \in \widehat{M}_Q$. We recall from Corollary 4.2 that if $(V^{\infty\prime})^{H_z} \neq \{0\}$, then $\xi|_{L_{Q,\mathrm{nc}}}$ is trivial, ξ is finite dimensional and unitarizable and $\xi|_M$ is irreducible. For $\xi \in \widehat{M}_Q$ and $\eta \in (V_\xi^*)^{M_Q \cap H_z}$ we define the meromorphic family of distributions

$$\mu_z(Q : \xi : \lambda : \eta) := \mu_z(P : \xi|_M : \lambda + \rho_P - \rho_Q : \eta)$$

with family parameter $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$.

Theorem 6.4. *For every z, ξ, η as above, the assignment*

$$(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}} \ni \lambda \to \mu_z(Q : \xi : \lambda : \eta)$$

defines a meromorphic family of distributions in $\mathcal{D}'(Z, Q : \xi : \lambda)$. The poles of the family lie on a locally finite union of complex affine hyperplanes of the form

$$\{\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}} : \lambda(Y) = a\} \quad \textit{for some } Y \in \mathfrak{a} \setminus \mathfrak{a}_{\mathfrak{h}} \textit{ and } a \in \mathbb{R}.$$

Let $\mathcal{O} \in P\backslash Z$ satisfy $\mathfrak{a}_{\mathcal{O}} = \mathfrak{a}_{\mathfrak{h}}$ and let $z \in \mathcal{O}$ be weakly adapted. Let $w \in N_G(\mathfrak{a})$ be so that (3.3) holds and $[\mathcal{O}] = w \cdot (P\backslash Z)_{\mathrm{open}}$. If

$$\lambda \in \rho_{wQw^{-1}} - \mathrm{Ad}^*(w)\Gamma + i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^* = \mathrm{Ad}(w)^*\big(\rho_Q - \Gamma + i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big),$$

then the distribution $\mu_z(Q : \xi : \lambda : \eta)$ is for $\phi \in \mathcal{D}(G, V_\xi)$ given by the absolutely convergent integral

$$\begin{aligned} &\mu_z(Q : \xi : \lambda : \eta)(\phi) &(6.8)\\ &\quad = \int_{N_Q \cap wN_Qw^{-1}} \int_{M/M\cap H_z} \int_{A/A\cap H_z} a^{-\lambda - \mathrm{Ad}^*(w)\rho_Q} \Big(\xi^\vee(m)\eta, \phi(nma \cdot z)\Big)\, da\, dm\, dn. \end{aligned}$$

Proof. We first prove that $\mu_z(Q : \xi : \lambda : \eta) \in \mathcal{D}'(Z, Q : \xi : \lambda)$. By Proposition 6.2 we have $\mu_z(Q : \xi : \lambda : \eta) \in \mathcal{D}'(P : \xi|_M : \lambda + \rho_P - \rho_Q)$. In view of Lemma 4.2 and meromorphicity it suffices to show that $\mu_z(Q : \xi : \lambda : \eta)$ is left-$M_Q \cap \overline{N}_P$-invariant for $\lambda \in \rho_{wQw^{-1}} - \mathrm{Ad}^*(w)\Gamma + i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$.

The fact that $\mathfrak{a}_{\mathcal{O}} = \mathfrak{a}_{\mathfrak{h}}$ implies $w \in \mathcal{N} = N_G(\mathfrak{a}) \cap N_G(\mathfrak{a}_{\mathfrak{h}})$. In view of Remark 3.2 the element w normalizes $L_{Q,\mathrm{nc}}$ and hence also M_Q. Since (3.3) is assumed to hold, w even normalizes $M_Q \cap \overline{N}_P$. The point $w^{-1} \cdot z$ is adapted. Therefore,

$$L_{Q,\mathrm{nc}} = wL_{Q,\mathrm{nc}}w^{-1} \subseteq wH_{w^{-1}z}w^{-1} = H_z. \tag{6.9}$$

In particular,

$$M_Q \cap \overline{N}_P \subseteq H_z.$$

We claim that $M_Q \cap \overline{N}_P$ normalizes $N_P \cap wN_Qw^{-1}$. In fact, M_Q has this property. As $N_P = (N_P \cap M_Q)N_Q$, we have

$$N_P \cap wN_Qw^{-1} = N_Q \cap wN_Qw^{-1}. \tag{6.10}$$

The claim is now proven by observing that M_Q normalizes N_Q and hence also wN_Qw^{-1}.

As M and A normalize $M_Q \cap \overline{N}_P$, $M_Q \cap \overline{N}_P$ normalizes $N_P \cap wN_Qw^{-1}$, it is follows from (6.9) and (6.3) that $\mu_z(Q : \xi : \lambda : \eta)$ is left-$M_Q \cap \overline{N}_P$-invariant. This concludes the proof that $\mu_z(Q : \xi : \lambda : \eta) \in \mathcal{D}'(Z, Q : \xi : \lambda)$.

The remaining assertions follow directly from Proposition 6.2 and (6.10). □

Remark 6.5. Let $\mathcal{O} \in P\backslash Z$ be of maximal rank and $z \in \mathcal{O}$ weakly adapted. If $\mathfrak{a}_{\mathcal{O}} = \mathfrak{a}_{\mathfrak{h}}$, then there exists a positive H_z-invariant Radon measure on $(H_z \cap Q)\backslash H_z$. To prove this it suffices to show that $H_z \cap Q$ is unimodular. The modular character of $H_z \cap Q$ is given by

$$\Delta : H_z \cap Q \to \mathbb{R}_{>0}; \quad h \mapsto \left| \det\left(\mathrm{Ad}(h)\big|_{\mathfrak{h}_z \cap \mathfrak{q}} \right) \right|$$

Let $w \in \mathcal{N}$ be so that (3.3) and $[\mathcal{O}] = w \cdot (P\backslash Z)_{\mathrm{open}}$. In view of (6.9), Proposition 3.1 and Theorem 3.2 we have

$$\begin{aligned} H_z \cap Q = L_{Q,\mathrm{nc}}(H_z \cap P) &= L_{Q,\mathrm{nc}}(M \cap H_z)(A \cap H)(N_P \cap H_z) \\ &= L_{Q,\mathrm{nc}}(M \cap H_z)(A \cap H)(N_Q \cap H_z). \end{aligned}$$

Since $L_{Q,\mathrm{nc}}$ is semisimple, $M \cap H_z$ is compact and $N_Q \cap H_z$ is unipotent, the restriction of Δ to each of these three subgroups is trivial. It thus remains to show that the restriction to $A \cap H$ is trivial as well.

As $A \cap H$ is contained in the center of L_Q it centralizes $L_{Q,\mathrm{nc}}(M \cap H_z)(A \cap H)$. Therefore,

$$\Delta(a) = \left| \det\left(\mathrm{Ad}(a)\big|_{\mathfrak{n}_Q \cap \mathfrak{h}_z} \right) \right| \qquad (a \in A \cap H_z).$$

It follows from Proposition 3.1 and (6.10) that the multiplication map

$$(N_Q \cap wN_Q w^{-1}) \times (N_Q \cap H_z) \to N_Q; \quad (n, n_H) \mapsto nn_H$$

is a diffeomorphism. Moreover, this map is $A \cap H_z$-equivariant, and hence

$$\Delta(a) = \left| \frac{\det\left(\mathrm{Ad}(a)\big|_{\mathfrak{n}_Q} \right)}{\det\left(\mathrm{Ad}(a)\big|_{\mathfrak{n}_Q \cap \mathrm{Ad}(w)\mathfrak{n}_Q} \right)} \right| = \frac{a^{2\rho_Q}}{a^{\rho_Q + \mathrm{Ad}^*(w)\rho_Q}} \qquad (a \in A \cap H_z).$$

As $A \cap H_z$ is connected, w centralizes this subgroup. It follows that Δ is the trivial character and hence that $H_z \cap Q$ is unimodular. This concludes the proof of the claim that $(H_z \cap Q)\backslash H_z$ admits a positive H_z-invariant Radon measure.

The invariant measure allows to describe the distributions $\mu_z(Q : \xi : \lambda : \eta)$ as a functional on $C^\infty(Q : \xi : \lambda)$. To do so, let $g \in G$ be so that $gH = z$. We use (4.2) to identify $\mathcal{D}'(Z, Q : \xi : \lambda)$ with $\mathcal{D}'(Q : \xi : \lambda)^H$. Recall the map $\omega^Q_{\xi,\lambda}$ from (4.1). A straightforward computation shows that for a suitable normalization of the invariant measure on $(H \cap g^{-1}Qg)\backslash H$

$$\Big(\omega^Q_{\xi,\lambda} \mu_z(Q : \xi : \lambda : \eta) \Big)(f) = \int_{(H \cap g^{-1}Qg)\backslash H} \big(\eta, f(gh) \big)\, dh \qquad \big(f \in C^\infty(Q : \xi : \lambda) \big).$$

The distributions $\mu_z(P : \sigma : \lambda : \eta)$ from Proposition 6.2 with z a weakly adapted point contained in a P-orbit $\mathcal{O}$ in Z with $\mathfrak{a}_{\mathcal{O}} \neq \mathfrak{a}_{\mathfrak{h}}$, can be similarly description as functionals on $C^\infty(P : \sigma : \lambda)$. However, in this generality not all homogeneous spaces $(H_z \cap P)\backslash H_z$ admit positive H_z-invariant Radon measures. To remedy this, one has to consider the elements in $C^\infty(P : \sigma : \lambda)$ as smooth densities; see [4, Lemma 3.1].

6.4 A description of $\mathcal{D}'(Q:\xi:\lambda)^H$

In this section we give a precise description of $\mathcal{D}'(Q:\xi:\lambda)^H$ for a finite dimensional unitary representation ξ of M_Q and $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ outside of a union of finitely many proper subspaces. We identify distributions on Z with H-invariant distributions on G via the map (4.2). We recall that the point $eH \in Z = G/H$ is chosen to be admissible.

We define

$$(P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}} := \{\mathcal{O} \in (P\backslash Z)_{\max} : \mathfrak{a}_{\mathcal{O}} = \mathfrak{a}_{\mathfrak{h}}\}.$$

First we choose a set of good representatives of the P-orbits in $(P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$.

By [39, Proposition 3.13] every open P-orbit contains a point $x \cdot z$ with

$$x \in G \cap \exp(i\mathfrak{a})H_{\mathbb{C}}.$$

For such a point the equality

$$\mathfrak{a} \cap \mathrm{Ad}(x)\mathfrak{h}^{\perp} = \mathfrak{a} \cap \mathfrak{h}^{\perp}$$

holds. For every $\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}$ we choose an $x_{\mathcal{O}} \in G \cap \exp(i\mathfrak{a})H_{\mathbb{C}}$ so that $x_{\mathcal{O}}H$ is an adapted point in $\mathcal{O} \subseteq Z = G/H$. We may and will choose x_{PH} to be e.

We recall the group $\mathcal{N} = N_G(\mathfrak{a}) \cap N_G(\mathfrak{a}_{\mathfrak{h}})$ and its subgroup $\mathcal{W}$ from (3.1) and (3.3). For every $\mathcal{N}/\mathcal{W}$ we choose a representative $v_w \in K \cap \mathcal{N}$ as follows. In view of Theorem 3.3 the set $\mathcal{N}/\mathcal{W}$ is in bijection with the set of equivalence classes of P-orbits in $(P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$, i.e., the map

$$\mathcal{N}/\mathcal{W} \to (P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}/_{\sim}; \quad v\mathcal{W} \mapsto v \cdot (P\backslash Z)_{\mathrm{open}}$$

is a bijection. We choose an order-regular element $X \in \mathfrak{a}^-$. For $w \in \mathcal{N}/\mathcal{W}$ we now choose $v_w \in K \cap \mathcal{N}$ so that

$$\mathfrak{h}_{z,X} = \mathrm{Ad}(v_w)\mathfrak{h}_{\emptyset}$$

for some weakly adapted point in an P-orbit $\mathcal{O}$ with $[\mathcal{O}] = w \cdot (P\backslash Z)_{\mathrm{open}}$. We note that this equation determines v_w up to right-multiplication by an element from $\mathcal{Z} \cap K$ and that $v_w\mathcal{Z}$ is independent of the choice of z. The elements v_w do however depend on the choice of X. The crucial property of the v_w is that they are representatives of the elements in $\mathcal{N}/\mathcal{W}$, i.e.,

$$v_w\mathcal{W} = w \qquad (w \in \mathcal{N}/\mathcal{W}).$$

We may and will choose the v_w so that they satisfy (3.3). The representative of $e\mathcal{W}$ we choose to be e.

By Theorem 3.3 (iii) and (iv) the points

$$v_w x_{\mathcal{O}'}H \qquad \big(w \in \mathcal{N}/\mathcal{W}, \mathcal{O}' \in (P\backslash Z)_{\mathrm{open}}\big)$$

form a set of weakly admissible representatives for the P-orbits in $(P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$. If $\mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$, then we write $x_{\mathcal{O}}$ for $v_w x_{\mathcal{O}'}$, where $w \in \mathcal{N}/\mathcal{W}$ and $\mathcal{O}' \in (P\backslash Z)_{\mathrm{open}}$ are so that $Pv_w x_{\mathcal{O}'}H$.

For a finite dimensional unitary representation (ξ, V_ξ) of M_Q we define the vector space

$$V^*(\xi) := \bigoplus_{\mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}} (V_\xi^*)^{M_Q \cap x_{\mathcal{O}} H x_{\mathcal{O}}^{-1}}$$

and equip it with the inner product induced from the inner product on V_ξ.

Proposition 6.1. *Let $\mathcal{O}, \mathcal{O}' \in (P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}$. If $\mathcal{O} \sim \mathcal{O}'$, then*

$$M_Q \cap x_\mathcal{O} H x_\mathcal{O}^{-1} = M_Q \cap x_{\mathcal{O}'} H x_{\mathcal{O}'}^{-1}.$$

In particular

$$M_Q \cap x_\mathcal{O} H x_\mathcal{O}^{-1} = M_Q \cap H \qquad \big(\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}\big).$$

Proof. Let $\mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}$. Let $w \in \mathcal{N}/\mathcal{W}$ be so that $[\mathcal{O}] = w \cdot (P\backslash Z)_{\mathrm{open}}$. Then $x_\mathcal{O} = v_w x_{\mathcal{O}_0}$ for some open P-orbit $\mathcal{O}_0$. It follows from Remark 3.2 that $\mathcal{N}$ normalizes M_Q. Therefore,

$$M_Q \cap x_\mathcal{O} H x_\mathcal{O}^{-1} = v_w \big(M_Q \cap x_{\mathcal{O}_0} H x_{\mathcal{O}_0}^{-1}\big) v_w^{-1}.$$

Since v_w only depends on the equivalence class $[\mathcal{O}]$, and not on the particular orbit $\mathcal{O}$ in it, it thus suffices to prove the assertion only for the open P-orbits $\mathcal{O}$.

We have $M_Q = M L_{Q,\mathrm{nc}}$ and $L_{Q,\mathrm{nc}} \subseteq x_\mathcal{O} H x_\mathcal{O}^{-1}$. Hence it suffices to prove

$$M \cap x_\mathcal{O} H x_\mathcal{O}^{-1} = M \cap H \qquad \big(\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}\big).$$

Note that $M \cap x_\mathcal{O} H x_\mathcal{O}^{-1} = M \cap x_\mathcal{O} H_\mathbb{C} x_\mathcal{O}^{-1}$. Let $t \in \exp(i\mathfrak{a})$ and $h \in H_\mathbb{C}$ be so that $x_\mathcal{O} = th$. Then

$$M \cap x_\mathcal{O} H_\mathbb{C} x_\mathcal{O}^{-1} = M \cap t H_\mathbb{C} t^{-1} = M \cap H_\mathbb{C}.$$

For the last equality we used that t centralizes M. The assertion now follows as $M \cap H_\mathbb{C} = M \cap H$. $\square$

As a corollary of the previous proposition we obtain that the group

$$M_{Q,[\mathcal{O}]} := M_Q \cap x_\mathcal{O} H x_\mathcal{O}^{-1}$$

only depends on the equivalence class $[\mathcal{O}] \in (P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}/\sim$, not on the choice of the P-orbit in $[\mathcal{O}]$. Therefore,

$$V^*(\xi) = \bigoplus_{[\mathcal{O}] \in (P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}/\sim} \Big((V_\xi^*)^{M_{Q,[\mathcal{O}]}}\Big)^{[\mathcal{O}]}$$

Here V^S for a vector space V and a finite set S denotes the vector space of functions $S \to V$. We write v_s for the s-component of a vector $v \in V^S$, i.e., $v_s = v(s)$.

Remark 6.2. The space $V^*(\xi)$ will serve as the multiplicity space in the Plancherel decomposition for the principal series representations $\mathrm{Ind}_Q^G(\xi \otimes \lambda \otimes 1)$ with $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. In case H is symmetric, i.e., H is an open subgroup of the fixed point subgroup G^σ of some involution σ of G, much information about these multiplicity spaces has been given in [5]. If H is equal to the full fixed point subgroup of an algebraic involution on G, then Proposition 6.1 coincides with [5, Lemma 7]. For symmetric spaces we have $(P\backslash Z)_{\mathfrak{a}_\mathfrak{h}} = (P\backslash Z)_{\mathrm{open}}$, see Appendix A.

Theorem 6.3. *There exists a finite union $\mathcal{S}$ of proper subspaces of $(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ so that for every finite dimensional unitary representation (ξ, V_ξ) of M_Q and every $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$ with $\operatorname{Im}\lambda \notin \mathcal{S}$ the map*

$$\mu(Q:\xi:\lambda): V^*(\xi) \to \mathcal{D}'(Q:\xi:\lambda)^H; \quad \eta \mapsto \sum_{\mathcal{O}\in(P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}} \mu_{x_{\mathcal{O}}H}(Q:\xi:\lambda:\eta_{\mathcal{O}})$$

is a linear isomorphism.

Proof. Let

$$\mathcal{S}_1 = (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^* \cap \bigcup_{\substack{\mathcal{O}\in P\backslash Z\\ \mathfrak{a}_{\mathcal{O}}\neq\mathfrak{a}_{\mathfrak{h}}}} (\mathfrak{a}/\mathfrak{a}_{\mathcal{O}})^*$$

If $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$ satisfies $\operatorname{Im}\lambda \notin \mathcal{S}_1$, then we have in view of Theorem 5.4

$$(P\backslash Z)_\mu \subseteq \big\{\mathcal{O} \in (P\backslash Z)_{\max} : \mathfrak{a}_{\mathcal{O}} = \mathfrak{a}_{\mathfrak{h}}\big\}$$

for every $\mu \in \mathcal{D}'(Z, Q:\xi:\lambda)$. Further, it follows from Theorem 5.2 that there exists a finite union $\mathcal{S}_2$ of proper subspaces of $(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ so that

$$\operatorname{trdeg}_{\mathcal{O}}(\mu) = 0 \qquad \big(\mathcal{O} \in (P\backslash Z)_\mu\big)$$

for all $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$ with $\operatorname{Im}\lambda \notin \mathcal{S}_1 \cup \mathcal{S}_2$ and all $\mu \in \mathcal{D}'(Z, Q:\xi:\lambda)$.

In view of Theorem 6.4 there exist a finite union $\mathcal{S}_3$ of hyperplanes in $(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ so that the poles of the meromorphic family of maps $\lambda \mapsto \mu(Q:\xi:\lambda)$ lie in $\mathcal{S}_3$. We set

$$\mathcal{S} := \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$$

We fix $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$ with $\operatorname{Im}\lambda \notin \mathcal{S}$. Let now $\mu \in \mathcal{D}'(Z, Q:\xi:\lambda)$. To prove the theorem, it suffices to show that μ is a sum of distributions $\mu_{x_{\mathcal{O}}H}(Q:\xi:\lambda:\eta_{\mathcal{O}})$ with $\mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$ and $\eta_{\mathcal{O}} \in (V_\xi^*)^{M\cap x_{\mathcal{O}}Hx_{\mathcal{O}}^{-1}}$.

The condition on λ assures that

$$(P\backslash Z)_\mu \subseteq (P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$$

and

$$\operatorname{trdeg}_{\mathcal{O}}(\mu) = 0 \qquad \big(\mathcal{O} \in (P\backslash Z)_\mu\big).$$

If $\mu \neq 0$, then there exists an $\mathcal{O} \in (P\backslash Z)_\mu$. Since $\operatorname{trdeg}_{\mathcal{O}}(\mu) = 0$, there exists a distribution $\mu_{\mathcal{O}}$ on $\mathcal{O}$ so that μ on

$$U = Z \setminus \Big(\partial\mathcal{O} \cup \bigcup_{\mathcal{O}'\in(P\backslash Z)_\mu\setminus\{\mathcal{O}\}} \overline{\mathcal{O}'}\Big)$$

is given by

$$\mu(\phi) = \mu_{\mathcal{O}}(\phi\big|_{\mathcal{O}}) \qquad \big(\phi \in \mathcal{D}(U, V_\xi)\big).$$

By [35, Lemma 5.5] $\mu_{\mathcal{O}}$ is in fact given by integrating against a smooth function. Moreover, since μ is right-H-invariant, also $\mu_{\mathcal{O}}$ is right-H-invariant. Likewise, $\mu_{\mathcal{O}}$ inherits the left-P-equivariance from μ. As $\mathcal{O}$ is a P-orbit in Z, $\mu_{\mathcal{O}}$ is fully determined by its value in any given point. In particular, we may evaluate $\mu_{\mathcal{O}}$ in $z_{\mathcal{O}} := x_{\mathcal{O}}H$. This results in a non-zero vector $\eta_{\mathcal{O}}$ in $(V_\xi^*)^{M_Q \cap H_{z_{\mathcal{O}}}} = (V_\xi^*)^{M_Q \cap x_{\mathcal{O}} H x_{\mathcal{O}}^{-1}}$. Let $w \in N_G(\mathfrak{a})$ satisfy (3.3) and $[\mathcal{O}] = w \cdot (P\backslash Z)_{\mathrm{open}}$. It follows from Theorem 3.2 that $\mu(\phi)$ is for every $\phi \in \mathcal{D}(U, V_\xi)$ given by

$$\begin{aligned}\int_{N_P \cap w N_Q w^{-1}} \int_{M/M\cap H_z} \int_{A/A\cap H_z} & a^{-\lambda + \mathrm{Ad}^*(w)\rho_Q} \Big(\xi^\vee(m)\eta_{\mathcal{O}}, \phi(nma \cdot z) \Big)\, da\, dm\, dn \\ &= \mu_{z_{\mathcal{O}}}(Q : \xi : \lambda : \eta_{\mathcal{O}})(\phi).\end{aligned}$$

Hence $\mu' := \mu - \mu_{z_{\mathcal{O}}}(Q : \xi : \lambda : \eta_{\mathcal{O}})$ is a distribution in $\mathcal{D}'(Z, Q : \xi : \lambda)$ with

$$(P\backslash Z)_{\mu'} \subseteq \Big((P\backslash Z)_\mu \setminus \{\mathcal{O}\} \Big) \cup \{\mathcal{O}' \in (P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}} : \mathcal{O}' \subseteq \partial \mathcal{O}\}$$

and

$$\operatorname{trdeg}_{\mathcal{O}'}\big(\mu'\big) = 0 \qquad \big(\mathcal{O}' \in (P\backslash Z)_{\mu'}\big).$$

We now replace μ by μ' and repeat this argument. After finitely many iterations of this process we obtain that μ is a sum of distributions $\mu_{x_{\mathcal{O}}H}(Q : \xi : \lambda : \eta_{\mathcal{O}})$ with $\mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$ and $\eta_{\mathcal{O}} \in (V_\xi^*)^{M \cap x_{\mathcal{O}} H x_{\mathcal{O}}^{-1}}$. □

From now on we fix a finite union $\mathcal{S}$ of proper subspaces of $(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$, so that the conclusions of Theorem 6.3 hold and for every $v \in \mathcal{N}$ and $\xi \in \widehat{M}_{Q,\mathrm{fu}}$ the intertwining operators $\mathcal{I}_v(Q : \xi : \lambda)$ and $\mathcal{I}_v^\circ(Q : \xi : \lambda)$ are isomorphisms for $\operatorname{Im} \lambda \notin \mathcal{S}$.

6.5 Action of A_E on $\mathcal{D}'(Q : \xi : \lambda)^H$

We recall the from (3.1) that $\mathfrak{a}_E$ denotes the edge of $\overline{\mathcal{C}}$, i.e.,

$$\mathfrak{a}_E = \overline{\mathcal{C}} \cap -\overline{\mathcal{C}}.$$

We write A_E for the connected subgroup of G with Lie algebra $\mathfrak{a}_E$, i.e.,

$$A_E := \exp(\mathfrak{a}_E).$$

The group A_E normalizes the Lie algebra $\mathfrak{h}$. In fact, it follows from the theory of smooth compactifications of Z that A_E normalizes $H_{\mathbb{C}}$, and hence also H. See [17, Theorem 4.1] where this is shown for an algebraic subgroup of $\underline{G}$ for which the identity component of the group of real points is equal to A_E. Therefore, A_E acts from the right on $Z = G/H$ by

$$gH \cdot a := gaH \qquad \big(g \in G, a \in A_E\big).$$

We now investigate the induced right action of A_E on the spaces $\mathcal{D}'(Q : \xi : \lambda)^H$.

If $\mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$, then we define

$$\iota_{\mathcal{O}} : (V_\xi^*)^{M_{Q,[\mathcal{O}]}} \hookrightarrow V^*(\xi) \tag{6.1}$$

to be the inclusion map determined by

$$\big(\iota_{\mathcal{O}}\eta\big)_{\mathcal{O}'} = \begin{cases} \eta & (\mathcal{O} = \mathcal{O}'), \\ 0 & (\mathcal{O} \neq \mathcal{O}'). \end{cases}$$

Proposition 6.1. *Let* (ξ, V_ξ) *be a finite dimensional unitary representation of* M_Q *and* $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ *with* $\operatorname{Im}\lambda \notin \mathcal{S}$. *If* $w \in \mathcal{N}/\mathcal{W}$ *and* $\mathcal{O} \in w \cdot (P\backslash Z)_{\mathrm{open}}$, *then*

$$R^\vee(a) \circ \mu(Q:\xi:\lambda) \circ \iota_\mathcal{O} = a^{-\operatorname{Ad}^*(v_w^{-1})\lambda + \rho_Q} \mu(Q:\xi:\lambda) \circ \iota_\mathcal{O} \qquad \big(a \in A_E\big).$$

We first prove a lemma.

Lemma 6.2. *Let* $w \in \mathcal{N}/\mathcal{W}$ *and* $\mathcal{O} \in w \cdot (P\backslash Z)_{\mathrm{open}}$. *Then*

$$x_\mathcal{O} a \in v_w a v_w^{-1} x_\mathcal{O} H \qquad \big(a \in A_E\big).$$

Proof. By [39, Lemma 12.1] the little Weyl group W_Z, and hence also $\mathcal{W}$, acts trivially on $\mathfrak{a}_E/\mathfrak{a}_\mathfrak{h}$. Let $\mathcal{O}' \in (P\backslash Z)_{\mathrm{open}}$ be so that $x_\mathcal{O} = v_w x_{\mathcal{O}'}$. Further let $t \in \exp(i\mathfrak{a})$ and $h \in H_\mathbb{C}$ be so that $x_{\mathcal{O}'} = th$. Now

$$x_\mathcal{O} a = \big(v_w a v_w^{-1}\big) v_w t\big(a^{-1} h a\big) = \big(v_w a v_w^{-1}\big) x_\mathcal{O} h^{-1}\big(a^{-1} h a\big) \qquad (a \in A_E).$$

The assertion now follows as

$$h^{-1}\big(a^{-1} h a\big) \in H_\mathbb{C} \cap G = H.$$

□

Proof of Proposition 6.1. By meromorphic continuation, it suffices to prove the assertion only for

$$\lambda \in \operatorname{Ad}(v)^*\big(\rho_Q - \Gamma + i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\big).$$

For these λ the distribution $\mu_{x_\mathcal{O} H}(Q:\xi:\lambda:\eta)$ is given by (6.8).

If $\mathfrak{s}$ is an A_E-stable subspace of $\mathfrak{g}$, then we write $\Delta_\mathfrak{s}$ for the character of A_E given by

$$\Delta_\mathfrak{s}(a) = \big|\det\nolimits_\mathfrak{s}\big(\operatorname{Ad}(a^{-1})\big|_\mathfrak{s}\big)\big| \qquad \big(a \in A_E\big).$$

If $\psi \in \mathcal{D}(G)$ and $a \in A_E$, then

$$\int_H \phi(ha^{-1})\,dh = \Delta_\mathfrak{h}(a) \int_H \phi(a^{-1}h)\,dh.$$

As $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_Q \oplus \mathfrak{m}' \oplus \mathfrak{a}'$ for suitable subspaces $\mathfrak{m}'$ and $\mathfrak{a}'$ of $\mathfrak{m}$ and $\mathfrak{a}$, respectively, we have

$$\Delta_\mathfrak{g} = \Delta_\mathfrak{h} \Delta_{\mathfrak{n}_Q} \Delta_{\mathfrak{m}' \oplus \mathfrak{a}'}.$$

Since G is reductive, the character $\Delta_\mathfrak{g}$ is trivial. As A_E centralizes $\mathfrak{m}'$ and $\mathfrak{a}'$, also $\Delta_{\mathfrak{m}' \oplus \mathfrak{a}'}$ is trivial. Furthermore,

$$\Delta_{\mathfrak{n}_Q}(a) = a^{-2\rho_Q} \qquad \big(a \in A_E\big).$$

We thus conclude that

$$\Delta_\mathfrak{h}(a) = a^{2\rho_Q} \qquad \big(a \in A_E\big).$$

The assertion now follows from Lemma 6.2, (6.8) and the invariance of the measure on $A/A \cap H$. □

6.6 B-matrices

We continue with the notation from the previous section.

The following is an immediate corollary of Theorem 6.3.

Corollary 6.1. *Let ξ be a finite dimensional unitary representation of M_Q, $v \in \mathcal{N}$, and $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ with $\operatorname{Im}\lambda \notin \mathcal{S}$. Then there exists a unique linear operator*

$$\mathcal{B}_v(Q:\xi:\lambda) : V^*(\xi) \to V^*(v\cdot\xi)$$

so that the diagram

$$\begin{tikzcd}
\mathcal{D}'(Q:\xi:\lambda)^H \arrow[r, "\mathcal{I}_v(Q:\xi:\lambda)"] & \mathcal{D}'(Q:v\cdot\xi:\mathrm{Ad}^*(v)\lambda)^H \\
V^*(\xi) \arrow[u, "\mu(Q:\xi:\lambda)"] \arrow[r, "\mathcal{B}_v(Q:\xi:\lambda)"] & V^*(v\cdot\xi) \arrow[u, "\mu\left(Q:v\cdot\xi:\mathrm{Ad}^*(v)\lambda\right)"']
\end{tikzcd} \tag{6.1}$$

commutes.

A similar map was first introduced in [1] in the setting of real reductive symmetric spaces, where it was called the $\mathcal{B}$-matrix, hence our notation.

We will prove a few properties of $\mathcal{B}$-matrices. Recall the maps $\iota_\mathcal{O}$ from (6.1) and the set of representatives $\{v_w : w \in \mathcal{N}/\mathcal{W}\}$ for $\mathcal{N}/\mathcal{W}$ in $\mathcal{N}\cap K$ from Section 6.4. If $w \in \mathcal{N}$, then by slight abuse of notation we write v_w for $v_{w\mathcal{W}}$. Every element $w \in \mathcal{N}$ defines a bijection

$$s_w : (P\backslash Z)_{\mathfrak{a}_\mathfrak{h}} \to (P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}; \quad \mathcal{O} \mapsto Pwx_\mathcal{O}H. \tag{6.2}$$

Note that

$$[s_w(\mathcal{O})] = w\cdot[\mathcal{O}] \qquad \big(w \in \mathcal{N}, \mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}\big),$$

and hence

$$M_{Q,[s_w\mathcal{O}]} = M_{Q,w\cdot[\mathcal{O}]} = wM_{Q,[\mathcal{O}]}w^{-1} \qquad \big(w \in \mathcal{N}, \mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}\big).$$

Proposition 6.2. *Let ξ be a finite dimensional unitary representation of M_Q and $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ with $\operatorname{Im}\lambda \notin \mathcal{S}$. Let $v, w \in \mathcal{N}$ and let $\mathcal{O} \in w\cdot(P\backslash Z)_{\mathrm{open}}$. Let further $\eta \in (V^*_\xi)^{M_{Q,[\mathcal{O}]}} = (V^*_\xi)^{M_Q\cap v_wHv_w^{-1}}$. Then $\mathcal{B}_v(Q:\xi:\lambda)\circ\iota_\mathcal{O}(\eta)$ satisfies the following assertions.*

(i) If $vw \notin Z_G(\mathfrak{a}_E/\mathfrak{a}_\mathfrak{h})$, then

$$\Big(\mathcal{B}_v(Q:\xi:\lambda)\circ\iota_\mathcal{O}(\eta)\Big)_{\mathcal{O}'} = 0 \qquad \big(\mathcal{O}' \in (P\backslash Z)_{\mathrm{open}}\big).$$

(ii) If $\dim\left(v^{-1}N_Qv\cap\overline{N_Q}\right) + \dim(\mathcal{O}) < \dim(Z)$, then

$$\Big(\mathcal{B}_v(Q:\xi:\lambda)\circ\iota_\mathcal{O}(\eta)\Big)_{\mathcal{O}'} = 0 \qquad \big(\mathcal{O}' \in (P\backslash Z)_{\mathrm{open}}\big).$$

(iii) If $\dim\big(v^{-1}N_Qv\cap\overline{N}_Q\big)+\dim(\mathcal{O})=\dim(Z)$ *and* $vw\notin\mathcal{W}$*, then*

$$\Big(\mathcal{B}_v(Q:\xi:\lambda)\circ\iota_{\mathcal{O}}(\eta)\Big)_{\mathcal{O}'}=0\qquad\big(\mathcal{O}'\in(P\backslash Z)_{\mathrm{open}}\big).$$

(iv) If $v=v_w^{-1}$*, then*

$$\Big(\mathcal{B}_v(Q:\xi:\lambda)\circ\iota_{\mathcal{O}}(\eta)\Big)_{\mathcal{O}'}=\begin{cases}\eta & \big(\mathcal{O}'=s_v(\mathcal{O})\big),\\[2ex] 0 & \big(\mathcal{O}'\in(P\backslash Z)_{\mathrm{open}},\mathcal{O}'\neq s_v(\mathcal{O})\big).\end{cases}$$

Proof. Let $\mu=\mathcal{I}_v(Q:\xi:\lambda)\circ\mu(Q:\xi:\lambda)\eta$.

By Proposition 6.1 we have for all $a\in A_E$

$$R^\vee(a)\mu=\mathcal{I}_v(Q:\xi:\lambda)\circ R^\vee(a)\circ\mu(Q:\xi:\lambda)(\eta)=a^{-\operatorname{Ad}^*(v_w^{-1})\lambda+\rho_Q}\mu.$$

Let $\eta'\in V^*(v\cdot\xi)$ be so that $\mu=\mu\big(Q:v\cdot\xi:\operatorname{Ad}^*(v)\lambda\big)\eta'$. Then for all $a\in A_E$

$$R^\vee(a)\mu=\sum_{w'\in\mathcal{N}/\mathcal{W}}\sum_{\mathcal{O}'\in w'\cdot(P\backslash Z)_{\mathrm{open}}}a^{-\operatorname{Ad}^*(v_{w'}^{-1}v)\lambda+\rho_Q}\mu(Q:\xi:\lambda)\circ\iota_{\mathcal{O}'}(\eta'_{\mathcal{O}'}).$$

Both identities are identities of meromorphic functions in the parameter λ. Therefore, the only terms in the sum on the right-hand side of the second identity that can be non-zero, are those for $w'\in\mathcal{N}/\mathcal{W}$ with $v_{w'}^{-1}vv_w\in Z_G(\mathfrak{a}_E/\mathfrak{a}_\mathfrak{h})$. Since $\mathcal{W}$ is a subgroup of $Z_G(\mathfrak{a}_E/\mathfrak{a}_\mathfrak{h})$, see [39, Lemma 12.1], the latter condition is equivalent to $v_{w'}^{-1}vw\in Z_G(\mathfrak{a}_E/\mathfrak{a}_\mathfrak{h})$. Assertion (i) now follows by taking $w'=e\mathcal{W}$.

We move on to prove (ii) and (iii). By meromorphic continuation it suffices to prove the assertion for $\lambda\in(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ for which the intertwining operator $\mathcal{A}(v^{-1}Qv:Q:\xi:\lambda)$ is given by a convergent integral over $v^{-1}N_Qv\cap\overline{N}_Q$. Since $\operatorname{supp}\big(\mu(Q:\xi:\lambda)\eta\big)\subseteq\overline{\mathcal{O}}$, we then have

$$\operatorname{supp}(\mu)=v\cdot\operatorname{supp}\big(\mathcal{A}(v^{-1}Qv:Q:\xi:\lambda)\mu\big)\subseteq v\cdot\overline{\big(v^{-1}N_Qv\cap\overline{N}_Q\big)\cdot\overline{\mathcal{O}}}.$$

If $\dim\big(v^{-1}N_Qv\cap\overline{N}_Q\big)+\dim(\mathcal{O})<\dim(Z)$, then the interior of the support of μ is empty. This proves (ii). Assume that $\dim\big(v^{-1}N_Qv\cap\overline{N}_Q\big)+\dim(\mathcal{O})=\dim(Z)$ and the support of μ contains an open P-orbit $\mathcal{O}'$. Then $v^{-1}\cdot\mathcal{O}'\subseteq(v^{-1}N_Qv\cap\overline{N}_Q)\cdot\mathcal{O}$, and hence $v^{-1}\cdot\mathcal{O}'\cap\mathcal{O}\neq\emptyset$. Let $z\in v^{-1}\cdot\mathcal{O}'\cap\mathcal{O}$. It follows from Proposition 3.1 that

$$v^{-1}N_Qv\to v^{-1}N_Qv\cdot z;\quad n\mapsto n\cdot z$$

is a diffeomorphism. Since

$$(v^{-1}N_Qv\cap\overline{N}_P)\times(v^{-1}N_Qv\cap N_P)\to v^{-1}N_Qv;\quad(\overline{n},n)\mapsto\overline{n}n$$

is a diffeomorphism as well, we obtain that

$$(v^{-1}N_Qv\cap N_P)\to(v^{-1}N_Qv\cap N_P)\cdot z;\quad n\mapsto n\cdot z$$

is a diffeomorphism. We note that v normalizes M_Q in view of Remark 3.2 as it is contained in $\mathcal{N}$. Therefore,

$$v^{-1}N_Q v \cap \overline{N}_P = v^{-1}N_Q v \cap \overline{N}_Q.$$

It follows from Proposition 3.1, Theorem 3.2 and the assumption on the dimension of $\mathcal{O}$ that

$$\begin{aligned}\dim(N_P \cdot z) &= \dim(N_Q) - \dim(Z) + \dim(\mathcal{O}) = \dim(N_Q) - \dim\left(v^{-1}N_Q v \cap \overline{N}_Q\right)\\ &= \dim\left(v^{-1}N_Q v \cap N_P\right) = \dim\Big(\left(v^{-1}N_Q v \cap N_P\right) \cdot z\Big)\end{aligned}$$

This implies that $(v^{-1}N_Q v \cap N_P) \cdot z$ is open in $N_P \cdot z$. in view of [44, Theorem 2] both $(v^{-1}N_Q v \cap N_P) \cdot z$ and $N_P \cdot z$ are closed. Therefore, $(v^{-1}N_Q v \cap N_P) \cdot z$ is also closed in $N_P \cdot z$. Moreover, both are connected. We thus conclude

$$(v^{-1}N_Q v \cap N_P) \cdot z = N_P \cdot z.$$

In particular we see that $N_P \cdot z \subseteq v^{-1} \cdot \mathcal{O}' \cap \mathcal{O}$. Since every N_P-orbit in $\mathcal{O}$ contains a weakly admissible point, we may without loss of generality assume that z is weakly admissible.

Now $v \cdot z$ is an admissible point in $\mathcal{O}'$. Hence if $X \in \mathfrak{a}^-$ is order-regular, then there exists an $u \in \mathcal{W}$ so that $\mathfrak{h}_{v\cdot z,\mathrm{Ad}(v)X} = \mathrm{Ad}(u)\mathfrak{h}_\emptyset$. We now have

$$\mathfrak{h}_{z,X} = \mathrm{Ad}(v^{-1})\mathfrak{h}_{v\cdot z,\mathrm{Ad}(v)X} = \mathrm{Ad}(v^{-1}u)\mathfrak{h}_\emptyset.$$

This implies that $\mathcal{O} \in v^{-1}u \cdot (P\backslash Z)_{\mathrm{open}}$. By assumption $\mathcal{O} \in w \cdot (P\backslash Z)_{\mathrm{open}}$. Therefore, $u^{-1}vw$ stabilizes $(P\backslash Z)_{\mathrm{open}}$. Since the stabilizer is equal to $\mathcal{W}$ by Theorem 3.3(ii), we may conclude that $vw \in \mathcal{W}$. This proves (iii).

Finally, we prove (iv). Let $v = v_w^{-1}$. In view of (6.2) we have

$$\mu = L^\vee(v)\mu_{x_\mathcal{O}H}(v^{-1}Pv : \xi : \lambda + \rho_P - \rho_Q : \eta).$$

Using meromorphic continuation, (6.3) and the fact that v_w satisfies (3.3) we obtain

$$\mu = \mu_{vx_\mathcal{O}H}\big(P : v\cdot\xi : \mathrm{Ad}^*(v)\lambda + \rho_P - \rho_Q : \eta\big) = \mu\big(Q : v\cdot\xi : \mathrm{Ad}^*(v)\lambda\big) \circ \iota_{s_v(\mathcal{O})}(\eta).$$

This proves (iv). □

We define the map

$$\beta(\xi : \lambda) : V^*(\xi) \to V^*(\xi)$$

for $\eta \in V^*(\xi)$, $\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}$ and $w \in \mathcal{N}/\mathcal{W}$ to be given by

$$\Big(\beta(\xi : \lambda)\eta\Big)_{s_{v_w}(\mathcal{O})} = \frac{1}{\gamma(v_w\overline{Q}v_w^{-1} : \overline{Q} : \xi : \lambda)}\Big(\mathcal{B}_{v_w^{-1}}(Q : \xi : \lambda)\eta\Big)_{\mathcal{O}}. \tag{6.3}$$

We will use $\beta(\xi : \lambda)$ for the normalization of the map $\mu(Q : \xi : \lambda)$ in the next section.

Let ev_g denote evaluation in a point $g \in G$. Since

$$\Big(\mathcal{B}_{v_w^{-1}}(Q:\xi:\lambda)\eta\Big)_{\mathcal{O}} = \mathrm{ev}_{v_w x_{\mathcal{O}}} \circ \mathcal{A}(v_w Q v_w^{-1} : Q : \xi : \lambda) \circ \mu(Q:\xi:\lambda)$$

depends meromorphically on $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$ for $w \in \mathcal{N}/\mathcal{W}$ and $\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}$, the assignment

$$(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}} \to \mathrm{End}\big(V^*(\xi)\big); \quad \lambda \mapsto \beta(\xi:\lambda)$$

is a meromorphic function. Moreover, $\lambda \mapsto \beta(\xi:\lambda)$ is holomorphic on

$$\{\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}} : \mathrm{Im}\,\lambda \notin \mathcal{S}\}.$$

If we order the orbits in $(P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$ by dimension and choose a basis of $V^*(\xi)$ subject to the decomposition

$$V^*(\xi) = \bigoplus_{\mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}} (V_\xi^*)^{M_{Q,[\mathcal{O}]}},$$

then in view of Proposition 6.2(ii – iv) the matrix of $\beta(\xi:\lambda)$ with respect to this basis is upper triangular and the diagonal entries are reciprocals of γ-factors. It follows that $\beta(\xi:\lambda)$ is invertible. Since $\lambda \mapsto \beta(\xi:\lambda)$ is meromorphic, it follows from Cramer's rule that also

$$(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}} \to \mathrm{End}\big(V^*(\xi)\big); \quad \lambda \mapsto \beta(\xi:\lambda)^{-1}$$

is meromorphic. This observation has the following corollary.

Corollary 6.3. *Let ξ be a finite dimensional unitary representation of M_Q. For every $v \in \mathcal{N}$ the $\mathcal{B}$-matrix $\mathcal{B}_v(Q:\xi:\lambda)$ depends meromorphically on $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$.*

Proof. The map $\beta(\xi:\lambda) \circ \mu(Q:\xi:\lambda)^{-1}$ is for $\mu \in \mathcal{D}'(Q:\xi:\lambda)^H$, $w \in \mathcal{N}/\mathcal{W}$ and $\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}$ given by

$$\begin{aligned}&\Big(\beta(\xi:\lambda) \circ \mu(Q:\xi:\lambda)^{-1}(\mu)\Big)_{s_{v_w}(\mathcal{O})}\\ &\quad = \frac{1}{\gamma(v_w \overline{Q} v_w^{-1} : \overline{Q} : \xi : \lambda)} \mathrm{ev}_{v_w x_{\mathcal{O}}} \circ \mathcal{A}(v_w Q v_w^{-1} : Q : \xi : \lambda)(\mu).\end{aligned}$$

It follows that for a meromorphic family of distributions $\mu_\lambda \in \mathcal{D}'(Q:\xi:\lambda)^H$ with family parameter $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$, the assignment $\lambda \mapsto \beta(\xi:\lambda) \circ \mu(Q:\xi:\lambda)^{-1}(\mu_\lambda)$ is meromorphic. We apply this to

$$\mu_\lambda = \mathcal{I}_v(Q:\xi:\lambda) \circ \mu(Q:\xi:\lambda)(\eta)$$

for $v \in \mathcal{N}$ and $\eta \in V^*(\xi)$ and thus conclude that

$$\begin{aligned}\mathcal{B}_v(Q:\xi:\lambda)\eta &= \mu(Q:\xi:\lambda)^{-1} \circ \mathcal{I}_v(Q:\xi:\lambda) \circ \mu(Q:\xi:\lambda)(\eta)\\ &= \beta(\xi:\lambda)^{-1} \circ \Big(\beta(\xi:\lambda) \circ \mu(Q:\xi:\lambda)^{-1}\Big)(\mu_\lambda)\end{aligned}$$

depends meromorphically on λ. $\square$

6.7 Normalization

We continue with the notation from the previous section. For a finite dimensional unitary representation ξ of M_Q and $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ with $\operatorname{Im}\lambda \notin \mathcal{S}$, we normalize our distributions $\mu(Q:\xi:\lambda)\eta$ using the map $\beta(\xi:\lambda)$ from (6.3) by defining

$$\mu^\circ(\xi:\lambda) := \mathcal{A}\big(Q:\overline{Q}:\xi:\lambda\big)^{-1} \circ \mu(Q:\xi:\lambda) \circ \beta(\xi:\lambda)^{-1} : V^*(\xi) \to \mathcal{D}'(\overline{Q}:\xi:\lambda)^H. \tag{6.1}$$

The reason for normalizing the distributions is to make sure that the composition of the constant term map and $\mu^\circ(\xi:\lambda)$ will have a desirable form, see Section 8.4.

We end this section with a reformulation of Theorem 6.3.

Theorem 6.1. *For every finite dimensional unitary representation (ξ, V_ξ) of M_Q and every $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ with $\operatorname{Im}\lambda \notin \mathcal{S}$ the map*

$$\mu^\circ(\xi:\lambda) : V^*(\xi) \to \mathcal{D}'(\overline{Q}:\xi:\lambda)^H \tag{6.2}$$

is a linear isomorphism. The assignment

$$\lambda \mapsto \mu^\circ(\xi:\lambda)\eta$$

defines for every $\eta \in V^*(\xi)$ a meromorphic family of distributions in $\mathcal{D}'(G, V_\xi)$ with family parameter $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$. The poles of the family lie on a locally finite union of complex affine hyperplanes of the form

$$\{\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C} : \lambda(X) = a\} \quad \textit{for some } X \in \mathfrak{a} \textit{ and } a \in \mathbb{R}. \tag{6.3}$$

Proof. The poles of standard intertwining operators, as well as the poles and zero's of γ-functions, all lie on a locally finite union of complex affine hyperplanes of the form (6.3). The proposition now follows from Theorem 6.3. □

For future reference we record here that in view of the corollary we may and will equip $\mathcal{D}'(\overline{Q}:\xi:\lambda)^H$ for $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ with $\operatorname{Im}\lambda \notin \mathcal{S}$ with an inner product so that the map (6.2) is an isometry.

6.8 The horospherical case

We call the real spherical homogeneous space Z horospherical if $\mathfrak{a}$ normalizes $\mathfrak{h}_z$ for one (and hence for every) adapted point $z \in Z$. We note that Z is horospherical if and only if the compression cone $\mathcal{C}$ equals $\mathfrak{a}$, which is equivalent to the little Weyl group of Z being trivial. In this case the stabilizer H_z of an adapted point $z \in Z$ is given by

$$H_z = (L_Q \cap H_z)\overline{N}_Q = (M \cap H_z)\exp(\mathfrak{a}_\mathfrak{h})L_{Q,\mathrm{nc}}\overline{N}_Q,$$

where $L_{Q,\mathrm{nc}}$ is the connected subgroup of G with Lie algebra $\mathfrak{l}_{Q,\mathrm{nc}}$. In this section we further explicate the description of $\mathcal{D}'(\overline{Q}:\xi:\lambda)^H$ from Theorem 6.4 and Theorem 6.1 under the assumption that H is horospherical.

We first note that $\overline{N}_P \subseteq L_{Q,\mathrm{nc}}\overline{N}_Q \subseteq H_z$ for every adapted point z. As P admits only one open orbit in $G/\overline{N}_P$, it follows that there exists precisely one open P-orbit $\mathcal{O}_0$ in Z. Recall that the point $eH \in Z = G/H$ is assumed to be admissible. As in [39, Example 3.5] it is easily seen that the set of adapted points in Z is equal to MAH/H. In particular, we are in the situation of Remark 3.4 (b) and (c), and hence the Weyl group W acts transitively on the set of P-orbits in Z of maximal rank. This action is given by (3.3) and (3.4). By Theorem 3.3 (ii) the stabilizer of the open orbit is equal to $\mathcal{Z}/MA$, where $\mathcal{Z} = N_{L_Q}(\mathfrak{a})$.

It follows from the Bruhat decomposition that each P-orbit in Z is of the form PwH, with $w \in N_G(\mathfrak{a})$. If $\mathcal{O} = PwH$, then

$$\mathfrak{a}_{\mathcal{O}} = \mathfrak{a} \cap \mathrm{Ad}(w)\mathfrak{h} = \mathrm{Ad}(w)\mathfrak{a}_{\mathfrak{h}}.$$

In particular, each orbit is of maximal rank. Therefore, the map

$$N_G(\mathfrak{a})/\mathcal{Z} \to P\backslash Z; \quad v\mathcal{Z} \mapsto PvH \tag{6.1}$$

is a bijection. We recall from (3.1) that $\mathcal{N}$ denotes the group $N_G(\mathfrak{a}) \cap N_G(\mathfrak{a}_{\mathfrak{h}})$. The image of $\mathcal{N}/\mathcal{Z}$ under the map (6.1) is equal to the set $(P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$ of P-orbits $\mathcal{O}$ in Z with $\mathfrak{a}_{\mathcal{O}} = \mathfrak{a}_{\mathfrak{h}}$. We complete the set $\{v_w : w \in \mathcal{N}/\mathcal{W}\}$ from Section 6.4 to a set of representatives $\mathfrak{N}$ of $\mathcal{N}/\mathcal{Z}$ in $\mathcal{N} \cap K$. Then

$$\mathfrak{N} \to (P\backslash Z)_{\max}; \quad v \mapsto PvH$$

is a bijection and the points $vH \in Z$ with $v \in \mathfrak{N}$ are weakly adapted. The $v \in \mathfrak{N}$ play the role of the elements $x_{\mathcal{O}} \in G$ from section 6.4.

Let ξ be a finite dimensional unitary representation of M_Q. Then

$$V^*(\xi) = \bigoplus_{v \in \mathfrak{N}} (V_\xi^*)^{M_Q \cap vHv^{-1}} \tag{6.2}$$

It follows from Proposition 6.2(i) that the map $\beta(\xi : \lambda)$ is diagonal with respect to a basis of $V^*(\xi)$ subject to the decomposition (6.2). Now Proposition 6.2 (iv) yields that $\mu^\circ(\xi : \lambda)$ for $\eta \in V^*(\xi)$ is given by

$$\mu^\circ(\xi : \lambda)\eta = \sum_{v \in \mathcal{N}} \gamma(v^{-1}\overline{Q}v : \overline{Q} : \xi : \lambda)\mathcal{A}(Q : \overline{Q} : \xi : \lambda)^{-1}\mu_{vH}(Q : \xi : \lambda : \eta_v).$$

For $v \in \mathfrak{N}$ we write ι_v for the inclusion map

$$\iota_v : (V_\xi^*)^{M_Q \cap vHv^{-1}} \hookrightarrow V^*(\xi). \tag{6.3}$$

For $v \in \mathcal{N}$ we write $\mathcal{I}_v^\circ(\xi : \lambda)$ for the normalized intertwining operator $\mathcal{I}_v^\circ(\overline{Q} : \xi : \lambda)$ from Section 4.3.

Corollary 6.1. *Let (ξ, V_ξ) be a finite dimensional unitary representation of M_Q. For all $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$ with $\mathrm{Im}\,\lambda \notin \mathcal{S}$ the map*

$$\mu^\circ(\xi : \lambda) : V^*(\xi) \to \mathcal{D}'(\overline{Q} : \xi : \lambda)^H; \tag{6.4}$$

is a linear isomorphism. The assignment

$$\lambda \mapsto \mu^\circ(\xi : \lambda)\eta$$

defines for every $\eta \in V^*(\xi)$ a meromorphic family of distributions in $\mathcal{D}'(G, V_\xi)$ with family parameter $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$. For all $v \in \mathfrak{N}$ the distributions

$$\mu \in \mathcal{D}'\big(Q : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big)^H$$

are smooth in e and may therefore be evaluated in e. The inverse of (6.4) is given by

$$\begin{aligned}&\mathcal{D}'(\overline{Q} : \xi : \lambda)^H \to V^*(\xi);\\ &\qquad\mu \mapsto \Big(\mathrm{ev}_e \circ \mathcal{A}\big(Q : \overline{Q} : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ \mathcal{I}^\circ_{v^{-1}}(\xi : \lambda)(\mu)\Big)_{v \in \mathfrak{N}},\end{aligned} \tag{6.5}$$

where ev_g denotes evaluation in a point $g \in G$.

Let $v \in \mathfrak{N}$. Then

$$R^\vee(a)\mu^\circ(\xi : \lambda) \circ \iota_v = a^{-\mathrm{Ad}^*(v^{-1})\lambda + \rho_Q}\mu^\circ(\xi : \lambda) \circ \iota_v \qquad (a \in A) \tag{6.6}$$

and

$$\mathcal{I}^\circ_v(\xi : \lambda) \circ \mu^\circ(\xi : \lambda) \circ \iota_e = \mu^\circ\big(v \cdot \xi : \mathrm{Ad}^*(v)\lambda\big) \circ \iota_v. \tag{6.7}$$

Finally, if λ satisfies $\mathrm{Re}\,\lambda(\alpha^\vee) > 0$ for all $\alpha \in -\Sigma(Q) \cap \Sigma(vQv^{-1})$, then for every $\eta \in (V^*_\xi)^{M_Q \cap vHv^{-1}}$ the distribution $\mu^\circ(\xi : \lambda)(\iota_v\eta)$ is for $\phi \in \mathcal{D}(G, V_\xi)$ given by

$$\begin{aligned}&\gamma(Q : vQv^{-1} : \xi : \lambda)\mu^\circ(\xi : \lambda)(\iota_v\eta)(\phi)\\ &\quad= \int_{\overline{N}_Q \cap vN_Qv^{-1}} \int_{M_Q} \int_A \int_{\overline{N}_Q} a^{-\lambda - \mathrm{Ad}^*(v)\rho_Q}\Big(\xi^\vee(m)\eta, \phi(nmav\overline{n})\Big)\, d\overline{n}\, da\, dm\, dn.\end{aligned} \tag{6.8}$$

Proof. Except for the identities (6.5), (6.7) and (6.8) all assertions follow directly from Theorem 6.4, Proposition 6.1 and Theorem 6.1.

Let $\eta \in V_\xi^{M_Q \cap H}$. It follows from (6.8) that $\mu(Q : \xi : \lambda)(\iota_e\eta)$ for $\phi \in \mathcal{D}(G, V^*_\xi)$ and $\lambda \in \rho_Q - \Gamma + i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ is given by

$$\big(\mu(Q : \xi : \lambda)(\iota_e\eta)\big)(\phi) = \int_{N_Q} \int_{M_Q} \int_A \int_{\overline{N}_Q} a^{-\lambda - \rho_Q}\Big(\xi^\vee(m)\eta, \phi(nma\overline{n})\Big)\, d\overline{n}\, da\, dm\, dn.$$

In view of (4.1) in Proposition 4.1 and (6.1) we have

$$\big(\mu^\circ(\xi : \lambda)(\iota_e\eta)\big)(\phi) = \int_{M_Q} \int_A \int_{\overline{N}_Q} a^{-\lambda - \rho_Q}\Big(\xi^\vee(m)\eta, \phi(ma\overline{n})\Big)\, d\overline{n}\, da\, dm. \tag{6.9}$$

The right-hand side is a convergent integral for all $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ and depends holomorphically on λ. Therefore, the identity holds for all $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$.

Let $v \in \mathfrak{N}$ and $\eta \in V_\xi^{M_Q \cap H}$. Since

$$\begin{aligned}
&\mathcal{A}\big(Q : \overline{Q} : v \cdot \xi : \mathrm{Ad}^*(v)\lambda\big) \circ \mathcal{I}_v^\circ(\xi : \lambda)\\
&\quad = \frac{1}{\gamma\big(v^{-1}\overline{Q}v : \overline{Q} : \xi : \lambda\big)} \mathcal{A}\big(Q : \overline{Q} : v \cdot \xi : \mathrm{Ad}^*(v)\lambda\big)\\
&\qquad\qquad\qquad\qquad \circ \mathcal{A}\big(\overline{Q} : v\overline{Q}v^{-1} : v \cdot \xi : \mathrm{Ad}^*(v)\lambda\big) \circ L^\vee(v)\\
&\quad = \gamma\big(v\overline{Q}v^{-1} : \overline{Q} : v \cdot \xi : \mathrm{Ad}^*(v)\lambda\big)\mathcal{A}\big(Q : v\overline{Q}v^{-1} : v \cdot \xi : \mathrm{Ad}^*(v)\lambda\big) \circ L^\vee(v),
\end{aligned}$$

we have by (6.9) and (4.1) for $\phi \in \mathcal{D}(G : V_\xi^*)$

$$\begin{aligned}
&\Big(\mathcal{A}(Q : \overline{Q} : v \cdot \xi : \mathrm{Ad}^*(v)\lambda) \circ \mathcal{I}_v^\circ(\xi : \lambda) \circ \mu^\circ(\xi : \lambda)(\iota_e\eta)\Big)(\phi)\\
&\quad = \gamma\big(v\overline{Q}v^{-1} : \overline{Q} : v \cdot \xi : \mathrm{Ad}^*(v)\lambda\big)\\
&\quad \times \int_{N_Q \cap vN_Qv^{-1}} \int_{M_Q} \int_A \int_{\overline{N}_Q} a^{-\mathrm{Ad}^*(v)\lambda - \mathrm{Ad}^*(v)\rho_Q} \Big((v \cdot \xi^\vee)(m)\eta, \phi(nmav\overline{n})\Big)\, d\overline{n}\, da\, dm\, dn
\end{aligned}$$

under the condition that λ satisfies $\mathrm{Re}\,\lambda(\alpha^\vee) > 0$ for all $\alpha \in \Sigma(Q) \cap -\Sigma(v^{-1}Qv)$. It follows from (6.8) and (6.1) that the right-hand side equals

$$\begin{aligned}
&\gamma\big(v\overline{Q}v^{-1} : \overline{Q} : v \cdot \xi : \mathrm{Ad}^*(v)\lambda\big)\mu_{vH}\big(Q : v \cdot \xi : \mathrm{Ad}^*(v)\lambda : \eta\big)(\phi)\\
&\qquad = \Big(\mathcal{A}\big(Q : \overline{Q} : v \cdot \xi : \mathrm{Ad}^*(v)\lambda\big) \circ \mu^\circ\big(v \cdot \xi : \mathrm{Ad}^*(v)\lambda\big)(\iota_v\eta)\Big)(\phi).
\end{aligned}$$

By meromorphic continuation we obtain (6.7).

If λ satisfies $\mathrm{Re}\,\lambda(\alpha^\vee) > 0$ for all $\alpha \in -\Sigma(Q) \cap \Sigma(vQv^{-1})$, then (6.8) follows from (4.1), (6.7) and (6.9).

Finally we move on to show (6.5). Let $v, w \in \mathfrak{N}$ and $\eta \in (V_\xi^*)^{M_Q \cap wHw^{-1}}$. We set

$$\mu := \mathcal{A}\big(Q : \overline{Q} : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ \mathcal{I}_{v^{-1}}^\circ(\xi : \lambda) \circ \mu^\circ(\xi : \lambda)(\iota_w\eta).$$

By (6.7) we have

$$\begin{aligned}
\mu &= \mathcal{A}\big(Q : \overline{Q} : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ \mathcal{I}_{v^{-1}w}^\circ\big(w^{-1}\xi : \mathrm{Ad}^*(w^{-1})\lambda\big)\\
&\qquad\qquad\qquad\qquad \circ \mu^\circ\big(w^{-1}\xi : \mathrm{Ad}^*(w^{-1})\lambda\big)(\iota_e\eta)\\
&= c\mathcal{A}\big(Q : v^{-1}w\overline{Q}w^{-1}v : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ L^\vee(v^{-1}w)\\
&\qquad\qquad\qquad\qquad \circ \mu^\circ\big(w^{-1}\xi : \mathrm{Ad}^*(w^{-1})\lambda\big)(\iota_e\eta)
\end{aligned}$$

for some $c \in \mathbb{C}$. By meromorphic continuation it follows from (6.8) and (4.1) that

$$\mathrm{supp}(\mu) \subseteq \overline{N_Q v^{-1} w\overline{Q}},$$

and hence $e \in \mathrm{supp}(\mu)$ if and only if $v = w$. Now suppose $v = w$. Then μ is smooth on the open subset $vQ\overline{N}_Q$. We may therefore evaluate μ in e. Now

$$\begin{aligned}
\mathrm{ev}_e(\mu) &= \mathrm{ev}_e \circ \mathcal{A}\big(Q : \overline{Q} : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ \mu^\circ\big(v^{-1}\xi : \mathrm{Ad}^*(v^{-1})\lambda\big)(\iota_e\eta)\\
&= \mathrm{ev}_e \circ \mu\big(Q : v^{-1}\xi : \mathrm{Ad}^*(v^{-1})\lambda\big)(\iota_e\eta) = \eta.
\end{aligned}$$

This proves (6.5). $\square$

7 Temperedness, the constant term and wave packets

We continue with the notation and choices from Section 6.4. If V is a finite dimensional vector space, $\mu \in \mathcal{D}'(G, V)$ and $\phi \in \mathcal{D}(G, V)$, then we write $m_{\phi,\mu}$ for the matrix coefficient

$$m_{\phi,\mu} : G \to \mathbb{C}; \quad g \mapsto \big(R^\vee(g)\mu\big)(\phi).$$

The aim of this section is to show that for every unitary representation ξ of M_Q and $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ outside of a finite union of hyperplanes all matrix-coefficients for $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ are tempered and that wave-packets of these matrix-coefficients are square integrable.

7.1 Temperedness

The space Z admits a polar decomposition, which was first given in [29]. The following version is a slight reformulation from [39, Proposition 8.6].

Recall that K is a maximal compact subgroup so that $\mathfrak{k}$ is Killing perpendicular to $\mathfrak{a}$.

Proposition 7.1. *There exists a compact subset $\Omega \subseteq G$ so that*

$$G = \bigcup_{\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}} \Omega \exp(\overline{\mathcal{C}}) x_\mathcal{O} H. \tag{7.1}$$

The set Ω can be chosen to be $\Omega = \bigcup_{j=1}^r f_j K_j$ with $r \in \mathbb{N}$, $f_j \in G$ and the K_j maximal compact subgroups of G.

Let ξ be a finite dimensional unitary representation of M_Q and $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$. We call a distribution $\eta \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ tempered if there exists an $N \in \mathbb{N}_0$ and a continuous seminorm p on $\mathcal{D}(G : V_\xi)$ so that for every $\phi \in \mathcal{D}(G : V_\xi)$

$$|m_{\phi,\eta}(\omega \exp(X) x_\mathcal{O})| \leq e^{\rho_Q(X)}(1 + \|X\|)^N p(\phi) \qquad \big(\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}, \omega \in \Omega, X \in \overline{\mathcal{C}}\big).$$

We recall that we have equipped the spaces $\mathcal{D}'(\overline{Q} : \xi : \lambda)^H$, with $\mathrm{Im}\,\lambda \notin \mathcal{S}$, with an inner product so that the map (6.2) is an isometry.

Theorem 7.2. *Let ξ be a finite dimensional unitary representation of M_Q and $\mathfrak{C}$ a compact subset of $\{\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C} : \mathrm{Im}\,\lambda \notin i\mathcal{S}\}$. There exist an $N \in \mathbb{N}_0$ and a continuous seminorm p on $\mathcal{D}(G, V_\xi)$ so that for every $\lambda \in \mathfrak{C}$, $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$, $\phi \in \mathcal{D}(G, V_\xi)$, $\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}$, $\omega \in \Omega$ and $X \in \overline{\mathcal{C}}$*

$$|m_{\phi,\mu}(\omega \exp(X) x_\mathcal{O})| \leq \max_{w \in W} e^{\rho_Q(X) + \mathrm{Re}\,\mathrm{Ad}^*(w)\lambda(X)}(1 + \|X\|)^N \|\mu\| p(\phi).$$

In particular, every distribution $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ with $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \setminus i\mathcal{S}$ is tempered.

The proof for the theorem is by induction on the faces $\mathcal{F}$ of $\overline{\mathcal{C}}$. In the Sections 7.2 we give an a priori estimate which accomplishes the initial step of the induction. In Section 7.3 we recall the notion of boundary degenerations and some of their properties. As the proof of the theorem relies heavily on the theory of the constant term map from [18], we have to recall the necessary definitions and results. We do so in Section 7.4. Finally the proof of the theorem will be given in Section 7.5.

7.2 An a priori estimate

We begin the proof of Theorem 7.1 with an a priori estimate on matrix-coefficients.

For $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$ we define $\langle\lambda\rangle : \mathfrak{a} \to \mathbb{R}$ by

$$\langle\lambda\rangle := \max\Big(\{0\} \cup \{\mathrm{Re}\ \mathrm{Ad}^*(w)\lambda : w \in W\}\Big).$$

Lemma 7.1. *Let ξ be a finite dimensional unitary representation of M_Q and $\mathfrak{C}$ a compact subset of $\{\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}} : \mathrm{Im}\,\lambda \notin \mathcal{S}\}$. There exists an $\zeta \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ with $\zeta|_{\mathfrak{a}_E} = \rho_Q|_{\mathfrak{a}_E}$, and a continuous seminorm p on $\mathcal{D}(G, V_\xi)$ so that for every $\lambda \in \mathfrak{C}$, $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ and $\phi \in \mathcal{D}(G, V_\xi)$ we have*

$$|m_{\phi,\mu}(\omega \exp(X))| \leq e^{\zeta(X)+\langle\lambda\rangle(X)}\|\mu\|p(\phi) \qquad \big(\omega \in \Omega, X \in \exp(\overline{\mathcal{C}})\big).$$

Proof. Let $\mathfrak{a}_0$ be a complementary subspace to $\mathfrak{a}_E$ in $\mathfrak{a}$. Let $k \in \mathbb{N}$ and let $\mathcal{D}'_k(\overline{Q} : \xi : \lambda)$ be the subspace of $\mathcal{D}'(\overline{Q} : \xi : \lambda)$ of distributions of order at most k. We recall the maximal compact subgroup K of G and note that $\mathcal{D}'_k(\overline{Q} : \xi : \lambda)$ is canonically isomorphic to the space $\mathcal{D}'_k(K; V_\xi)^M$ of left-M-invariant distributions in $\mathcal{D}'(K, V_\xi)$ of order at most k. Note that $\mathcal{D}'_k(K; V_\xi)^M$ is a Banach space. The same proof as for [2, Lemma 10.1] yields the existence of constants $C > 0$ and $r > 0$, independent of $\lambda \in \mathfrak{C}$, such that for every $X \in \mathfrak{a}_0$, the operator $R\big(\exp(X)\big)$ maps $\mathcal{D}'_k(\overline{Q} : \xi : \lambda)$ to itself with operator norm

$$\|R\big(\exp(X)\big)\| \leq Ce^{r\|X\|}.$$

We recall that the spherical root system Σ_Z in $(\mathfrak{a}/\mathfrak{a}_E)^*$ admits the image $\overline{\mathcal{C}}/\mathfrak{a}_E$ of $\overline{\mathcal{C}}$ under the projection $\mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_E$ as a Weyl chamber. By taking a sum of positive roots, we find a functional $\zeta_0 \in (\mathfrak{a}/\mathfrak{a}_E)^*$ that is strictly positive on $\overline{\mathcal{C}} \setminus \mathfrak{a}_E$. Note that $\zeta_0|_{\mathfrak{a}_E} = 0$.

By Proposition 6.1 we have for all $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$

$$|m_{\phi,\mu}\big(g \exp(Y)\big)| \leq \max_{w\in W} e^{\rho_Q(Y)+\mathrm{Re}\ \mathrm{Ad}^*(w)\lambda(Y)}|m_{\phi,\mu}(g)| \qquad \big(g \in G, Y \in \mathfrak{a}_E\big).$$

We recall the isometry $\mu^\circ(\xi : \lambda)$ from Theorem 6.1. Since the distributions $\mu^\circ(\xi : \lambda)\eta$ depend smoothly on $\lambda \in \mathfrak{C}$ and linearly on η, the assertion therefore follows, after rescaling ζ_0 if necessary, with $\zeta = \zeta_0 + \rho_Q$. ☐

7.3 Boundary degenerations

To improve the a priori estimate from the previous section we will use the theory of the constant term as developed in [18]. In this theory certain degenerations of Z play an important role. We recall here the necessary definitions and results.

The closure of the compression cone $\overline{\mathcal{C}}$ is finitely generated and hence polyhedral as $-\mathcal{C}^\vee$ is finitely generated. We call a subset $\mathcal{F} \subseteq \overline{\mathcal{C}}$ a face of $\overline{\mathcal{C}}$ if $\mathcal{F} = \overline{\mathcal{C}}$ or there exists a closed half-space $\mathcal{H}$ so that

$$\mathcal{F} = \overline{\mathcal{C}} \cap \mathcal{H} \quad \text{and} \quad \mathcal{C} \cap \partial\mathcal{H} = \emptyset.$$

There exist finitely many faces of $\overline{\mathcal{C}}$ and each face is polyhedral cone. For a face $\mathcal{F}$ of $\overline{\mathcal{C}}$ we define

$$\mathfrak{a}_{\mathcal{F}} := \operatorname{span}(\mathcal{F})$$

and denote the interior of $\mathcal{F}$ in $\mathfrak{a}_{\mathcal{F}}$ by $\mathcal{F}^{\circ}$.

Let $z \in Z$ be adapted and let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. By [39, Lemma 8.1] the limits $\mathfrak{h}_{z,X}$ are the same for all X in the interior of $\mathcal{F}$. We may thus define

$$\mathfrak{h}_{z,\mathcal{F}} := \mathfrak{h}_{z,X},$$

where X is any element in the interior of $\mathcal{F}$.

The following lemma is [39, Lemma 8.3].

Lemma 7.1. *Let $z \in Z$ be adapted and let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. The Lie algebra $\mathfrak{h}_{z,\mathcal{F}}$ is a real spherical subalgebra of $\mathfrak{g}$. Moreover,*

$$N_{\mathfrak{g}}(\mathfrak{h}_{z,\mathcal{F}}) = \mathfrak{h}_{z,\mathcal{F}} + \mathfrak{a}_{\mathcal{F}} + N_{\mathfrak{m}}(\mathfrak{h}_{z,\mathcal{F}}).$$

Finally,

$$\mathfrak{h}_{z,\mathcal{F}} \cap \mathfrak{a} = \mathfrak{a}_{\mathfrak{h}}.$$

For an adapted point $z \in Z$ and a face $\mathcal{F}$ of $\overline{\mathcal{C}}$ we define $H_{z,\mathcal{F}}$ to be the connected subgroup of G with Lie algebra $\mathfrak{h}_{z,\mathcal{F}}$. Each subgroup $H_{z,\mathcal{F}}$ equals the connected component of a group of real points of an algebraic subgroup of $\underline{G}$, namely the subgroups $H_{I,c}$ defined in [17, Section 4.5]. We write $Z_{z,\mathcal{F}}$ for the homogeneous space $G/H_{z,\mathcal{F}}$. These spaces are called the boundary degenerations of Z. Since $\mathfrak{a}_{\mathcal{F}}$ normalizes $\mathfrak{h}_{z,\mathcal{F}}$, the group $A_{\mathcal{F}} := \exp(\mathfrak{a}_{\mathcal{F}})$ normalizes $H_{z,\mathcal{F}}$.

One boundary degeneration will be of particular interest to us when we come to Section 8: the boundary degeneration for the face $\mathcal{F} = \overline{\mathcal{C}}$. If $z \in Z$ is an adapted point so that $M \cap H_z = M \cap H$, then $\mathfrak{h}_{z,X} = \mathfrak{h}_{\emptyset}$ for all $X \in \mathcal{C}$. Therefore, the group $H_{z,\overline{\mathcal{C}}}$ is in this case the connected component of the subgroup

$$H_{\emptyset} := (L_Q \cap H)\overline{N}_Q.$$

7.4 Preparation for the proof of Theorem 7.2

The proof of Theorem 7.2 relies heavily on the theory of the constant term as developed in [18]. In this section we recall the necessary objects and results, which we will then use in the next section to prove the theorem. We first discuss the algebras of invariant differential operators on Z and its boundary degenerations and some relations between them. We then give the differential equations satisfied by the matrix-coefficients. Finally, we introduce the notion of $\mathcal{F}$-piece-wise linear functionals and construct an $\mathcal{F}$-piece-wise linear functional $\beta_{\mathcal{F},\lambda}$, which will be used to improve the a priori estimate on the matrix-coefficients from Lemma 7.1.

We fix an adapted point $z \in Z$. In this and the next section we will suppress the indices z and simply write $Z_{\mathcal{F}}$, $H_{\mathcal{F}}$ and $\mathfrak{h}_{\mathcal{F}}$ for $Z_{z,\mathcal{F}}$, $H_{z,\mathcal{F}}$ and $\mathfrak{h}_{z,\mathcal{F}}$, respectively.

We now follow [18, Section 5].

Let $\mathfrak{b} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_Q$ and $\mathfrak{b}_H = (\mathfrak{m} \cap \mathfrak{h}_z) \oplus \mathfrak{a}_\mathfrak{h}$. Now $\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ is a two-sided ideal of $\mathcal{U}(\mathfrak{b})$. We recall from [18, (5.4)] that the rings $\mathbb{D}(Z)$ and $\mathbb{D}(Z_\mathcal{F})$ of G-invariant differential operators on Z and $Z_\mathcal{F}$, respectively, may be identified with subalgebras of $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$. By [18, Lemma 5.2] the limit

$$\lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)D$$

exists for every $D \in \mathbb{D}(Z)$ and $X \in \mathcal{F}^\circ$ and defines a G-invariant differential operator on $Z_\mathcal{F}$. The limit does not depend on the choice of X. Moreover, the map

$$\delta_\mathcal{F} : \mathbb{D}(Z) \to \mathbb{D}(Z_\mathcal{F}); \quad D \mapsto \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)D$$

is an injective algebra morphism. The $\mathfrak{a}$-weights occurring in $\delta_\mathcal{F}(D) - D$, with $D \in \mathbb{D}(Z)$, considered as an element of $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$, are strictly negative on $\mathcal{F}^\circ$ and the $\mathfrak{a}$-weights occurring in $\delta_\mathcal{F}(D)$ are non-positive on $\overline{\mathcal{C}}$.

Every element u of the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ determines a differential operator $D_u \in \mathbb{D}(Z_\mathcal{F})$. We write $\mathbb{D}_0(Z_\mathcal{F})$ for the image of $\mathcal{Z}(\mathfrak{g})$ in $\mathbb{D}(Z_\mathcal{F})$. By [18, Lemma 5.6] the ring $\mathbb{D}(Z_\mathcal{F})$ is finitely generated as a $\mathbb{D}_0(Z_\mathcal{F})$-module. Let $V_\mathcal{F}$ be a finite dimensional vector subspace of $\mathbb{D}(Z_\mathcal{F})$ so that the linear map

$$\mathbb{D}_0(Z_\mathcal{F}) \otimes V_\mathcal{F} \to \mathbb{D}(Z_\mathcal{F}); \quad \sum_j D_j \otimes u_j \mapsto \sum_j D_j u_j$$

is surjective.

For $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ we write $\mathcal{I}_\lambda$ and $\mathcal{I}_{\mathcal{F},\lambda}$ for the ideals of $\mathbb{D}(Z)$ and $\mathbb{D}(Z_\mathcal{F})$, respectively, generated by the elements of the form $D_u - \chi_\lambda(u)$ with $u \in \mathcal{Z}(\mathfrak{g})$, where $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$ is the infinitesimal character of $\mathcal{D}'(\overline{Q} : \xi : \lambda)$. Now $\mathcal{I}_{\mathcal{F},\lambda} = \delta_\mathcal{F}(\mathcal{I}_\lambda)$. As

$$\mathbb{D}_0(Z_\mathcal{F}) = \mathbb{C} + \mathcal{I}_{\mathcal{F},\lambda},$$

we have

$$\mathbb{D}(Z_\mathcal{F}) = (\mathbb{C} + \mathcal{I}_{\mathcal{F},\lambda})V_\mathcal{F}.$$

Since

$$\mathcal{I}_{\mathcal{F},\lambda}V_\mathcal{F} = \mathcal{I}_{\mathcal{F},\lambda}\mathbb{D}_0(Z_\mathcal{F})V_\mathcal{F} = \mathcal{I}_{\mathcal{F},\lambda}\mathbb{D}(Z_\mathcal{F}) = \mathbb{D}(Z_\mathcal{F})\mathcal{I}_{\mathcal{F},\lambda}, \tag{7.1}$$

we find

$$\mathbb{D}(Z_\mathcal{F}) = V_\mathcal{F} + \mathbb{D}(Z_\mathcal{F})\mathcal{I}_{\mathcal{F},\lambda}.$$

For every $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ there exists a subspace $U_\mathcal{F}$ of $V_\mathcal{F}$ so that the sum

$$\mathbb{D}(Z_\mathcal{F}) = U_\mathcal{F} \oplus \mathbb{D}(Z_\mathcal{F})\mathcal{I}_{\mathcal{F},\lambda} \tag{7.2}$$

is direct sum of vector spaces. As $V_\mathcal{F}$ is finite dimensional and $V_\mathcal{F} \cap \mathbb{D}(Z_\mathcal{F})\mathcal{I}_{\mathcal{F},\lambda}$ depends continuously on λ, the subspace $U_\mathcal{F}$ can in fact be chosen locally uniformly with respect to λ, i.e., every $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ has an open neighborhood $\mathfrak{B}$ in $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ so that there exists a subspace $U_\mathcal{F}$ of $V_\mathcal{F}$ for which (7.2) holds for all $\lambda \in \mathfrak{B}$.

We define

$$\rho_{\mathcal{F},\lambda} : \mathbb{D}(Z_\mathcal{F}) \to \mathrm{End}(U_\mathcal{F})$$

to be the map determined by

$$Du \in \rho_{\mathcal{F},\lambda}(D)u + \mathbb{D}(Z_{\mathcal{F}})\mathcal{I}_{\mathcal{F},\lambda} \qquad \big(D \in \mathbb{D}(Z_{\mathcal{F}}), u \in \mathcal{U}_{\mathcal{F}}\big). \tag{7.3}$$

Then $\rho_{\mathcal{F},\lambda}$ defines a representation of $\mathbb{D}(Z_{\mathcal{F}})$ on $U_{\mathcal{F}}$ which is isomorphic to the canonical representation of $\mathbb{D}(Z_{\mathcal{F}})$ on $\mathbb{D}(Z_{\mathcal{F}})/\mathbb{D}(Z_{\mathcal{F}})\mathcal{I}_{\mathcal{F},\lambda}$, and $\rho_{\mathcal{F},\lambda}$ depends polynomially on λ.

There exists a natural injective algebra homomorphism

$$S(\mathfrak{a}_{\mathcal{F}}) \hookrightarrow \mathbb{D}(Z_{\mathcal{F}}); \quad X \mapsto D_X, \tag{7.4}$$

which is determined by

$$D_X f(z) = \frac{d}{dt} f\big(z \cdot \exp(tX)\big)\Big|_{t=0} \qquad \big(X \in \mathfrak{a}_{\mathcal{F}}, f \in \mathcal{E}(Z_{\mathcal{F}}), z \in Z_{\mathcal{F}}\big).$$

In view of (7.4) the $\mathbb{D}(Z_{\mathcal{F}})$-representation $\rho_{\mathcal{F},\lambda}$ induces a Lie algebra homomorphism

$$\Gamma_{\mathcal{F},\lambda} : \mathfrak{a}_{\mathcal{F}} \to \mathrm{End}(U_{\mathcal{F}}^*); \quad X \mapsto \rho_{\mathcal{F},\lambda}(D_X)^t.$$

We note that $\Gamma_{\mathcal{F},\lambda}$ depends polynomially on λ.

We use this machinery to analyse the matrix-coefficients $m_{\phi,\mu}$ for $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ and $\phi \in \mathcal{D}(G, V_\xi)$. To do so we define the map

$$\Phi_{\mu,\phi} : A \to U_{\mathcal{F}}^*$$

by setting

$$\Big(\Phi_{\mu,\phi}(a)\Big)(u) = \big(R(u)m_{\phi,\mu}\big)(a) \qquad (a \in A, u \in U_{\mathcal{F}}).$$

The function $\Phi_{\mu,\phi}$ satisfies the system of differential equations

$$\partial_X \Phi_{\mu,\phi} = \Gamma_{\mathcal{F},\lambda}(X)\Phi_{\mu,\phi} + \Psi_{\mu,\phi,X} \qquad (X \in \mathfrak{a}_{\mathcal{F}}) \tag{7.5}$$

with

$$\Psi_{\mu,\phi,X} : A \to U_{\mathcal{F}}^*$$

given by

$$\Psi_{\mu,\phi,X}(a)(u) = \Big(R\big(Xu - \rho_{\mathcal{F},\lambda}(X)u\big)m_{\phi,\mu}\Big)(a) \qquad (a \in A, u \in U_{\mathcal{F}}).$$

As in [18, Lemma 5.7] we may solve the ordinary differential equation (7.5) and obtain that for all $a \in A$ and $X \in \mathfrak{a}_{\mathcal{F}}$,

$$\Phi_{\mu,\phi}(a\exp(X)) = e^{\Gamma_{\mathcal{F},\lambda}(X)}\Phi_{\mu,\phi}(a) + \int_0^1 e^{(1-s)\Gamma_{\mathcal{F},\lambda}(X)}\Psi_{\mu,\phi,X}\big(a\exp(sX)\big)\,ds. \tag{7.6}$$

Let $\mathcal{Q}_{\mathcal{F},\lambda}$ be the set of generalized $\mathfrak{a}_{\mathcal{F}}$-weights of $\Gamma_{\mathcal{F},\lambda}$. For $\nu \in \mathcal{Q}_{\mathcal{F},\lambda}$ we define $E_\nu \in \mathrm{End}(\mathcal{U}_{\mathcal{F}}^*)$ to be the projection onto the generalized eigenspace with eigenvalue ν. We further define

$$\Phi_{\mu,\phi}^\nu := E_\nu \circ \Phi_{\mu,\phi} \qquad \big(\phi \in \mathcal{D}(G, V_\xi)\big).$$

In view of (7.6) we have

$$\Phi^{\nu}_{\mu,\phi}(a\exp(X)) = e^{\Gamma_{\mathcal{F},\lambda}(X)}\Phi^{\nu}_{\mu,\phi}(a) + \int_0^1 E_\nu e^{(1-s)\Gamma_{\mathcal{F},\lambda}(X)}\Psi_{\mu,\phi,X}\big(a\exp(sX)\big)\,ds$$

for all $\phi \in \mathcal{D}(G,V_\xi)$, $\nu \in \mathcal{Q}_{\mathcal{F},\lambda}$, $a \in A$ and $X \in \mathfrak{a}_{\mathcal{F}}$.

If $X \in \mathfrak{a}_{\mathcal{F}}$ and $u \in U_{\mathcal{F}}$, then in view of (7.3) and (7.1) the element $Xu - \rho_{\mathcal{F},\lambda}(X)u$ is contained in $V_{\mathcal{F}}\mathcal{I}_{\mathcal{F},\lambda} = V_{\mathcal{F}}\delta_{\mathcal{F}}(\mathcal{I}_\lambda)$. Therefore, if $u_1,\dots,u_n$ is a basis of $V_{\mathcal{F}}$, then there exist bilinear maps

$$\omega^i_{\mathcal{F},\lambda} : \mathfrak{a}_{\mathcal{F}} \times U_{\mathcal{F}} \to \mathcal{I}_\lambda \qquad (1 \leq i \leq n)$$

so that

$$Xu - \rho_{\mathcal{F},\lambda}(X)u = \sum_{i=1}^n u_i\delta_{\mathcal{F}}\big(\omega^i_{\mathcal{F},\lambda}(X,u)\big) \qquad \big(X \in \mathfrak{a}_{\mathcal{F}}, u \in U_{\mathcal{F}}\big).$$

We denote by $\Xi_{\mathcal{F},\lambda}$ the finite set of all $\mathfrak{a}$-weights that occur in

$$\big\{\delta_{\mathcal{F}}\big(\omega^i_{\mathcal{F},\lambda}(X,u)\big) - \omega^i_{\mathcal{F},\lambda}(X,u) : 1 \leq i \leq n, X \in \mathfrak{a}_{\mathcal{F}}, u \in U_{\mathcal{F}}\big\}.$$

We recall from Section 3.9 that the image of $\overline{\mathcal{C}}$ under the projection $\mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_E$ is a Weyl chamber of the spherical root system Σ_Z in $(\mathfrak{a}/\mathfrak{a}_E)^*$. Let Σ_Z^+ be the positive system of Σ_Z so that this Weyl chamber is the negative one. We then define $\beta_{\mathcal{F},\lambda}$ on $\mathfrak{a}$ by

$$\beta_{\mathcal{F},\lambda}(X) := \max_{\nu \in \Xi_{\mathcal{F},\lambda}\cup\Sigma_Z^+} \nu(X), \quad (X \in \mathfrak{a}).$$

Because of the signs of the $\mathfrak{a}$-weights occurring in $\mathbb{D}(Z_{\mathcal{F}})$ and elements of the form $\delta_{\mathcal{F}}(D) - D$ with $D \in \mathbb{D}(Z)$, we have $\beta_{\mathcal{F},\lambda}\big|_{\overline{\mathcal{C}}} \leq 0$ and $\beta_{\mathcal{F},\lambda}\big|_{\mathcal{F}^\circ} < 0$. The maximum in the definition of $\beta_{\mathcal{F},\lambda}$ also runs over the set of positive roots in Σ_Z; this is to ensure that $\beta_{\mathcal{F},\lambda}$ vanishes on the edge of $\mathcal{F}$. We do not actually need this, but we simply follow the definition in [18, (5.23)]. Note that $\beta_{\mathcal{F},\lambda}$ depends polynomially on λ.

7.5 Proof of Theorem 7.2

We continue with the notation from the previous section. We start with the a priori estimate from Lemma 7.1 and use the theory of the constant term to improve this estimate recursively. The proof is by induction on the faces $\mathcal{F}$ of $\overline{\mathcal{C}}$.

Let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. We call a function $\zeta : \mathfrak{a} \to \mathbb{R}$ an $\mathcal{F}$-piecewise linear functional on $\mathfrak{a}$ if it is piecewise linear and satisfies

$$\zeta\big|_{\partial\mathcal{F}} = \rho_Q\big|_{\partial\mathcal{F}}.$$

Here $\partial\mathcal{F}$ denotes the union of the faces $\mathcal{F}'$ of $\overline{\mathcal{C}}$ with $\mathcal{F}' \subsetneq \mathcal{F}$. Note that for any two $\mathcal{F}$-piecewise linear functionals ζ and ζ', also the functions

$$\mathfrak{a} \ni X \mapsto \max\big(\zeta(X),\zeta'(X)\big) \quad \text{and} \quad \mathfrak{a} \ni X \mapsto \min\big(\zeta(X),\zeta'(X)\big)$$

are $\mathcal{F}$-piecewise linear functionals on $\mathfrak{a}$. Moreover,

$$\mathfrak{a} = \mathfrak{a}_{\mathcal{F}} \times \left(\mathfrak{a} \cap \mathfrak{a}_{\mathcal{F}}^{\perp}\right) \ni (X, Y) \mapsto \zeta(X) + \zeta'(Y)$$

defines an $\mathcal{F}$-piecewise linear functional as well.

For an $\mathcal{F}$-piece-wise linear functional ζ we decompose $\mathcal{Q}_{\mathcal{F},\lambda}$ as

$$\mathcal{Q}_{\mathcal{F},\lambda} = \mathcal{Q}_{\mathcal{F},\lambda}^{\zeta,+} \cup \mathcal{Q}_{\mathcal{F},\lambda}^{\zeta,0} \cup \mathcal{Q}_{\mathcal{F},\lambda}^{\zeta,-},$$

with

$$\begin{aligned}
\mathcal{Q}_{\mathcal{F},\lambda}^{\zeta,+} &:= \left\{\nu \in \mathcal{Q}_{\mathcal{F},\lambda} : \operatorname{Re}\nu(X) > \zeta(X) + \langle\lambda\rangle(X) \text{ for some } X \in \mathcal{F}^{\circ}\right\},\\
\mathcal{Q}_{\mathcal{F},\lambda}^{\zeta,-} &:= \left\{\nu \in \mathcal{Q}_{\mathcal{F},\lambda} : \operatorname{Re}\nu < \zeta + \langle\lambda\rangle\big|_{\mathcal{F}^{\circ}}\right\},\\
\mathcal{Q}_{\mathcal{F},\lambda}^{\zeta,0} &:= \mathcal{Q}_{\mathcal{F},\lambda} \setminus \left(\mathcal{Q}_{\mathcal{F},\lambda}^{\zeta,+} \cup \mathcal{Q}_{\mathcal{F},\lambda}^{\zeta,-}\right)\\
&= \Big\{\nu \in \mathcal{Q}_{\mathcal{F},\lambda} : \operatorname{Re}\nu \leq \zeta + \langle\lambda\rangle\big|_{\mathcal{F}}\\
&\qquad\qquad \text{and } \operatorname{Re}\nu(X) = \zeta(X) + \langle\lambda\rangle(X) \text{ for some } X \in \mathcal{F}^{\circ}\Big\}.
\end{aligned}$$

The following lemma is needed for the induction step.

Lemma 7.1. *Let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$ and ξ a finite dimensional unitary representation of M_Q. Let further $\mathfrak{C}$ be a compact subset of $\{\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}} : \operatorname{Im}\lambda \notin \mathcal{S}\}$ such that there exists a subspace $U_{\mathcal{F}}$ of $V_{\mathcal{F}}$ for which (7.2) holds for all $\lambda \in \mathfrak{C}$. Let $\zeta : \mathfrak{C} \times \mathfrak{a} \to \mathbb{R}$ be a continuous function so that $\zeta_\lambda := \zeta(\lambda, \cdot\,)$ is an $\mathcal{F}$-piecewise linear functional on $\mathfrak{a}$ for all $\lambda \in \mathfrak{C}$. Assume that there exists an $N \in \mathbb{N}_0$ and a continuous seminorm p on $\mathcal{D}(G, V_\xi)$ so that for every $\lambda \in \mathfrak{C}$, $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ and $\phi \in \mathcal{D}(G, V_\xi)$*

$$\left|m_{\phi,\mu}\big(\omega \exp(X)\big)\right| \leq e^{\left(\zeta_\lambda + \langle\lambda\rangle\right)(X)}(1 + \|X\|)^N \|\mu\| p(\phi) \qquad \big(\omega \in \Omega, X \in \overline{\mathcal{C}}\big). \quad (7.1)$$

Then there exists an $N' \in \mathbb{N}_0$, a continuous semi-norm p' on $\mathcal{D}(G, V_\xi)$, and a continuous function $\zeta' : \mathfrak{C} \times \mathfrak{a} \to \mathbb{R}$ so that

(i) $\zeta'_\lambda := \zeta'(\lambda, \cdot\,)$ is an $\mathcal{F}$-piecewise linear functional for all $\lambda \in \mathfrak{C}$

(ii) $\zeta'_\lambda\big|_{\mathcal{F}} = \max\Big(\big\{\zeta_\lambda + \frac{1}{2}\beta_{\mathcal{F},\lambda}, \rho_Q\big\} \cup \big\{\operatorname{Re}\nu - \langle\lambda\rangle : \nu \in \mathcal{Q}_{\mathcal{F},\lambda}^{\zeta_\lambda,-}\big\}\Big)\Big|_{\mathcal{F}}$ for all $\lambda \in \mathfrak{C}$

(iii) For all $\lambda \in \mathfrak{C}$, $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ and $\phi \in \mathcal{D}(G, V_\xi)$

$$\left|m_{\phi,\mu}\big(\omega \exp(X)\big)\right| \leq e^{\left(\zeta'_\lambda + \langle\lambda\rangle\right)(X)}(1 + \|X\|)^{N'} \|\mu\| p'(\phi) \qquad \big(\omega \in \Omega, X \in \overline{\mathcal{C}}\big).$$

Proof. We apply Lemma 7.1 to $Z_{\mathcal{F}}$ instead of Z, and find that there exists an $N_{\mathcal{F}} \in \mathbb{N}_0$, a continuous semi-norm $p_{\mathcal{F}}$ on $\mathcal{D}(G, V_\xi)$ and a $\zeta_{\mathcal{F}} \in \mathfrak{a}^*$ so that

$$\zeta_{\mathcal{F}}\big|_{\mathfrak{a}_{\mathcal{F}}} = \rho_Q\big|_{\mathfrak{a}_{\mathcal{F}}}$$

and for every $\mu_{\mathcal{F}} \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^{H_{\mathcal{F}}}$ and $\phi \in \mathcal{D}(G, V_\xi)$

$$\left|m_{\phi,\mu_{\mathcal{F}}}\big(\exp(X)\big)\right| \leq e^{\left(\zeta_{\mathcal{F}} + \langle\lambda\rangle\right)(X)}(1 + \|X\|)^{N_{\mathcal{F}}} \|\mu_{\mathcal{F}}\| p_{\mathcal{F}}(\phi) \qquad \big(\lambda \in \mathfrak{C}, X \in \overline{\mathcal{C}}\big). \quad (7.2)$$

We define $\zeta' : \mathfrak{C} \times \mathfrak{a} \to \mathbb{R}$ by requiring that for all $X \in \mathfrak{a}_{\mathcal{F}}$ and $Y \in \mathfrak{a} \cap \mathfrak{a}_{\mathcal{F}}^{\perp}$

$$\zeta'(\lambda, X) = \max\Big(\big\{\zeta_\lambda(X) + \frac{1}{2}\beta_{\mathcal{F},\lambda}(X), \rho_Q(X)\big\} \cup \big\{\mathrm{Re}\,\nu(X) - \langle\lambda\rangle(X) : \nu \in \mathcal{Q}_{\mathcal{F},\lambda}^{\zeta_\lambda,-}\big\}\Big),$$
$$\zeta'(\lambda, Y) = \max\big(\zeta_\lambda(Y), \zeta_{\mathcal{F}}(Y)\big),$$
$$\zeta'(\lambda, X+Y) = \zeta'(\lambda, X) + \zeta'(\lambda, Y).$$

Observe that ζ' is continuous and $\zeta'(\lambda, \cdot\,)$ defines for every $\lambda \in \mathfrak{C}$ an $\mathcal{F}$-piecewise linear functional on $\mathfrak{a}$. Therefore, it suffices to prove the existence of an $N' \in \mathbb{N}_0$ and a continuous semi-norm p' on $\mathcal{D}(G, V_\xi)$ such that for every $\lambda \in \mathfrak{C}$, $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$, $\phi \in \mathcal{D}(G, V_\xi)$ and $\nu \in \mathcal{Q}_{\mathcal{F},\lambda}$

$$\|\Phi_{\mu,\phi}^{\nu}\big(\exp(X)\big)\| \leq e^{\big(\zeta'_\lambda + \langle\lambda\rangle\big)(X)}(1+\|X\|)^{N'}\|\mu\|p'(\phi) \qquad (X \in \overline{\mathcal{C}}).$$

For this we follow [18, Sections 5.3, 5.4, 6.2 & 6.3].

If one uses the estimate (7.1) instead of the tempered estimates, then in the same way as in the proof for [18, Lemma 5.8] it follows that there exists a continuous semi-norm q on $\mathcal{D}(G, V_\xi)$ such that for all $\lambda \in \mathfrak{C}$, $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$, and $\phi \in \mathcal{D}(G, V_\xi)$

$$\|L(v)\Phi_{\mu,\phi}\big(\exp(X)\big)\| \leq e^{\big(\zeta_\lambda + \langle\lambda\rangle\big)(X)}(1+\|X\|)^{N}\|\mu\| q\big(L(v)\phi\big) \qquad \big(v \in \mathcal{U}(\mathfrak{a}), X \in \overline{\mathcal{C}}\big), \tag{7.3}$$

and for every compact subset $B \subseteq \mathfrak{a}$, there exists a constant $C > 0$ so that

$$\|L(v)\Psi_{\mu,\phi,X}(\exp(Y))\| \leq Ce^{\big(\zeta_\lambda + \beta_{\mathcal{F},\lambda} + \langle\lambda\rangle\big)(Y)}(1+\|Y\|)^{N}\|\mu\|\|X\| q\big(L(v)\phi\big) \tag{7.4}$$

for all $v \in \mathcal{U}(\mathfrak{a})$, $X \in \mathfrak{a}_{\mathcal{F}}$ and $Y \in B + \overline{\mathcal{C}}$. Let

$$E_\nu(\lambda, X) := e^{-\nu(X)}E_\nu \circ \Gamma_{\mathcal{F},\lambda}(X) \qquad \big(\lambda \in \mathfrak{C}, \nu \in \mathcal{Q}_{\mathcal{F},\lambda}, X \in \mathfrak{a}_{\mathcal{F}}\big).$$

The same arguments as the ones for [18, Lemma 5.9] show the existence of constants $c > 0$ and $n \in \mathbb{N}_0$ so that

$$\|E_\nu(\lambda, X)\| \leq c(1+\|X\|)^n \qquad \big(\lambda \in \mathfrak{C}, \nu \in \mathcal{Q}_{\mathcal{F},\lambda}, X \in \mathfrak{a}_{\mathcal{F}}\big). \tag{7.5}$$

We define $\delta_\lambda : \mathfrak{a}_{\mathcal{F}} \to [0, \frac{1}{2}]$ by

$$\delta_\lambda(X) = \min\left(\left\{\frac{\mathrm{Re}\,\nu(X) - \zeta_\lambda(X) - \langle\lambda\rangle(X)}{\beta_{\mathcal{F},\lambda(X)}} : \nu \in \mathcal{Q}_{\mathcal{F},\lambda}^{\zeta_\lambda,-}\right\} \cup \left\{\frac{1}{2}\right\}\right) \qquad (X \in \mathcal{F}^\circ).$$

In view of (7.3), (7.4) and (7.5) it suffices to prove that for every $\phi \in \mathcal{D}(G, V_\xi)$, $\lambda \in \mathfrak{C}$, $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ and $\nu \in \mathcal{Q}_{\mathcal{F},\lambda}$ there exists a function $\Phi_{\mu,\phi}^{\nu,\infty} : A \to U_{\mathcal{F}}^*$ so that

$$\begin{aligned}
&\big\|\Phi_{\mu,\phi}^{\nu}\big(a\exp(tX)\big) - \Phi_{\mu,\phi}^{\nu,\infty}\big(a\exp(tX)\big)\big\| \qquad (7.6)\\
&\leq e^{\big(\zeta_\lambda + \delta_\lambda\beta_{\mathcal{F},\lambda} + \langle\lambda\rangle\big)(tX)}\Bigg(\|E_\nu(\lambda, tX)\|\|\Phi_{\mu,\phi}(a)\|\\
&\qquad + \int_0^\infty e^{-\big(\zeta_\lambda + \frac{1}{2}\beta_{\mathcal{F},\lambda} + \langle\lambda\rangle\big)(sX)}\big\|E_\nu\big(\lambda, (t-s)X\big)\big\|\big\|\Psi_{\mu,\phi,X}\big(a\exp(sX)\big)\big\|\,ds\Bigg)
\end{aligned}$$

for all $a \in A$, $X \in \mathcal{F}^\circ$ and $t \geq 0$, and

$$\big\|\Phi^{\nu,\infty}_{\mu,\phi}\big(\exp(X)\big)\big\| \leq e^{\big(\zeta_{\mathcal{F}}+\langle\lambda\rangle\big)(X)}(1+\|X\|)^{N_\infty}\|\mu\|q_\infty(\phi) \qquad (X \in \overline{\mathcal{C}}) \tag{7.7}$$

for some $N_\infty \in \mathbb{N}_0$ and a continuous semi-norm q_∞ on $\mathcal{D}(G, V_\xi)$.

If $\nu \in \mathcal{Q}^{\zeta_\lambda,+}_{\mathcal{F},\lambda}$ or $\nu \in \mathcal{Q}^{\zeta_\lambda,-}_{\mathcal{F},\lambda}$, then we may take $\Phi^{\nu,\infty}_{\mu,\phi} = 0$. The estimate (7.6) is the analogue of [18, Lemma 5.11] and is obtained as in [18, Corollary 5.16 & Lemma 5.17] and [18, Lemma 5.18], respectively.

Let $\nu \in \mathcal{Q}^{\zeta_\lambda,0}_{\mathcal{F},\lambda}$. If $Y \in \mathcal{F}^\circ$ with $\operatorname{Re}\nu(Y) > \zeta_\lambda(Y)+\beta_{\mathcal{F},\lambda}(Y)+\langle\lambda\rangle(Y)$, then it follows as in [18, Section 5.4.1] that the limit

$$\Phi^{\nu,\infty}_{\mu,\phi}(a) := \lim_{t\to\infty} e^{-t\Gamma_{\mathcal{F},\lambda}(Y)}\Phi^{\nu}_{\mu,\phi}\big(a\exp(tY)\big) \qquad (a \in A)$$

exists and is independent of the choice of Y. Note that such Y exist because $\nu \in \mathcal{Q}^{\zeta_\lambda,0}_{\mathcal{F},\lambda}$ and $\beta_{\mathcal{F},\lambda}|_{\mathcal{F}^\circ}<0$. We first show that (7.7) is satisfied. For $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ we define

$$\mu_{\mathcal{F},\nu} : \mathcal{D}(G, V_\xi) \to \mathbb{C}; \quad \phi \mapsto \Phi^{\nu,\infty}_{\mu,\phi}(e)(1).$$

From the definitions it is easily seen that $\mu_{\mathcal{F},\zeta_\lambda}$ is a distribution in $\mathcal{D}'(\overline{Q} : \xi : \lambda)$. We claim that $\mu_{\mathcal{F},\nu}$ is right $H_{\mathcal{F}}$-invariant. To prove the claim we choose $Y \in \mathcal{F}^\circ$ so that $\zeta_\lambda(Y) + \langle\lambda\rangle(Y) = \operatorname{Re}\nu(Y)$. From (7.6) it follows that

$$\lim_{t\to\infty} e^{-t\nu(Y)}\|\Phi^{\nu}_{\mu,\phi}\big(\exp(tY)\big) - \Phi^{\nu,\infty}_{\mu,\phi}\big(\exp(tY)\big)\| = 0.$$

Moreover, if we use this Y in the proof of [18, Lemma 6.2], then we obtain

$$\Phi^{\nu,\infty}_{\mu,\phi}\big(a\exp(X)\big) = e^{\Gamma_{\mathcal{F},\lambda}(X)}\Phi^{\nu,\infty}_{\mu,\phi}(a) \qquad (a \in A, X \in \mathfrak{a}_{\mathcal{F}}).$$

Now the claim follows with the same arguments as in the proof for [18, Lemma 6.5(iii)]. The estimate (7.7) follows from the claim and (7.2).

We finish the proof by showing that (7.6) holds also in this case. If $\nu \in \mathcal{Q}^{\zeta_\lambda,0}_{\mathcal{F},\lambda}$ and $X \in \mathcal{F}^\circ$ with $\operatorname{Re}\nu(X) \leq \zeta_\lambda(X) + \frac{1}{2}\beta_{\mathcal{F},\lambda}(X) + \langle\lambda\rangle(X)$, then the estimate follows from (7.7) and the estimate on $\Phi^{\nu}_{\mu,\phi}\big(a\exp(tX)\big)$ that one obtains analogous to [18, Lemma 5.17]. If $X \in \mathcal{F}^\circ$ with

$$\zeta_\lambda(X) + \frac{1}{2}\beta_{\mathcal{F},\lambda}(X) + \langle\lambda\rangle(X) \leq \operatorname{Re}\nu(X) \leq \zeta_\lambda(X) + \langle\lambda\rangle(X),$$

then we find as in the proof for [18, Lemma 5.19] that

$$\Phi^{\nu,\infty}_{\mu,\phi}\big(a\exp(tX)\big) = \Phi^{\nu}_{\mu,\phi}\big(a\exp(tX)\big) + \int_t^\infty E_\nu e^{(t-s)\Gamma_{\mathcal{F},\lambda}(X)}\Psi_{\mu,\phi,X}\big(a\exp(sX)\big)\,ds.$$

Now (7.6) follows from (7.4) and (7.5). (The fact that δ_λ is not a constant like in [18, (5.37)] is irrelevant for the proof.) □

Proof of Theorem 7.2. We prove the theorem using the principle of induction on the faces $\mathcal{F}$ of $\overline{\mathcal{C}}$. In particular we will show that for every face $\mathcal{F}$ of $\overline{\mathcal{C}}$ there exists an $N \in \mathbb{N}_0$, a continuous seminorm p on $\mathcal{D}(G, V_\xi)$ and a continuous function $\zeta : \mathfrak{C} \times \mathfrak{a} \to \mathbb{R}$ so that

(i) $\zeta_\lambda := \zeta(\lambda, \cdot\,)$ is an $\mathcal{F}$-piecewise linear functional for all $\lambda \in \mathfrak{C}$,

(ii) $\zeta_\lambda\big|_{\mathcal{F}} = \rho_Q\big|_{\mathcal{F}}$,

(iii) For all $\lambda \in \mathfrak{C}$, $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ and $\phi \in \mathcal{D}(G, V_\xi)$

$$\big|m_{\phi,\mu}\big(\omega \exp(X)\big)\big| \leq e^{\big(\zeta_\lambda + \langle\lambda\rangle\big)(X)}(1 + \|X\|)^N \|\mu\| p(\phi) \qquad \big(\omega \in \Omega, X \in \overline{\mathcal{C}}\big).$$

Lemma 7.1 serves as the initial step with $\mathcal{F} = \mathfrak{a}_E$. Let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$ with $\mathcal{F} \neq \mathfrak{a}_E$, and assume that for every face $\mathcal{F}'$ of $\overline{\mathcal{C}}$ with $\mathcal{F}' \subsetneq \mathcal{F}$ there exists an $N_{\mathcal{F}'} \in \mathbb{N}_0$, a continuous seminorm $p_{\mathcal{F}'}$ on $\mathcal{D}(G, V_\xi)$ and a continuous function $\zeta_{\mathcal{F}'} : \mathfrak{C} \times \mathfrak{a} \to \mathbb{R}$ so that (i) – (iii) hold with $\mathcal{F}'$ in place of $\mathcal{F}$. Without loss of generality may assume that $\zeta_{\mathcal{F}'}(\lambda, X) \geq \rho_Q(X)$ for all $\lambda \in \mathfrak{C}$ and $X \in \mathcal{F}$. We define

$$\zeta_0 : \mathfrak{C} \times \mathfrak{a} \to \mathbb{R}; \quad (\lambda, X) \mapsto \min_{\mathcal{F}' \subsetneq \mathcal{F}} \zeta_{\mathcal{F}'}(\lambda, X)$$

Then ζ_0 is a continuous function with the property that $\zeta_0(\lambda, \cdot\,)$ is an $\mathcal{F}$-piecewise linear functional on $\mathfrak{a}$ and there exists an $N_0 \in \mathbb{N}_0$ and a continuous seminorm p_0 on $\mathcal{D}(G, V_\xi)$ so that for all $\lambda \in \mathfrak{C}$, $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ and $\phi \in \mathcal{D}(G, V_\xi)$

$$\big|m_{\phi,\mu}\big(\omega \exp(X)\big)\big| \leq e^{\zeta_0(\lambda, X) + \langle\lambda\rangle(X)}(1 + \|X\|)^{N_0} \|\mu\| p_0(\phi) \qquad \big(\omega \in \Omega, X \in \overline{\mathcal{C}}\big).$$

We use Lemma 7.1 to improve this estimate.

After passing to a finite cover of $\mathfrak{C}$ of sufficiently small compact subsets of

$$\{\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C} : \operatorname{Im}\lambda \notin \mathcal{S}\},$$

we may assume that $\mathfrak{C}$ satisfies the condition in Lemma 7.1. We now apply Lemma 7.1 repeatedly and obtain sequences $(N_k)_{k\in\mathbb{N}_0}$, $(p_k)_{k\in\mathbb{N}_0}$ and $(\zeta_k)_{k\in\mathbb{N}_0}$ of natural numbers, continuous seminorms on $\mathcal{D}(G, V_\xi)$ and continuous functions on $\mathfrak{C} \times \mathfrak{a} \to \mathbb{R}$, respectively, so that the above assertions (i) and (iii) hold with N_k, p_k and ζ_k in place of N, p and ζ. The sequence $(\zeta_k)_{k\in\mathbb{N}_0}$ satisfies for $\lambda \in \mathfrak{C}$ and $X \in \mathcal{F}$

$$\begin{aligned}&\zeta_{k+1}(\lambda, X) \\ &\quad= \max\Big(\big\{\zeta_k(\lambda, X) + \frac{1}{2}\beta_{\mathcal{F},\lambda}(X), \rho_Q(X)\big\} \cup \big\{\operatorname{Re}\nu(X) - \langle\lambda\rangle(X) : \nu \in \mathcal{Q}_{\mathcal{F},\lambda}^{\zeta_k(\lambda,\,\cdot\,),-}\big\}\Big).\end{aligned} \tag{7.8}$$

We claim that there exists an $n \in \mathbb{N}$ so that $\zeta_k(\lambda, \cdot\,)|_{\mathcal{F}} = \rho_Q|_{\mathcal{F}}$ for every $k \geq n$ and $\lambda \in \mathfrak{C}$. To see this, we first note that the subsequence $(\zeta_k)_{k\in\mathbb{N}}$ is decreasing. This implies that the sets $\mathcal{Q}_{\mathcal{F},\lambda}^{\zeta_k(\lambda,\,\cdot\,),0}$ and $\mathcal{Q}_{\mathcal{F},\lambda}^{\zeta_k(\lambda,\,\cdot\,),-}$ are decreasing with k. Note that the cardinality of $\mathcal{Q}_{\mathcal{F},\lambda}^{\zeta_k(\lambda,\,\cdot\,),-}$ is bounded by the dimension of $U^*_{\mathcal{F}}$. Furthermore, the fact that $\beta_{\mathcal{F},\lambda}$ is a piece-wise linear functional that depends continuously on λ implies that

$$n' := \sup_{\lambda\in\mathfrak{C}, X\in\mathcal{F}^\circ} -2\frac{\zeta_0(\lambda, X) - \rho_Q(X)}{\beta_{\mathcal{F},\lambda}(X)} < \infty$$

The claim now follows from (7.8) with $n = n' + \dim(U^*_{\mathcal{F}})$. The above assertions (i) – (iii) now follow with $N = N_n$, $p = p_n$ and $\zeta = \zeta_n$. ☐

7.6 Constant term approximation

We now give a version of the constant term approximation (see [18, Theorem 1.2] and [36, Theorem 6.2]) which is applicable to our setting.

Theorem 7.1. *Let ξ be a finite dimensional unitary representation of M_Q. Let further $z \in Z$ be admissible and let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. There exists an open neighborhood $\mathfrak{U}$ of $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \setminus i\mathcal{S}$ in $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ and for every $\lambda \in \mathfrak{U}$ a linear map*

$$\mathrm{CT}_{z,\mathcal{F}}(\xi : \lambda) : \mathcal{D}'(\overline{Q} : \xi : \lambda)^H \to \mathcal{D}'(\overline{Q} : \xi : \lambda)^{H_{z,\mathcal{F}}}; \quad \mu \mapsto \mu_{z,\mathcal{F}}$$

with the following properties.

(i) For every $\eta \in V^*(\xi)$ the map

$$\mathfrak{U} \to \mathcal{D}'(\overline{Q} : \xi : \lambda)^{H_{z,\mathcal{F}}}; \quad \lambda \mapsto \mathrm{CT}_{z,\mathcal{F}}(\xi : \lambda) \circ \mu^\circ(\xi : \lambda)(\eta)$$

is a holomorphic family of distributions.

(ii) Let $x \in G$ be so that $z = xH \in G/H = Z$ and let $X \in \mathcal{F}^\circ$. If $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \setminus i\mathcal{S}$ and $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$, then

$$\lim_{t\to\infty} e^{-t\rho_Q(X)}\Big(R^\vee\big(\exp(tX)x\big)\mu - R^\vee\big(\exp(tX)\big)\mu_{z,\mathcal{F}}\Big) = 0 \qquad (7.1)$$

with convergence in $\mathcal{D}'(G, V_\xi)$.

(iii) For every compact subset $\mathfrak{C}$ of $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \setminus i\mathcal{S}$, every compact subset B of G and every closed cone $\Upsilon \subset \mathcal{F}^\circ \cup \{0\}$, there exists a $\gamma \in \mathfrak{a}^*$ with $\gamma|_{\overline{\mathcal{C}}} \leq 0$ and $\gamma|_{\Upsilon\setminus\{0\}} < 0$, an $N \in \mathbb{N}_0$, and a continuous seminorm p on $\mathcal{D}(G, V_\xi)$, so that

$$\begin{aligned} e^{-\rho_Q(Y+X)}&\Big|m_{\phi,\mu}\big(g\exp(Y+X)x\big) - m_{\phi,\mu_{z,\mathcal{F}}}\big(g\exp(Y+X)\big)\Big| \\ &\leq e^{\gamma(X)}(1+\|Y\|)^N\|\mu\|p(\phi) \end{aligned}$$

for all $\lambda \in \mathfrak{C}$, $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$, $\phi \in \mathcal{D}(G, V_\xi)$, $Y \in \overline{\mathcal{C}}$, $X \in \Upsilon$ and $g \in B$.

(iv) Let $z \in Z$ be an admissible point so that $M \cap H_z = M \cap H$. For the face $\mathcal{F} = \overline{\mathcal{C}}$ the image of $\Gamma_{z,\overline{\mathcal{C}}}(\xi : \lambda)$ lies for all $\lambda \in \mathfrak{U}$ in $\mathcal{D}'(\overline{Q} : \xi : \lambda)^{H_\emptyset}$, where

$$H_\emptyset = (L_Q \cap H)\overline{N}_Q.$$

The distribution $\mu_{z,\mathcal{F}}$ is called the constant term of μ with respect to the adapted point z and the face $\mathcal{F}$.

Proof. Without loss of generality we may assume that $z = eH$. We fix $\lambda_0 \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \setminus i\mathcal{S}$ and set

$$\begin{aligned} \mathcal{Q}_\mathcal{F}^+ &:= \big\{\nu \in \mathcal{Q}_{\mathcal{F},\lambda_0} : \operatorname{Re}\nu(X) > \rho_Q(X) \text{ for some } X \in \mathcal{F}^\circ\big\}, \\ \mathcal{Q}_\mathcal{F}^- &:= \big\{\nu \in \mathcal{Q}_{\mathcal{F},\lambda_0} : (\operatorname{Re}\nu - \rho_Q)\big|_{\mathcal{F}^\circ} < 0\big\}, \\ \mathcal{Q}_\mathcal{F}^0 &:= \mathcal{Q}_{\mathcal{F},\lambda_0} \setminus \big(\mathcal{Q}_\mathcal{F}^+ \cup \mathcal{Q}_\mathcal{F}^-\big) = \big\{\nu \in \mathcal{Q}_{\mathcal{F},\lambda_0} : \operatorname{Re}\nu|_\mathcal{F} = \rho_Q|_\mathcal{F}\big\}. \end{aligned}$$

Let $\mathfrak{A} \subseteq \mathfrak{a}_{\mathcal{F}}^*$ be an open polydisc centered at $\rho_Q|_{\mathfrak{a}_{\mathcal{F}}}$ so that

$$\big(\mathcal{Q}_{\mathcal{F}}^+ \cup \mathcal{Q}_{\mathcal{F}}^-\big) \cap \big(\overline{\mathfrak{A}} + i\mathfrak{a}_{\mathcal{F}}\big) = \emptyset.$$

There exits an open neighborhood $\mathfrak{U}_0$ of λ_0 in $\{\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}} : \operatorname{Im}\lambda \notin \mathcal{S}\}$ so that $\mathcal{Q}_{\mathcal{F},\lambda}$ does not intersect with the boundary of $\mathfrak{A}$. We may choose $\mathfrak{U}_0$ so small that there exists a subspace $U_{\mathcal{F}}$ of $V_{\mathcal{F}}$ for which (7.2) holds for all $\lambda \in \mathfrak{U}_0$. We define

$$E : \mathfrak{U}_0 \to \operatorname{End}(U_{\mathcal{F}})$$

for $\lambda \in \mathfrak{U}_0$ to be the projection onto the generalized eigenspaces of $\Gamma_{\mathcal{F},\lambda}$ with eigenvalues in $\mathfrak{A}$, i.e.,

$$E(\lambda) := \sum_{\nu \in \mathfrak{A} \cap \mathcal{Q}_{\mathcal{F},\lambda}} E_\nu.$$

It follows from the Cauchy integral formula for spectral projections from functional calculus that E is holomorphic.

We fix an element $X \in \mathcal{F}^\circ$. After shrinking $\mathfrak{U}_0$ we may assume that

$$\operatorname{Re}\nu(X) > \rho_Q(X) + \langle\lambda\rangle(X) + \beta_{\mathcal{F},\lambda}(X) \qquad \big(\lambda \in \mathfrak{U}_0, \nu \in \mathcal{Q}_{\mathcal{F},\lambda}^0\big).$$

From the estimate (7.4) with $\zeta = \rho_Q$, we obtain that for $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$, with $\lambda \in \mathfrak{U}_0$, and $\phi \in \mathcal{D}(G, V_\xi)$, the $U_{\mathcal{F}}^*$-valued integral

$$\int_0^\infty E(\lambda) e^{-s\Gamma_{\mathcal{F},\lambda}(X)} \Psi_{\mu,\phi,X}\big(\exp(sX)\big)\, ds$$

converges uniformly on any compact subset of $\mathfrak{U}_0$. We may thus define $\operatorname{CT}(\xi : \lambda)\mu \in \mathcal{D}'(G, V_\xi)$ for $\phi \in \mathcal{D}(G, V_\xi)$ by

$$\big(\operatorname{CT}_{z,\mathcal{F}}(\xi : \lambda)\mu\big)(\phi) = \Big(E(\lambda) \circ \Phi_{\mu,\phi}(e) + \int_0^\infty E(\lambda) e^{-s\Gamma_{\mathcal{F},\lambda}(X)} \Psi_{\mu,\phi,X}\big(\exp(sX)\big)\, ds\Big)(1).$$

For every $\eta \in V^*(\xi)$ the family of distributions

$$\mathfrak{U}_0 \ni \lambda \mapsto \operatorname{CT}_{z,\mathcal{F}}(\xi : \lambda) \circ \mu^\circ(\xi : \lambda)\eta$$

is holomorphic.

In view of Theorem 7.2 all distributions in $\mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ with $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^* \setminus i\mathcal{S}$ are tempered. We may therefore apply [18, Theorem 6.9] to these distributions. It follows from [18, (5.36) & (6.1)] that the constant term-map coincides with $\operatorname{CT}_{z,\mathcal{F}}(\xi : \lambda)$ for $\lambda \in \mathfrak{U}_0 \cap i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$. It follows that $\operatorname{CT}_{z,\mathcal{F}}(\xi : \lambda)$ maps $\mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ to $\mathcal{D}'(\overline{Q} : \xi : \lambda)^{H_{z,\mathcal{F}}}$ for these λ. By analytic continuation the same holds for all $\lambda \in \mathfrak{U}_0$.

With the above construction we find for every $\lambda_0 \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^* \setminus i\mathcal{S}$ an open neighborhood $\mathfrak{U}_0$ so that the constant term map $\mathcal{D}'(\overline{Q} : \xi : \lambda)^H \to \mathcal{D}'(\overline{Q} : \xi : \lambda)^{H_{z,\mathcal{F}}}$ from [18] extends holomorphically to $\lambda \in \mathfrak{U}_0$. It follows that there exists an open neighborhood $\mathfrak{U}$ of $i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^* \setminus i\mathcal{S}$ so that the constant term map extends holomorphically to $\mathfrak{U}$.

The remaining assertions in (ii) and (iii) are a reformulation of [18, Theorem 6.9] with uniformity in the estimate in $\lambda \in \mathfrak{C}$. The uniform estimates are obtained by using the estimates (7.3), (7.4) and (7.5), which are uniform in $\lambda \in \mathfrak{C}$, instead of the estimates in [18, Lemmas 5.8 & 5.9].

Finally, we turn to (iv). By holomorphicity it suffices to prove the assertion for $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \setminus i\mathcal{S}$. For these λ it follows from [18, Lemma 6.5] and Corollary 6.1 that $\mathrm{CT}_{z,\overline{\mathcal{C}}}(\xi : \lambda)\mu$ for a $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ is uniquely determined by (7.1). Since M centralizes $\mathfrak{a}$, it is easily seen that

$$R^\vee(m) \circ \mathrm{CT}_{z,\overline{\mathcal{C}}}(\xi : \lambda) = \mathrm{CT}_{z,\overline{\mathcal{C}}}(\xi : \lambda) \circ R^\vee(m) \qquad (m \in M).$$

In particular, for every $m \in M \cap H$ and $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$

$$R^\vee(m)\big(\mathrm{CT}_{z,\overline{\mathcal{C}}}(\xi : \lambda)\mu\big) = \mathrm{CT}_{z,\overline{\mathcal{C}}}(\xi : \lambda)\big(R^\vee(m)\mu\big) = \mathrm{CT}_{z,\overline{\mathcal{C}}}(\xi : \lambda)\mu.$$

As

$$H_\emptyset = (M \cap H)(H_\emptyset)_e = (M \cap H)H_{z,\overline{\mathcal{C}}},$$

this proves (iv). $\square$

7.7 Construction of wave packets

We use the notation from Section 6.4, and recall the finite union $\mathcal{S}$ of hyperplanes in $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ from the end of Section 6.4. For a finite dimensional unitary representation ξ of M_Q we define the wave packet transform

$$\mathcal{WP}_\xi : \mathcal{D}\big(i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \setminus i\mathcal{S}\big) \otimes V^*(\xi) \otimes \mathcal{D}(G, V_\xi) \to \mathcal{E}(Z),$$

to be given by

$$\mathcal{WP}_\xi(\psi \otimes \eta \otimes \phi)(gH) = \int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*} \psi(\lambda)\Big(R^\vee(g) \circ \mu^\circ(\xi : \lambda)(\eta)\Big)(\phi)\, d\lambda$$

for $\psi \in \mathcal{D}\big(i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \setminus i\mathcal{S}\big)$, $\eta \in V^*(\xi)$, $\phi \in \mathcal{D}(G, V_\xi)$ and $g \in G$. The following is the main result in this section.

Theorem 7.1. *Let ξ be a finite dimensional unitary representation of M_Q. The image of $\mathcal{WP}_\xi$ is consists of square integrable functions on Z.*

Remark 7.2. Theorem 7.1 has an important consequence for the multiplicity spaces in (1.3). Each multiplicity space $\mathcal{M}_{\xi,\lambda}$ is a subspace of the space of H-fixed functionals on $C^\infty(\overline{Q} : \xi : \lambda)$. In view of the topological isomorphism (4.2) we may view $\mathcal{M}_{\xi,\lambda}$ as a subspace of $\mathcal{D}'(\overline{Q} : \xi : \lambda)^H$. From the theorem it follows that for almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ the multiplicity space $\mathcal{M}_{\xi,\lambda}$ equals $\mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ and the map

$$\mu^\circ(\xi : \lambda) : V^*(\xi) \to \mathcal{M}_{\xi,\lambda}$$

is a linear isomorphism. We will prove this in Corollary 8.2.

Proof of Theorem 7.1. In view of (7.1) we have every integrable function χ on Z

$$\int_Z \chi(z)\,dz = \sum_{j=1}^{r} \sum_{\mathcal{O}\in(P\backslash Z)_{\mathrm{open}}} \int_{K_j}\int_{\overline{\mathcal{C}}} \chi\big(f_j k \exp(X) x_{\mathcal{O}} H\big) J_{j,\mathcal{O}}(k,X)\,dX\,dk.$$

Here dX is the Lebesgue-measure on $\mathfrak{a}$, dk denotes for each j the Haar measure on K_j. The Jacobian

$$J_{j,\mathcal{O}} : K_j \times \overline{\mathcal{C}} \to \mathbb{R}_{\geq 0}$$

are readily seen to be constant in the first variable. We therefore consider these functions as functions on $\overline{\mathcal{C}}$ only. Important for our consideration is the estimate

$$J_{j,\mathcal{O}}(X) \leq C e^{-2\rho_Q(X)} \qquad \big(f \in F, \mathcal{O} \in (P\backslash Z)_{\mathrm{open}}, X \in \overline{\mathcal{C}}\big)$$

for some constant $C > 0$. This estimate follows from [30, Proposition 4.3].

We will decompose the integral over $\overline{\mathcal{C}}$ as a sum of integrals over suitable subsets which allow to apply Theorem 7.1. We recall from Section 3.9 that the little Weyl group is the Weyl group of the spherical root system Σ_Z in $(\mathfrak{a}/\mathfrak{a}_E)^*$. The faces of $\overline{\mathcal{C}}$ are in bijection with the power set of the simple system Π_Z of Σ_Z whose corresponding positive system consists of all roots that are strictly negative on $\mathcal{C}$. To be more precise, to a face $\mathcal{F}$ of $\overline{\mathcal{C}}$ a subset $S_{\mathcal{F}}$ of Π_Z is attached with the property

$$\mathcal{F} = \big\{X \in \mathfrak{a} : \sigma(X) = 0 \text{ for all } \sigma \in S_{\mathcal{F}} \text{ and } \sigma(X) < 0 \text{ for all } \sigma \in \Pi_Z \setminus S_{\mathcal{F}}\big\}.$$

The assignment $\mathcal{F} \mapsto S_{\mathcal{F}}$ is a bijection between the faces of $\overline{\mathcal{C}}$ and the power set of Π_Z. Let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. If $\mathcal{F}'$ is the unique face of $\overline{\mathcal{C}}$ with $S_{\mathcal{F}'} = \Pi_Z \setminus S_{\mathcal{F}}$, then $\mathcal{F} \cap \mathcal{F}' = \overline{\mathcal{C}} \cap (-\overline{\mathcal{C}}) = \mathfrak{a}_E$ is the edge of $\overline{\mathcal{C}}$. We then define the cone

$$\mathcal{F}_\perp := \mathcal{F}' \cap \mathfrak{a}_E^\perp$$

and set

$$\mathfrak{a}_{\mathcal{F},\perp} := \mathrm{span}(\mathcal{F}_\perp).$$

Now

$$\mathfrak{a} = \mathfrak{a}_{\mathcal{F}} \oplus \mathfrak{a}_{\mathcal{F}_\perp} \tag{7.1}$$

and

$$\overline{\mathcal{C}} = \mathcal{F} + \mathcal{F}_\perp.$$

We write $p_{\mathcal{F}}$ to be the projection $\mathfrak{a} \to \mathfrak{a}_{\mathcal{F}}$ along the decomposition (7.1). We fix a $\delta > 0$ and define cones $C_{\mathcal{F}}$ in $\overline{\mathcal{C}}$ by setting

$$C_{\mathfrak{a}_E} := \{X \in \overline{\mathcal{C}} : \|X\| \leq (1+\delta)\big\|p_{\mathcal{F}}(X)\big\|\}$$

if $\mathcal{F} = \mathfrak{a}_E$, and then recursively by

$$C_{\mathcal{F}} := \Big\{X \in \overline{\mathcal{C}} \setminus \bigcup_{\substack{\mathcal{F}' \text{ face of } \overline{\mathcal{C}} \\ \mathcal{F}' \subsetneq \mathcal{F}}} C_{\mathcal{F}'} : \|X\| \leq (1+\delta)\big\|p_{\mathcal{F}}(X)\big\|\Big\}$$

for the remaining faces $\mathcal{F}$. Then for every face $\mathcal{F}$ of $\overline{\mathcal{C}}$ there exists a closed cone $\Upsilon_{\mathcal{F}}$ in $\mathcal{F}^\circ \cup \{0\}$ so that

$$\overline{C_{\mathcal{F}}} \subseteq \Big\{X \in \overline{\mathcal{C}} : p_{\mathcal{F}}(X) \in \Upsilon_{\mathcal{F}}, \|X\| \leq (1+\delta)\big\|p_{\mathcal{F}}(X)\big\|\Big\}.$$

Moreover,

$$\overline{\mathcal{C}} = \bigsqcup_{\mathcal{F} \text{ face of } \overline{\mathcal{C}}} C_{\mathcal{F}}.$$

Now

$$\begin{aligned}\int_Z \chi(z)\,dz &= \sum_{j=1}^r \sum_{\mathcal{O}\in(P\backslash Z)_{\mathrm{open}}} \sum_{\mathcal{F} \text{ face of } \overline{\mathcal{C}}} \int_{K_j}\int_{C_{\mathcal{F}}} \chi\big(f_j k \exp(X) x_{\mathcal{O}} H\big) J_{j,\mathcal{O}}(X)\,dX\,dk \\ &\leq C\sum_{j=1}^r \sum_{\mathcal{O}\in(P\backslash Z)_{\mathrm{open}}} \sum_{\mathcal{F} \text{ face of } \overline{\mathcal{C}}} \int_{K_j}\int_{C_{\mathcal{F}}} \chi\big(f_j k \exp(X) x_{\mathcal{O}} H\big) e^{-2\rho_Q(X)}\,dX\,dk\end{aligned}$$

for every non-negative measurable function χ on Z.

Let K' be a maximal compact subgroup, $\mathcal{F}$ a face of $\overline{\mathcal{C}}$ and $z \in Z$ an adapted point. To prove the theorem, it suffices to prove that for every $\psi \in \mathcal{D}\big(i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^* \setminus i\mathcal{S}\big)$, $\eta \in V^*(\xi)$ and $\phi \in \mathcal{D}(G, V_\xi)$ the function

$$K' \times C_{\mathcal{F}} \to \mathbb{C}; \quad (k, X) \mapsto e^{-\rho_Q(X)} \mathcal{WP}_\xi(\psi \otimes \eta \otimes \phi)\big(k\exp(X)\cdot z\big)$$

is square integrable.

It follows from Proposition 6.1 that for each $w \in W$ there exists a linear map

$$\mu_{\mathcal{F},w}(\xi : \lambda) : V^*(\xi) \to \mathcal{D}'(G, V_\xi)^{H_{z,\mathcal{F}}}$$

so that

$$\mathrm{CT}_{z,\mathcal{F}}(\xi : \lambda) \circ \mu^\circ(\xi : \lambda) = \sum_{w\in W} \mu_{\mathcal{F},w}(\xi : \lambda)$$

and

$$R^\vee\big(\exp(X)\big) \circ \mu_{\mathcal{F},w}(\xi : \lambda) = e^{\big(-\mathrm{Ad}^*(w^{-1})\lambda + \rho_Q\big)(X)} \mu_{\mathcal{F},w}(\xi : \lambda) \qquad \big(w \in W, X \in \mathfrak{a}_{\mathcal{F}}\big).$$

The family $\lambda \mapsto \mu_{\mathcal{F},w}(\xi : \lambda)$ is meromorphic. Moreover, the family is holomorphic on $\{\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}} : \mathrm{Im}\,\lambda \notin \mathcal{S}\}$.

We now fix a face $\mathcal{F}$ of $\overline{\mathcal{C}}$ and a $w \in W$. Let $\gamma_{\mathcal{F}} \in \mathfrak{a}^*$ and $p = p_{z,\mathcal{F}}$ satisfy the properties of Theorem 7.1 (iii) with the closed cone Υ taken to be $\Upsilon_{\mathcal{F}}$ and the compact subset B equal to K'. As $\gamma_{\mathcal{F}}|_{\Upsilon_{\mathcal{F}}} < 0$, we have

$$\int_{C_{\mathcal{F}}} \Big(e^{\gamma_{\mathcal{F}}\big(p_{\mathcal{F}}(X)\big)}(1 + \|X - p_{\mathcal{F}}(X)\|)^N\Big)^2\,dX < \infty$$

for every $N \in \mathbb{N}$. Therefore, it suffices to prove that for every $\psi \in \mathcal{D}\big(i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^* \setminus i\mathcal{S}\big)$, $\eta \in V^*(\xi)$ and $\phi \in \mathcal{D}(G, V_\xi)$ the function

$$\begin{aligned}&\Omega_{\psi,\eta,\phi} : G \times (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}) \to \mathbb{C};\\ &\qquad (g, X) \mapsto e^{-\rho_Q(X)} \int_{i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*} \psi(\lambda)\Big(R^\vee\big(g\exp(X)\big)\mu_{\mathcal{F},w}(\xi : \lambda)\eta\Big)(\phi)\,d\lambda\end{aligned}$$

is square integrable on $K' \times C_{\mathcal{F}}$.

Let $\mathscr{F}_{\mathrm{eucl}}$ be the Euclidean Fourier transform on $\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}$, i.e., the transform

$$\mathscr{F}_{\mathrm{eucl}} : \mathcal{S}\big(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}\big) \to \mathcal{S}\big(i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big)$$

given by

$$\mathscr{F}_{\mathrm{eucl}}\psi(\xi) = \int_{\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}} \psi(X)e^{\xi(X)}\,dX \qquad \big(\psi \in \mathcal{S}(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}}), \xi \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big).$$

For every $X \in \mathfrak{a}_{\mathcal{F}}$ and $\eta \in V^*(\xi)$

$$e^{-\rho_Q(X)}R^\vee\big(\exp(X)\big)\mu_{\mathcal{F},w}(\xi : \lambda)\eta = e^{-\lambda\big(\mathrm{Ad}(w)X\big)}\mu_{\mathcal{F},w}(\xi : \lambda)\eta,$$

and hence

$$\begin{aligned}\Omega_{\psi,\eta,\phi}(g, X) &= e^{-\rho_Q(X)} \int_{i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*} \psi(\lambda)\Big(R^\vee\big(g\exp(X)\big)\mu_{\mathcal{F},w}(\xi : \lambda)\eta\Big)(\phi)\,d\lambda\\ &= \mathscr{F}_{\mathrm{eucl}}^{-1}\Big(\lambda \mapsto \psi(\lambda)\big(R^\vee(g)\mu_{\mathcal{F},w}(\xi : \lambda)\eta\big)(\phi)\Big)\big(\mathrm{Ad}(w)X\big).\end{aligned}$$

Since $\lambda \mapsto \psi(\lambda)\big(R^\vee(g)\mu_{\mathcal{F},w}(\xi : \lambda)\eta\big)(\phi)$ is compactly supported and smooth, the function $X \mapsto \Omega_{\psi,\eta,\phi}(g, X)$ is contained in $\mathcal{S}\big(\mathfrak{a}_{\mathcal{F}}\big)$. Moreover, the continuity of $\mathscr{F}_{\mathrm{eucl}}$ implies that for every continuous seminorm p on $\mathcal{S}(\mathfrak{a}_{\mathcal{F}})$ there exists a continuous seminorm q on $\mathcal{D}\big(i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big)$, independent of $g \in G$ and $\phi \in \mathcal{D}(G, V_\xi)$, so that

$$p\big(\Omega_{\psi,\eta,\phi}(g,\,\cdot\,)\big) \leq q\Big(\lambda \mapsto \psi(\lambda)\big(R^\vee(g)\mu_{\mathcal{F},w}(\xi : \lambda)\eta\big)(\phi)\Big). \tag{7.2}$$

Let $\mathfrak{C} \subseteq i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^* \setminus i\mathcal{S}$ be a compact subset. We claim that for every differential operator D on $i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ with constant coefficients there exists an $N \in \mathbb{N}_0$ and a continuous seminorm r on $\mathcal{D}(G, V_\xi)$ so that for all $\lambda \in \mathfrak{C}$, $\phi \in \mathcal{D}(G, V_\xi)$, $k \in K'$ and $Y \in \mathfrak{a}$ the estimate

$$\Big|D\big(R^\vee\big(k\exp(Y)\big)\mu_{\mathcal{F},w}(\xi : \lambda)\eta\big)(\phi)\Big| \leq e^{\rho_Q(Y)}(1 + \|Y\|)^N r(\phi) \tag{7.3}$$

holds. It suffices to prove the claim for $D = \partial_\lambda^\alpha$, with α a multi-index. We first note that it follows from Theorem 7.1 that $\lambda \mapsto \mu_{\mathcal{F},w}(\xi : \lambda)\eta$ extends to a holomorphic family of distributions with family parameter λ in an open neighborhood $\mathfrak{U}$ of $i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^* \setminus i\mathcal{S}$. Let $\epsilon > 0$ be so small that the polydisc Δ with radius ϵ and center λ is contained in $\mathfrak{U}$. Let now Δ_δ be the polydisc centered at λ of radius $\delta > 0$. For every $\delta \leq \epsilon$ we obtain from Cauchy's integral formula the estimate

$$\Big|\partial_\lambda^\alpha\big(R^\vee\big(k\exp(Y)\big)\mu_{\mathcal{F},w}(\xi : \lambda)\eta\big)(\phi)\Big| \leq \frac{\alpha!}{\delta^{|\alpha|}} \sup_{\lambda' \in \Delta_\delta} \Big|\big(R^\vee\big(k\exp(Y)\big)\mu_{\mathcal{F},w}(\xi : \lambda')\eta\big)(\phi)\Big|.$$

We now invoke Theorem 7.2. This yields the existence of an $N' \in \mathbb{N}_0$ and a continuous seminorm r' on $\mathcal{D}(G, V_\xi)$ so that

$$\begin{aligned}&\sup_{\lambda' \in \Delta_\delta} \Big|\big(R^\vee\big(k\exp(Y)\big)\mu_{\mathcal{F},w}(\xi : \lambda')\eta\big)(\phi)\Big|\\ &\quad\leq \sup_{\lambda' \in \Delta_\delta} \max_{w \in W} e^{\rho_Q(Y) + \mathrm{Re}\,\mathrm{Ad}^*(w)\lambda'(Y)}\big(1 + \|Y\|\big)^{N'} r'(\phi) \leq e^{\rho_Q(Y) + \delta\|Y\|}\big(1 + \|Y\|\big)^{N'} r'(\phi).\end{aligned}$$

The claim now follows with $N = N' + |\alpha|$ and $r = e\alpha!\,r'$ by taking δ equal to the minimum of ϵ and $(1 + \|Y\|)^{-1}$.

We now consider the space $\mathcal{D}_{\mathfrak{C}}\big(i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big)$ of functions $\psi \in \mathcal{D}\big(i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big)$ satisfying $\mathrm{supp}(\psi) \subseteq \mathfrak{C}$. Every continuous seminorm on $\mathcal{D}_{\mathfrak{C}}\big(i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big)$ can be dominated by a sum of seminorms of the sort $\psi \mapsto \sup|D\phi|$, where D is a differential operator with constant coefficients. It follows from (7.2), the Leibnitz rule and (7.3) that for every continuous seminorm p on $\mathcal{S}(\mathfrak{a}_{\mathcal{F}})$ there exist continuous seminorms r and s on $\mathcal{D}_{\mathfrak{C}}\big(i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big)$ and $\mathcal{D}(G, V_\xi)$, respectively, and an $N \in \mathbb{N}_0$, so that for all $\psi \in \mathcal{D}_{\mathfrak{C}}\big(i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big)$, $\eta \in V^*(\xi)$, $\phi \in \mathcal{D}(G, V_\xi)$, $\lambda \in \mathfrak{C}$, $k \in K'$ and $Y \in \mathfrak{a}$ the estimate

$$p\Big(\mathfrak{a}_{\mathcal{F}} \ni X \mapsto \Omega_{\psi,\eta,\phi}\big(k, X + Y\big)\Big) \leq (1 + \|Y\|)^N r(\psi)\|\eta\| s(\phi).$$

holds. In particular, for every $n \in \mathbb{N}_0$ there exist continuous seminorms r_n on $\mathcal{D}_{\mathfrak{C}}\big(i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big)$ and s_n on $\mathcal{D}(G, V_\xi)$ so that

$$\sup_{X \in \mathfrak{a}_{\mathcal{F}}} (1 + \|X\|)^n \big|\Omega_{\psi,\eta,\phi}(k, X + Y)\big| \leq (1 + \|Y\|)^N r_n(\psi)\|\eta\| s_n(\phi).$$

for every $\psi \in \mathcal{D}_{\mathfrak{C}}\big(i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\big)$, $\eta \in V^*(\xi)$, $\phi \in \mathcal{D}(G, V_\xi)$, $k \in K'$ and $Y \in \mathfrak{a}$.

From the definition of $C_{\mathcal{F}}$ it follows that there exists a constant $c > 0$, so that if $X \in \mathfrak{a}_{\mathcal{F}}$, $Y \in \mathfrak{a}_{\mathcal{F}_\perp}$ and $X + Y \in C_{\mathcal{F}}$, then $\|Y\| \leq c\|X\|$. For $n > N + \dim(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})/2$ the integral

$$\int_{K'} \int_{C_{\mathcal{F}}} \big|\Omega_{\psi,\eta,\phi}(k, X)\big|^2 \, dX \, dk$$

is therefore absolutely convergent and bounded by

$$\mathrm{vol}(K') r_n(\psi)\|\eta\| s_n(\phi) \int_{\Upsilon_{\mathcal{F}}} (1 + \|X\|)^{2N + \dim(\mathfrak{a}_{\mathcal{F}_\perp}) - 2n} \, dX.$$

This proves the theorem. □

8 The most continuous part of $L^2(Z)$

8.1 Abstract Plancherel decomposition

In this section we describe the abstract Plancherel theorem for the space Z. We denote by $\widehat{G}$ the unitary dual of G. For each equivalence class $[\pi] \in \widehat{G}$ we choose a representative $(\pi, \mathcal{H}_\pi)$, i.e., $\mathcal{H}_\pi$ is a Hilbert space and π is a unitary representation of G on $\mathcal{H}_\pi$ in the equivalence class $[\pi]$. We denote the space of smooth vectors of π by $\mathcal{H}_\pi^\infty$.

Let $[\pi] \in \widehat{G}$. Since Z is real spherical, the space $(\mathcal{H}_{\pi^\vee}^{\infty}{}')^H$ is finite dimensional. See [33, Theorem C] and [37]. For every $\mu \in (\mathcal{H}_{\pi^\vee}^{\infty}{}')^H$ and $f \in \mathcal{D}(Z)$ the functional

$$\mathcal{H}_{\pi^\vee}^\infty \ni v \mapsto \int_Z f(gH)\Big(\pi(g)\mu\Big)(v) \, dgH$$

actually defines a smooth vector for π. We define the Fourier transform

$$\mathscr{F} f(\pi) \in \mathrm{Hom}_{\mathbb{C}}\left((\mathcal{H}_{\pi^\vee}^{\infty}{}')^H, \mathcal{H}_\pi^\infty\right)$$

of a function $f \in \mathcal{D}(Z)$ and $\mu \in (\mathcal{H}_{\pi^\vee}^{\infty}{}')^H$ by

$$\mathscr{F} f(\pi)\mu = \int_Z f(gH)\pi(g)\mu \, dgH.$$

By the abstract Plancherel Theorem there exists a Radon measure $d_{\mathrm{Pl}}[\pi]$ on $\widehat{G}$ and for every $[\pi] \in \widehat{G}$ a Hilbert space

$$\mathcal{M}_\pi \subseteq (\mathcal{H}_\pi^{\infty\prime})^H,$$

depending measurably on $[\pi]$, so that the Fourier transform

$$\mathscr{F} : \mathcal{D}(Z) \to \int_{\widehat{G}}^{\oplus} \mathrm{Hom}_{\mathbb{C}}\left(\mathcal{M}_{\pi^\vee}, \mathcal{H}_\pi\right) d_{\mathrm{Pl}}[\pi]$$

with the induced Hilbert space structure on $\mathrm{Hom}_{\mathbb{C}}\left(\mathcal{M}_{\pi^\vee}, \mathcal{H}_\pi\right)$ extends to a unitary G-isomorphism

$$\mathscr{F} : L^2(Z) \to \int_{\widehat{G}}^{\oplus} \mathrm{Hom}_{\mathbb{C}}\left(\mathcal{M}_{\pi^\vee}, \mathcal{H}_\pi\right) d_{\mathrm{Pl}}[\pi]. \tag{8.1}$$

The measure class of the Plancherel measure $d_{\mathrm{Pl}}[\pi]$ is uniquely determined by Z. Once $d_{\mathrm{Pl}}[\pi]$ has been fixed, the multiplicity spaces $\mathcal{M}_\pi$, including their inner products, are uniquely determined for almost all $[\pi] \in \widehat{G}$. By dualizing (8.1) we obtain that the dual space of $\mathcal{M}_{\pi^\vee}$ is equal to $\mathcal{M}_\pi$. Therefore, the abstract Plancherel decomposition may also be written in its more common form

$$L^2(Z) \simeq \int_{\widehat{G}}^{\oplus} \mathcal{M}_\pi \otimes \mathcal{H}_\pi \, d_{\mathrm{Pl}}[\pi].$$

We recall the Bernstein morphisms B_I with $I \subseteq \Pi_Z$ from (1.1) and (1.2). In the remainder of Section 8 we will derive the decomposition of the most continuous part of $L^2(Z)$

$$L^2_{\mathrm{mc}}(Z) := \mathrm{Im}\,(B_\emptyset) \cap L^2(Z)$$

into a direct integral of irreducible unitary representations of G.

8.2 Plancherel decomposition for $Z_\emptyset$

We recall that $Z_\emptyset = G/H_\emptyset$, where

$$H_\emptyset = (L_Q \cap H)\overline{N}_Q.$$

In this section we determine the Plancherel decomposition for $Z_\emptyset$.

We choose a set of representatives $\mathfrak{N}$ of $\mathcal{N}/\mathcal{Z}$ in $\mathcal{N} \cap K$ as in Section 6.8 and define

$$V_\emptyset^*(\xi) := \bigoplus_{v \in \mathfrak{N}} (V_\xi^*)^{M_Q \cap vHv^{-1}}.$$

We write

$$\mu_\emptyset^\circ(\xi : \lambda) : V_\emptyset^*(\xi) \to \mathcal{D}'(\overline{Q} : \xi : \lambda)^{H_\emptyset}$$

for the map from Corollary 6.1 for the space $Z_\emptyset$. Now we define the Fourier transform

$$\mathscr{F}_\emptyset f(\xi : \lambda) \in \operatorname{Hom}_{\mathbb{C}}\big(V_\emptyset^*(\xi^\vee), C^\infty(\overline{Q} : \xi : \lambda)\big) = V_\emptyset^*(\xi) \otimes C^\infty(\overline{Q} : \xi : \lambda).$$

of a function $f \in \mathcal{D}(Z_\emptyset)$ by

$$\mathscr{F}_\emptyset f(\xi : \lambda)\eta = \int_{Z_\emptyset} f(gH_\emptyset) R^\vee(g)\Big(\mu_\emptyset^\circ(\xi^\vee : -\lambda)\eta\Big)\, dgH_\emptyset.$$

Let $\langle\cdot,\cdot\rangle_{\emptyset,\xi^\vee}$ be the inner product on $V_\emptyset^*(\xi^\vee)$ induced by the inner product on $V_{\xi^\vee}$, and let $\langle\cdot,\cdot\rangle_{\emptyset,\xi,\lambda}$ be the inner product on $V_\emptyset^*(\xi) \otimes \operatorname{Ind}_{\overline{Q}}^G(\xi \otimes \lambda \otimes \mathbf{1})$ induced by the inner products $\langle\cdot,\cdot\rangle_{\emptyset,\xi}$ and $\langle\cdot,\cdot\rangle_{\overline{Q},\xi,\lambda}$ on $V_\emptyset^*(\xi)$ and $\operatorname{Ind}_{\overline{Q}}^G(\xi \otimes \lambda \otimes \mathbf{1})$, respectively.

Recall that $\widehat{M}_{Q,\mathrm{fu}}$ denotes the set of equivalence classes of finite dimensional unitary representations of M_Q. Let $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ be a fundamental domain for the action of $\mathcal{N}$ on $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. We recall that we normalize Lebesgue measure on $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ by requiring that

$$\phi(e) = \int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*} \int_{A/(A\cap H)} \phi(a) a^\lambda \, da\, d\lambda \qquad \Big(\phi \in \mathcal{D}\big(A/(A\cap H)\big)\Big).$$

We then have the following Plancherel decomposition.

Theorem 8.1. *The Fourier transform $f \mapsto \mathscr{F}_\emptyset f$ extends to a continuous linear operator*

$$L^2(Z_\emptyset) \to \widehat{\bigoplus_{\xi\in\widehat{M}_{Q,\mathrm{fu}}}} \int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*}^{\oplus} V_\emptyset^*(\xi) \otimes \operatorname{Ind}_{\overline{Q}}^G(\xi \otimes \lambda \otimes \mathbf{1})\, d\lambda. \tag{8.1}$$

Moreover, for every $f_1, f_2 \in L^2(Z_\emptyset)$

$$\int_{Z_\emptyset} f_1(z)\overline{f_2(z)}\, dz = \sum_{[\xi]\in\widehat{M}_{Q,\mathrm{fu}}} \dim(V_\xi) \int_{(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+} \Big\langle \mathscr{F}_\emptyset f_1(\xi : \lambda), \mathscr{F}_\emptyset f_2(\xi : \lambda)\Big\rangle_{\emptyset,\xi,\lambda} d\lambda. \tag{8.2}$$

Remark 8.2. In view of the following assertions the decomposition (8.1) is in fact the Plancherel decomposition for $Z_\emptyset$.

(a) Let $\xi \in \widehat{M}_Q$. For every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ the representation $\operatorname{Ind}_{\overline{Q}}^G(\xi\otimes\lambda\otimes\mathbf{1})$ is irreducible. See [12, p. 203, Théorème 4] and [34, Theorem 4.11].

(b) Let $\xi, \xi' \in \widehat{M}_Q$. For almost all $\lambda, \lambda' \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ the representations $\operatorname{Ind}_{\overline{Q}}^G(\xi \otimes \lambda \otimes \mathbf{1})$ and $\operatorname{Ind}_{\overline{Q}}^G(\xi' \otimes \lambda' \otimes \mathbf{1})$ are equivalent if and only if there exists a $w \in \mathcal{N}$ so that $\xi = w \cdot \xi'$ and $\lambda = \operatorname{Ad}^*(w)\lambda'$. This assertion follows from the same arguments as those in the proof of [9, Theorem 10.7].

We first prove a lemma. Recall the inclusions ι_v for $v \in \mathcal{N}$ from (6.3).

Lemma 8.3. *Let ξ be a finite dimensional unitary representation of M_Q, $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\backslash i\mathcal{S}$ and $f \in \mathcal{D}(Z_\emptyset)$. Then for every $\eta \in V_\xi^{M_Q\cap H}$*

$$\mathscr{F}_\emptyset f(\xi : \lambda)\big(\iota_e\eta\big) : g \mapsto \int_{M_Q/(M_Q\cap H)} \int_{A/(A\cap H)} f(g^{-1}amH_\emptyset)a^{\lambda-\rho_Q}\xi(m)\eta \, da\, dm. \quad (8.3)$$

Furthermore, for every $v \in \mathfrak{N}$

$$\begin{aligned}&\mathscr{F}_\emptyset f\big(v\cdot\xi : \mathrm{Ad}^*(v)\lambda\big)\circ\iota_v &(8.4)\\ &\quad= \frac{1}{\gamma(v^{-1}\overline{Q}v : \overline{Q} : \xi : \lambda)} L(v)\circ A(v^{-1}\overline{Q}v : \overline{Q} : \xi : \lambda)\circ \mathscr{F}_\emptyset f(\xi : \lambda)\circ\iota_e.\end{aligned}$$

Finally, for every $v \in \mathfrak{N}$ and $\eta \in V_{v\cdot\xi}^{M_Q\cap vHv^{-1}} = V_\xi^{M_Q\cap H}$ we have

$$\big\|\mathscr{F}_\emptyset f\big(v\cdot\xi : \mathrm{Ad}^*(v)\lambda\big)\big(\iota_v\eta\big)\big\|_{\overline{Q},\xi,\lambda} = \big\|\mathscr{F}_\emptyset f\big(\xi : \lambda\big)\big(\iota_e\eta\big)\big\|_{\overline{Q},\xi,\lambda}. \quad (8.5)$$

Proof. Let $\eta \in V_\xi^{M_Q\cap H}$. By (6.8) the distribution $\mu_\emptyset^\circ(\xi^\vee : -\lambda)(\iota_e\eta)$ is for $\phi \in \mathcal{D}(G, V_\xi^*)$ and $\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ given by

$$\big(\mu_\emptyset^\circ(\xi^\vee : -\lambda)(\iota_e\eta)\big)(\phi) = \int_{M_Q}\int_A\int_{\overline{N}_Q} a^{\lambda-\rho_Q}\Big(\xi(m)\eta, \phi(ma\overline{n})\Big)\, d\overline{n}\, da\, dm\, dn.$$

Let $f \in \mathcal{D}(Z_\emptyset)$. Then

$$\begin{aligned}&\Big(\mathscr{F}_\emptyset f(\xi : \lambda)\big(\iota_e\eta\big)\Big)(\phi)\\ &\quad= \int_{Z_\emptyset} f(gH_\emptyset)R^\vee(g)\Big(\mu_\emptyset^\circ(\xi^\vee : -\lambda)(\iota_e\eta)\Big)(\phi)\, dgH_\emptyset\\ &\quad= \int_{Z_\emptyset}\int_{M_Q}\int_A\int_{\overline{N}_Q} f(gH_\emptyset)a^{\lambda-\rho_Q}\Big(\xi(m)\eta, \phi(ma\overline{n}g^{-1})\Big)\, d\overline{n}\, da\, dm\, dgH_\emptyset\\ &\quad= \int_{Z_\emptyset}\int_{M_Q/(M_Q\cap H)}\int_{A/(A\cap H)}\int_{H_\emptyset} f(gH_\emptyset)a^{\lambda-\rho_Q}\Big(\xi(m)\eta, \phi(mah^{-1}g^{-1})\Big)\, dh\, da\, dm\, dgH_\emptyset.\end{aligned}$$

Let M_0 be a submanifold of M_Q so that

$$M_0 \to M_Q/(M_Q\cap H); \quad m_0 \mapsto m_0(M_Q\cap H)$$

is a diffeomorphism onto an open and dense subset of $M_Q/(M_Q\cap H)$ and let $d\mu$ be the pull back of the invariant measure on $M_Q/(M_Q\cap H)$ along this map. Let further A_0 be a closed subgroup of A so that

$$A_0 \to A/(A\cap H); \quad a_0 \mapsto a_0(A\cap H)$$

is a diffeomorphism. Then

$$\begin{aligned}
&\Big(\mathscr{F}_\emptyset f(\xi:\lambda)\big(\iota_e\eta\big)\Big)(\phi)\\
&\qquad= \int_{Z_\emptyset}\int_{H_\emptyset}\int_{M_0}\int_{A_0} f(ghH_\emptyset)a^{\lambda-\rho_Q}\Big(\xi(m)\eta, \phi(mah^{-1}g^{-1})\Big)\,da\,d\mu(m)\,dh\,dgH_\emptyset\\
&\qquad= \int_{G}\int_{M_0}\int_{A_0} f(gH_\emptyset)a^{\lambda-\rho_Q}\Big(\xi(m)\eta, \phi(mag^{-1})\Big)\,da\,d\mu(m)\,dg\\
&\qquad= \int_{G}\bigg(\int_{M_0}\int_{A_0} f(g^{-1}amH_\emptyset)a^{\lambda-\rho_Q}\xi(m)\eta\,da\,d\mu(m), \phi(g)\bigg)\,dg\\
&\qquad= \int_{G}\bigg(\int_{M_Q/(M_Q\cap H)}\int_{A/(A\cap H)} f(g^{-1}amH_\emptyset)a^{\lambda-\rho_Q}\xi(m)\eta\,da\,dm, \phi(g)\bigg)\,dg.
\end{aligned}$$

This proves (8.3).

The identity (8.4) follows from (6.7) as the intertwining operator $\mathcal{I}_v^\circ(\xi^\vee:-\lambda)$ acts on the subspace $C^\infty(\overline{Q}:\xi:\lambda)$ of $\mathcal{D}'(\overline{Q}:\xi^\vee:-\lambda)$ by

$$\frac{1}{\gamma(v^{-1}\overline{Q}v:\overline{Q}:\xi:\lambda)}L(v)\circ A(v^{-1}\overline{Q}v:\overline{Q}:\xi:\lambda). \tag{8.6}$$

Finally, (8.5) follows from (8.4) as (8.6) is a unitary map. □

Proof of Theorem 8.1. Let $f\in\mathcal{D}(Z_\emptyset)$. In view of the decomposition polar $G=KAH_\emptyset$, we have

$$\int_{Z_\emptyset}|f(z)|^2\,dz = \int_K\int_{A/(A\cap H)} a^{-2\rho_Q}|f(kaH_\emptyset)|^2\,da\,dk.$$

By Fubini's theorem the function

$$A/A\cap H\ni a\mapsto a^{-\rho_Q}f(kaH_\emptyset)$$

is square integrable for almost every $k\in K$. We now apply the Plancherel theorem for the euclidean Fourier transform on $A/A\cap H$ to the inner integral and obtain

$$\int_{Z_\emptyset}|f(z)|^2\,dz = \int_K\int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*}\bigg|\int_{A/(A\cap H)} a^{\lambda-\rho_Q}f(kaH_\emptyset)\,da\bigg|^2\,d\lambda\,dk.$$

Since $M\subseteq K$, we have for every $\theta\in\mathcal{D}(G/H_\emptyset)$ and $a\in A$

$$\int_K\theta(kaH_\emptyset)\,dk = \int_K\int_{M/(M\cap H)}\theta(kmaH_\emptyset)\,dm\,dk.$$

It follows that

$$\int_{Z_\emptyset}|f(z)|^2\,dz = \int_K\int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*}\int_{M/(M\cap H)}\bigg|\int_{A/(A\cap H)} a^{\lambda-\rho_Q}f(kmaH_\emptyset)\,da\bigg|^2\,dm\,d\lambda\,dk.$$

By Fubini's theorem the function

$$M/(M\cap H)\ni m\mapsto \int_{A/(A\cap H)} a^{\lambda-\rho_Q} f(kmaH_\emptyset)\,da$$

is square integrable for almost every $k\in K$ and $\lambda\in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. For every finite dimensional representation σ of M we choose an orthonormal basis E_σ of $V_\sigma^{M\cap H}$. The set of equivalence classes of irreducible unitary representations of M we denote by $\widehat{M}$. We now apply the Peter-Weyl theorem for $M/M\cap H$ and obtain

$$\begin{aligned}\int_{M/(M\cap H)} &\left|\int_{A/(A\cap H)} a^{\lambda-\rho_Q} f(kmaH_\emptyset)\,da\right|^2 dm\\ &=\sum_{[\sigma]\in\widehat{M}}\dim(V_\sigma)\sum_{\eta\in E_\sigma}\left\|\int_{M/(M\cap H)}\int_{A/(A\cap H)} a^{\lambda-\rho_Q} f(kmaH_\emptyset)\sigma(m)\eta\,da\,dm\right\|_\sigma^2.\end{aligned}$$

In view of Lemma 4.1 and Corollary 4.3 we may replace M by M_Q, hence the right-hand side equals

$$\sum_{[\xi]\in\widehat{M}_{Q,\mathrm{fu}}}\dim(V_\xi)\sum_{\eta\in E_{\xi,e}}\left\|\int_{M_Q/(M_Q\cap H)}\int_{A/(A\cap H)} a^{\lambda-\rho_Q} f(kmaH_\emptyset)\xi(m)\eta\,da\,dm\right\|_\xi^2.$$

Here $E_{\xi,e}$ denotes a choice of an orthonormal basis of $V_\xi^{M_Q\cap H}$. By (8.3) in Lemma 8.3

$$\begin{aligned}\int_{Z_\emptyset}|f(z)|^2\,dz&=\sum_{[\xi]\in\widehat{M}_{Q,\mathrm{fu}}}\dim(V_\xi)\sum_{\eta\in E_{\xi,e}}\int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*}\int_K\big\|\mathscr{F}_\emptyset f(\xi:\lambda)\big(\iota_e\eta\big)(k)\big\|_\xi^2\,dk\,d\lambda\\ &=\sum_{[\xi]\in\widehat{M}_{Q,\mathrm{fu}}}\dim(V_\xi)\sum_{\eta\in E_{\xi,e}}\int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*}\big\|\mathscr{F}_\emptyset f(\xi:\lambda)\big(\iota_e\eta\big)\big\|_{\overline{Q},\xi,\lambda}^2\,d\lambda.\end{aligned}$$

Since $\mathfrak{N}$ is a set of representatives of $\mathcal{N}/\mathcal{Z}$ in $K\cap\mathcal{N}$ and $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ a fundamental domain for the action of $\mathcal{N}/\mathcal{Z}$ on $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$, the right-hand side equals the sum over $v\in\mathfrak{N}$ of

$$\sum_{[\xi]\in\widehat{M}_{Q,\mathrm{fu}}}\dim(V_{v^{-1}\cdot\xi})\sum_{\eta\in E_{v^{-1}\cdot\xi,e}}\int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+}\big\|\mathscr{F}_\emptyset f(v^{-1}\cdot\xi:\mathrm{Ad}^*(v^{-1})\lambda)\big(\iota_e\eta\big)\big\|_{\overline{Q},v^{-1}\cdot\xi,\mathrm{Ad}^*(v^{-1})\lambda}^2\,d\lambda.$$

Since $V_{v^{-1}\cdot\xi}^{M_Q\cap H}=V_\xi^{M_Q\cap vHv^{-1}}$, the set $E_{\xi,v}:=E_{v^{-1}\cdot\xi,e}$ is an orthonormal basis of $V_\xi^{M_Q\cap vHv^{-1}}$. Therefore, $\bigcup_{v\in\mathfrak{N}}E_{\xi,v}$ is an orthonormal basis of $V_\emptyset^*(\xi)$. We now apply (8.5). This yields

$$\begin{aligned}\int_{Z_\emptyset}|f(z)|^2\,dz&=\sum_{v\in\mathfrak{N}}\sum_{[\xi]\in\widehat{M}_{Q,\mathrm{fu}}}\dim(V_\xi)\sum_{\eta\in E_{\xi,v}}\int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+}\big\|\mathscr{F}_\emptyset f(\xi:\lambda)\big(\iota_v\eta\big)\big\|_{\overline{Q},\xi,\lambda}^2\,d\lambda\\ &=\sum_{[\xi]\in\widehat{M}_{Q,\mathrm{fu}}}\dim(V_\xi)\sum_{\eta\in E_\xi}\int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+}\big\|\mathscr{F}_\emptyset f(\xi:\lambda)\eta\big\|_{\overline{Q},\xi,\lambda}^2\,d\lambda\\ &=\sum_{[\xi]\in\widehat{M}_{Q,\mathrm{fu}}}\dim(V_\xi)\int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+}\big\|\mathscr{F}_\emptyset f(\xi:\lambda)\big\|_{\emptyset,\xi,\lambda}^2\,d\lambda.\end{aligned}$$

This proves (8.2) for $f \in \mathcal{D}(Z_\emptyset)$.

From (8.2) and the density of $\mathcal{D}(Z_\emptyset)$ in $L^2(Z_\emptyset)$ it follows that $f \mapsto \mathscr{F}_\emptyset f$ extends uniquely to a continuous linear operator (8.1) and the identity (8.2) holds for $f \in L^2(Z_\emptyset)$ as well. $\square$

8.3 Multiplicity spaces

We recall that $\widehat{M}_{Q,\mathrm{fu}}$ denotes the set of equivalence classes of finite dimensional unitary representations of M_Q and that $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ is a fundamental domain for the action of $\mathcal{N}$ on $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. Further, we recall that the Lebesgue measure on $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ is normalized by requiring that

$$\phi(e) = \int_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*} \int_{A/(A\cap H)} \phi(a)a^\lambda \, da\, d\lambda \qquad \Big(\phi \in \mathcal{D}\big(A/(A\cap H)\big)\Big).$$

Theorem 8.1 and [17, Theorem 11.1] have the following direct corollary.

Corollary 8.1. *For every $[\xi] \in \widehat{M}_{Q,\mathrm{fu}}$ there exists a measurable family of Hilbert spaces $\mathcal{M}_{\xi,\lambda}$, with family parameter $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ so that $L^2_{\mathrm{mc}}(Z)$ decomposes G-equivariantly as*

$$L^2_{\mathrm{mc}}(Z) \simeq \widehat{\bigoplus_{[\xi]\in\widehat{M}_{Q,\mathrm{fu}}}} \int^\oplus_{i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+} \mathcal{M}_{\xi,\lambda} \otimes \mathrm{Ind}_Q^G\big(\xi\otimes\lambda\otimes\mathbf{1}\big)\, d\lambda. \tag{8.1}$$

Each multiplicity space $\mathcal{M}_{\xi,\lambda}$ is as a vector space naturally identified with a subspace of the space of H-fixed functionals on $C^\infty(\overline{Q} : \xi : \lambda)$, and hence in view of the topological isomorphism (4.2) we may view $\mathcal{M}_{\xi,\lambda}$ as a subspace of $\mathcal{D}'(\overline{Q} : \xi : \lambda)^H$. The Theorems 6.1 and 7.1 now have the following corollary.

Corollary 8.2. *Let ξ be a finite dimensional unitary representation of M_Q. For almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ the multiplicity space $\mathcal{M}_{\xi,\lambda}$ is equal to $\mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ and the map*

$$\mu^\circ(\xi : \lambda) : V^*(\xi) \to \mathcal{M}_{\xi,\lambda}$$

is a linear isomorphism.

Proof. It suffices to prove that for almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ the dimensions of $\mathcal{M}_{\xi,\lambda}$ and $V^*(\xi)$ coincide.

For $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$, $\eta \in V^*(\xi)$ and $\phi \in \mathcal{D}(G, V_\xi)$, the function

$$Z \to \mathbb{C};\quad gH \mapsto \Big(R^\vee(g)\circ\mu^\circ(\xi:\lambda)(\eta)\Big)(\phi)$$

is in view of the identification (4.1) a generalized matrix coefficient for $\mathrm{Ind}_Q^G\big(\xi\otimes\lambda\otimes\mathbf{1}\big)$. Since the representation $\mathrm{Ind}_Q^G\big(\xi\otimes\lambda\otimes\mathbf{1}\big)$ is irreducible for almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$, such a generalized matrix coefficient does not vanishes for almost all λ if $\eta \neq 0$ and $\phi \neq 0$. By Theorem 7.1 all wave packets of generalized matrix coefficients are square integrable. For almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ the representation $\mathrm{Ind}_Q^G\big(\xi\otimes\lambda\otimes\mathbf{1}\big)$ is inequivalent to any representation $\mathrm{Ind}_Q^G\big(\xi\otimes\lambda'\otimes\mathbf{1}\big)$ with $\lambda' \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ and $\lambda \neq \lambda'$. It thus follows that the dimension of the multiplicity space $\mathcal{M}_{\xi,\lambda}$ is for almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ at least as large as the dimension of $V^*(\xi)$. On the other hand, the dimension of $\mathcal{M}_{\xi,\lambda}$ is bounded by the dimension of $\mathcal{D}'(\overline{Q} : \xi : \lambda)^H$, which is for almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_+$ equal to the dimension of $V^*(\xi)$ by Theorem 6.1. This proves the corollary. □

In view of Corollary 8.2 the multiplicity space $\mathcal{M}_{\xi,\lambda}$ can for almost every λ be identified with $V^*(\xi)$. The multiplicity spaces are Hilbert spaces and thus equipped with an inner product. We write $\langle\cdot,\cdot\rangle_{\mathrm{Pl},\xi,\lambda}$ for the inner product on $V^*(\xi)$ that is induced by the inner product on the multiplicity space $\mathcal{M}_{\xi,\lambda}$. To make the unitary equivalence (8.1) explicit, we have to determine $\langle\cdot,\cdot\rangle_{\mathrm{Pl},\xi,\lambda}$. This we do in Section 8.6 using a refinement of the Maaß-Selberg relations from [17, §9.4].

8.4 Constant Term

For the application of the Maaß-Selberg relations from [17, §9.4] in Section 8.6 we need a description of the constant term map from Section 7.6 for $\mathcal{F} = \overline{\mathcal{C}}$.

We recall the finite union of proper subspaces $\mathcal{S} \subsetneq (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ and the set of elements $\{x_\mathcal{O} : \mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_\mathfrak{h}}\}$ from Section 6.4. For a finite dimensional unitary representation ξ of M_Q, $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \setminus i\mathcal{S}$ and $\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}$ we write

$$\mathrm{CT}_\mathcal{O}(\xi : \lambda) : \mathcal{D}'(\overline{Q} : \xi : \lambda)^H \to \mathcal{D}'(\overline{Q} : \xi : \lambda)^{H_\emptyset}; \quad \mu \mapsto \mu_{x_\mathcal{O} H, \overline{\mathcal{C}}}$$

for the constant term map for the adapted point $z = x_\mathcal{O} H$ and face $\mathcal{F} = \overline{\mathcal{C}}$. Since we only consider $\mathcal{F} = \overline{\mathcal{C}}$ we have dropped the subscript $\mathcal{F}$.

Our description of the constant term map will be given in terms of the intertwining operators from Section 4.3. We recall the choice of a set of representatives $\mathfrak{N}$ of $\mathcal{N}/\mathcal{Z}$ in $\mathcal{N} \cap K$ and the space $V^*_\emptyset(\xi)$ from Section 8.2.

Proposition 8.1. *Let $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \backslash i\mathcal{S}$, let ξ be a finite dimensional unitary representation of M_Q and $\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}$. For every $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ the distribution*

$$\mathcal{A}\big(Q : \overline{Q} : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ \mathcal{I}^\circ_{v^{-1}}(\xi : \lambda)(\mu)$$

is smooth in the point $x_\mathcal{O}$. We have

$$\mathrm{CT}_\mathcal{O}(\xi : \lambda)\mu = \mu^\circ_\emptyset(\xi : \lambda)\eta,$$

where $\eta \in V^*_\emptyset(\xi)$ is given by

$$\eta_v = \mathrm{ev}_{x_\mathcal{O}} \circ \mathcal{A}\big(Q : \overline{Q} : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ \mathcal{I}^\circ_{v^{-1}}(\xi : \lambda)(\mu) \qquad (v \in \mathfrak{N}).$$

Proof. Let $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$. Since $\mathrm{CT}_\mathcal{O}(\xi : \lambda)\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^{H_\emptyset}$, it follows from Corollary 6.1 that there exists an $\eta \in V^*_\emptyset(\xi)$ so that $\mathrm{CT}_\mathcal{O}(\xi : \lambda)\mu = \mu^\circ_\emptyset(\xi : \lambda)\eta$. Moreover, η is given by

$$\eta_v = \mathrm{ev}_e \circ \mathcal{A}\big(Q : \overline{Q} : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ \mathcal{I}^\circ_{v^{-1}}(\xi : \lambda) \circ \mathrm{CT}_\mathcal{O}(\xi : \lambda)\mu \qquad (v \in \mathfrak{N}).$$

We set

$$\mu_v := \mathcal{A}\big(Q : \overline{Q} : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ \mathcal{I}^{\circ}_{v^{-1}}(\xi : \lambda)(\mu)$$

and

$$\mu_{v,\emptyset} := \mathcal{A}\big(Q : \overline{Q} : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ \mathcal{I}^{\circ}_{v^{-1}}(\xi : \lambda)(\mu) \circ \mathrm{CT}_{\mathcal{O}}(\xi : \lambda)(\mu).$$

It then suffices to prove that for every $v \in \mathfrak{N}$

$$\mathrm{ev}_{x_{\mathcal{O}}}(\mu_v) = \mathrm{ev}_e(\mu_{v,\emptyset}). \tag{8.1}$$

It follows from Theorem 7.1 that for every $X \in \mathcal{C}$ the limit

$$\lim_{t\to\infty} e^{-t\rho_Q(X)}\Big(R^\vee\big(\exp(tX)x_{\mathcal{O}}\big)\mu_v - R^\vee\big(\exp(tX)\big)\mu_{v,\emptyset}\Big)$$

exists and equals 0. Since μ_v is contained in $\mathcal{D}'\big(Q : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big)^H$, it is given by a smooth function on the open subset $\mathcal{O}$. Let $\chi = \mathrm{ev}_{x_{\mathcal{O}}}(\mu_v)$. Then

$$\mu_v(manx_{\mathcal{O}}h) = a^{-\lambda+\rho_Q}\xi^\vee(m)\chi \qquad \big(m \in M, a \in A, n \in N_P, h \in H\big).$$

By Lemma 6.1 there exists a $\nu \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ and a regular function $f_\chi : G \to V_\xi^*$ so that

$$f_\chi(manx_{\mathcal{O}}h) = a^\nu\xi^\vee(m)\chi \qquad (m \in M, a \in A, n \in N_P, h \in H).$$

Let $\nu_1, \ldots, \nu_r \in \Lambda$ be a basis of $(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$. Then there exist regular functions

$$f_1, \ldots, f_r : G \to \mathbb{R}$$

so that

$$f_j(manx_{\mathcal{O}}h) = a^{\nu_j} \qquad (1 \leq j \leq r, m \in M, a \in A, n \in N_P, h \in H).$$

Note that each f_j is real valued and thus f_j^2 is non-negative. Now

$$\mu_v\big|_{\mathcal{O}} = \Big(\prod_{j=1}^r \big(f_j^2\big)^{u_j} f_\chi\Big)\big|_{\mathcal{O}}$$

where $u_j \in \mathbb{C}$ is determined by

$$-\lambda + \rho_Q - \nu = 2\sum_{j=1}^r u_j\nu_j.$$

Let V be the span of $R(G)f_\chi$. Then V is finite dimensional and the restriction of R to V has lowest weight ν. Note that f_χ is an H-fixed vector in V. The limit of

$$e^{-t\nu(X)}R\big(\exp(tX)x_{\mathcal{O}}\big)f_\chi$$

for $t \to \infty$ exist and is a non-zero lowest weight vector in V. In fact,

$$\mathrm{ev}_e\Big(\lim_{t\to\infty} e^{-t\nu(X)} R\big(\exp(tX)x_{\mathcal{O}}\big)f_\chi\Big) = f_\chi(x_{\mathcal{O}}) = \chi.$$

Likewise, for every $1 \leq j \leq r$ the span V_j of $R(G)f_j$ is finite dimensional and the restriction of R to V_j has lowest weight ν_j. The limits of

$$e^{-t\nu_j(X)} R\big(\exp(tX)x_{\mathcal{O}}\big)f_j \qquad (1 \leq j \leq r)$$

for $t \to \infty$ exist, and

$$\mathrm{ev}_e\Big(\lim_{t\to\infty} e^{-t\nu_j(X)} R\big(\exp(tX)x_{\mathcal{O}}\big)f_j\Big) = f_j(x_{\mathcal{O}}) = 1 \qquad (1 \leq j \leq r).$$

It follows that

$$e^{-t\nu(X)} R^\vee\big(\exp(tX)x_{\mathcal{O}}\big)\mu_v$$

converges for $t \to \infty$ uniformly on a neighborhood of e in G, and the limit $\mu_{v,\emptyset}$ satisfies

$$\mathrm{ev}_e(\mu_{v,\emptyset}) = \mathrm{ev}_e\Big(\lim_{t\to\infty} e^{-t\nu(X)} R^\vee\big(\exp(tX)x_{\mathcal{O}}\big)\mu_v\Big) = \chi = \mathrm{ev}_{x_{\mathcal{O}}}(\mu_v).$$

This establishes (8.1). $\square$

In view of Theorem 6.1 and Corollary 6.1 there exists for every $\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}$ a unique linear map

$$\Gamma_{\mathcal{O}}(\xi : \lambda) : V^*(\xi) \to V^*_\emptyset(\xi)$$

so that the diagram

$$\begin{tikzcd}
\mathcal{D}'(\overline{Q} : \xi : \lambda)^H \arrow[r, "\mathrm{CT}_{\mathcal{O}}(\xi:\lambda)"] & \mathcal{D}'(\overline{Q} : \xi : \lambda)^{H_\emptyset} \\
V^*(\xi) \arrow[u, "\mu^\circ(\xi:\lambda)"] \arrow[r, "\Gamma_{\mathcal{O}}(\xi:\lambda)"] & V^*_\emptyset(\xi) \arrow[u, "\mu^\circ_\emptyset(\xi:\lambda)"']
\end{tikzcd}$$

commutes. We end this section with a description of this map $\Gamma_{\mathcal{O}}(\xi : \lambda)$ in terms of the B-matrices and the map $\beta(\xi : \lambda)$ from Section 6.6.

We recall the maps s_w for $w \in \mathcal{N}$ from (6.2).

Proposition 8.2. *Let $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*\backslash i\mathcal{S}$, let ξ be a finite dimensional unitary representation of M_Q and let $\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}$. Then for every $\eta \in V^*(\xi)$ and $v \in \mathfrak{N}$*

$$\Big(\Gamma_{\mathcal{O}}(\xi : \lambda)\eta\Big)_v = \frac{1}{\gamma(v\overline{Q}v^{-1} : \overline{Q} : \xi : \lambda)}\Big(\mathcal{B}_{v^{-1}}(Q : \xi : \lambda) \circ \beta(\xi : \lambda)^{-1}\eta\Big)_{\mathcal{O}}. \tag{8.2}$$

In particular, if $\eta \in V^*(\xi)$, $\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}$ and v_w is the representative in $K \cap \mathcal{N}$ of an element $w \in \mathcal{N}/\mathcal{W}$ from Section 6.4, then

$$\Big(\Gamma_{\mathcal{O}}(\xi : \lambda)\eta\Big)_{v_w} = \eta_{s_{v_w}(\mathcal{O})}. \tag{8.3}$$

Proof. Let $\eta \in V^*(\xi)$, $\mathcal{O} \in (P\backslash Z)_{\rm open}$ and $v \in \mathfrak{N}$. Then by Proposition 8.1

$$\Big(\Gamma_{\mathcal{O}}(\xi : \lambda)\eta\Big)_v = \mathrm{ev}_{x_{\mathcal{O}}} \circ \mathcal{A}\big(Q : \overline{Q} : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ \mathcal{I}^{\circ}_{v^{-1}}(\xi : \lambda) \circ \mu^{\circ}(\xi : \lambda)(\eta).$$

Using (6.1) and the identity

$$\begin{aligned}\mathcal{A}\big(Q : \overline{Q} : v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ \mathcal{I}^{\circ}_{v^{-1}}(\xi : \lambda) \circ \mathcal{A}\big(Q : \overline{Q} : \xi : \lambda\big)^{-1} \\ = \frac{1}{\gamma\big(v\overline{Q}v^{-1} : \overline{Q} : \xi : \lambda\big)} L^{\vee}(v^{-1}) \circ \mathcal{A}\big(vQv^{-1} : Q : \xi : \lambda\big) \\ = \frac{1}{\gamma\big(v\overline{Q}v^{-1} : \overline{Q} : \xi : \lambda\big)} I_{v^{-1}}(Q : \xi : \lambda),\end{aligned}$$

we find

$$\begin{aligned}&\Big(\Gamma_{\mathcal{O}}(\xi : \lambda)\eta\Big)_v \\ &\quad = \frac{1}{\gamma(v\overline{Q}v^{-1} : \overline{Q} : \xi : \lambda)} \mathrm{ev}_{x_{\mathcal{O}}} \circ I_{v^{-1}}(Q : \xi : \lambda) \circ \mu(Q : \xi : \lambda) \circ \beta(\xi : \lambda)^{-1}(\eta).\end{aligned}$$

By (6.1) we thus have

$$\begin{aligned}&\Big(\Gamma_{\mathcal{O}}(\xi : \lambda)\eta\Big)_v \\ &\quad = \frac{1}{\gamma(v\overline{Q}v^{-1} : \overline{Q} : \xi : \lambda)} \mathrm{ev}_{x_{\mathcal{O}}} \circ \mu(Q : \xi : \lambda) \circ \mathcal{B}_{v^{-1}}(Q : \xi : \lambda) \circ \beta(\xi : \lambda)^{-1}(\eta) \\ &\quad = \frac{1}{\gamma(v\overline{Q}v^{-1} : \overline{Q} : \xi : \lambda)} \Big(\mathcal{B}_{v^{-1}}(Q : \xi : \lambda) \circ \beta(\xi : \lambda)^{-1}(\eta)\Big)_{\mathcal{O}}.\end{aligned}$$

This proves (8.2). The identity (8.3) follows from (8.2) and the definition (6.3) of the function $\beta(\xi : \lambda)$. $\square$

For a finite dimensional unitary representation ξ of M_Q and $v \in \mathfrak{N}$ we define the space

$$V^*_{\emptyset,v}(\xi) := \bigoplus_{\mathcal{O} \in (P\backslash Z)_{\rm open}} (V^*_{\xi})^{M_Q \cap vHv^{-1}}. \tag{8.4}$$

We view $V^*_{\emptyset,v}(\xi)$ as a subspace of $\bigoplus_{\mathcal{O} \in (P\backslash Z)_{\rm open}} V^*_{\emptyset}(\xi)$. We now reorder the components of the constant term maps and thus define

$$\Gamma_v(\xi : \lambda) : V^*(\xi) \to V^*_{\emptyset,v}(\xi) \subseteq \bigoplus_{\mathcal{O} \in (P\backslash Z)_{\rm open}} V^*_{\emptyset}(\xi)$$

by setting

$$\Big(\Gamma_v(\xi : \lambda)\eta\Big)_{\mathcal{O}} := \mathrm{pr}_v \circ \Gamma_{\mathcal{O}}(\xi : \lambda) \qquad \big(\eta \in V^*(\xi), \mathcal{O} \in (P\backslash Z)_{\rm open}\big),$$

where

$$\mathrm{pr}_v : V^*_{\emptyset}(\xi) \to (V^*_{\xi})^{M_Q \cap vHv^{-1}}; \quad \eta \mapsto \eta_v. \tag{8.5}$$

Now Proposition 8.2 has the following corollary.

Corollary 8.3. *Let ξ be a finite dimensional unitary representation of M_Q and $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^* \setminus i\mathcal{S}$. Then for every $w \in \mathcal{N}/\mathcal{W}$*

$$\Gamma_{v_w}(\xi : \lambda)\eta = \big(\eta_{s_{v_w}(\mathcal{O})}\big)_{\mathcal{O}\in(P\backslash Z)_{\mathrm{open}}} \in V^*_{\emptyset,v_w}(\xi).$$

Proof. The identity is a reformulation of (8.3). □

8.5 Invariant differential operators

Let ξ be a finite dimensional unitary representation of M_Q. For $w \in \mathcal{N}/\mathcal{W}$ define the subspace of $V^*(\xi)$

$$V^*_w(\xi) := \bigoplus_{\mathcal{O}\in w\cdot(P\backslash Z)_{\mathrm{open}}} \big(V^*_\xi\big)^{M_Q\cap v_w H v_w^{-1}}. \tag{8.1}$$

We view $V^*_w(\xi)$ as a subspace of $V^*(\xi)$. In this section we show that the subspaces $\mu^\circ(\xi : \lambda)\big(V^*_w(\xi)\big)$ of $\mathcal{D}'(\overline{Q} : \xi : \lambda)^H$ for $w \in \mathcal{N}/\mathcal{W}$ are spectrally separated by the invariant differential operators on Z. Recall the maps $\iota_{\mathcal{O}}$ for $\mathcal{O} \in (P\backslash Z)_{\mathfrak{a}_{\mathfrak{h}}}$ from (6.1).

Proposition 8.1. *Let ξ be a finite dimensional unitary representation of M_Q. For every $w \in \mathcal{N}/\mathcal{W}$ and for $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ outside of a finite union of proper subspaces there exists a differential operator D_w in the center of $\mathbb{D}(Z)$ so that*

$$D_w \circ \mu^\circ(\xi : \lambda) \circ \iota_{\mathcal{O}} = \begin{cases} \mu^\circ(\xi : \lambda) \circ \iota_{\mathcal{O}} & (\mathcal{O} \in w\cdot(P\backslash Z)_{\mathrm{open}}),\\ 0 & (\mathcal{O} \notin w\cdot(P\backslash Z)_{\mathrm{open}}).\end{cases}$$

Before we prove the proposition we first give a corollary, which we will use in Section 8.6. By Corollary 8.2 the multiplicity spaces $\mathcal{M}_{\xi,\lambda}$ in (8.1) can for almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ be identified with $V^*(\xi)$ via the map $\mu^\circ(\xi : \lambda)$. Recall that $\langle\cdot,\cdot\rangle_{\mathrm{Pl},\xi,\lambda}$ denotes the inner product on $V^*(\xi)$ that is induced by the natural inner product on the multiplicity space $\mathcal{M}_{\xi,\lambda}$.

Corollary 8.2. *The decomposition*

$$V^*(\xi) = \bigoplus_{w\in\mathcal{N}/\mathcal{W}} V^*_w(\xi).$$

is for almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathrm{Pl},\xi,\lambda}$.

Proof. Every differential operator $D \in \mathbb{D}(Z)$ defines a operator on $L^2(Z)$ with domain $\mathcal{D}(Z)$. To every $D \in \mathbb{D}(Z)$ we can associate a formal adjoint D^* which is defined by

$$\int_Z D^*\phi(z)\overline{\psi(z)}\,dz = \int_Z \phi(z)\overline{D\psi(z)}\,dz \qquad \big(\phi,\psi \in \mathcal{D}(Z)\big).$$

It is easy to see that D^* is a G-invariant differential operators and thus is contained in $\mathbb{D}(Z)$. Furthermore, if D is contained in the center of $\mathbb{D}(Z)$, then for all $\phi,\psi \in \mathcal{D}(Z)$ and $D' \in \mathbb{D}(Z)$

$$\begin{aligned}\int_Z D'D^*\phi(z)\overline{\psi(z)}\,dz &= \int_Z \phi(z)\overline{DD'^*\psi(z)}\,dz = \int_Z \phi(z)\overline{D'^*D\psi(z)}\,dz\\ &= \int_Z D^*D'\phi(z)\overline{\psi(z)}\,dz,\end{aligned}$$

and hence D^* is contained in the center of $\mathbb{D}(Z)$ as well.

Since the differential operators in $\mathbb{D}(Z)$ commute with the regular representation of G, each $D \in \mathbb{D}(Z)$ induces for $\xi \in \widehat{M}_{Q,\mathrm{fu}}$ and $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ an operator $r(\xi : \lambda)(D)$ on the multiplicity space $\mathcal{M}_{\xi,\lambda} = V^*(\xi)$. The operator is given by

$$D \circ \mu^\circ(\xi : \lambda) = \mu^\circ(\xi : \lambda) \circ r(\xi : \lambda)(D) \qquad \big(D \in \mathbb{D}(Z)\big). \tag{8.2}$$

Furthermore, if † denotes hermitian conjugation with respect to $\langle \cdot, \cdot, \rangle_{\mathrm{Pl},\xi,\lambda}$, then

$$r(\xi : \lambda)(D^*) = \big(r(\xi : \lambda)(D)\big)^\dagger \qquad \big(D \in \mathbb{D}(Z)\big).$$

If D is contained in the center of $\mathbb{D}(Z)$, then $r(\xi, \lambda)(D)$ commutes with $r(\xi, \lambda)(D)^\dagger$, and hence $r(\xi, \lambda)(D)$ is normal. In particular, eigenspaces corresponding to different eigenvalues are orthogonal to each other. The assertion now follows from Proposition 8.1. $\square$

In the remainder of this section we give the proof of Proposition 8.1. Part of the proof is based on ideas of Delorme and Beuzart-Plessis, in particular our Lemma 8.5. We begin by recalling the Harish-Chandra homomorphism of Knop from [26]. For a smooth complex $G_\mathbb{C}$-variety $\mathcal{X}$ let $\mathcal{U}_\mathcal{X} = \mathcal{O}_\mathcal{X} \otimes \mathcal{U}(\mathfrak{g})$, where $\mathcal{O}_\mathcal{X}$ denotes the structure ring of $\mathcal{X}$. We equip $\mathcal{U}_\mathcal{X}$ with the structure of an algebra by equipping it with the multiplication determined by

$$(f \otimes \xi) \cdot (g \otimes \eta) = fg \otimes \xi\eta + f(\xi g) \otimes \eta \qquad (f, g \in \mathcal{O}_\mathcal{X}, \xi, \eta \in \mathfrak{g}).$$

Since elements of $\mathcal{U}(\mathfrak{g})$ naturally define differential operators on $\mathcal{X}$, we may view $\mathcal{U}_\mathcal{X}$ as a subsheaf of the sheaf of differential operators on $\mathcal{X}$. If $\tilde{\mathcal{X}}$ is pseudo-free (see the definition at the bottom of [26, page 259]) and $\phi : \tilde{\mathcal{X}} \to \mathcal{X}$ is an equivariant, birational, proper morphism, then we define $\overline{\mathcal{U}}_\mathcal{X} := \phi_* \mathcal{U}_{\tilde{\mathcal{X}}}$. The sheaf $\overline{\mathcal{U}}_\mathcal{X}$ does not depend on the choice of $\tilde{\mathcal{X}}$ or ϕ. We set $\mathcal{U}(\mathcal{X}) := H^0(\overline{\mathcal{X}}, \overline{\mathcal{U}}_{\overline{\mathcal{X}}})$, where $\overline{\mathcal{X}}$ is any smooth $G_\mathbb{C}$-equivariant completion of $\mathcal{X}$. The differential operators in $\mathcal{U}(\mathcal{X})$ are called completely regular. Finally, let $\mathcal{Z}(\mathcal{X}) := \mathcal{U}(\mathcal{X})^{G_\mathbb{C}}$. By the [26, Corollaries 7.6 & 9.2] the algebra $\mathcal{Z}(\mathcal{X})$ is equal to the center of $\mathcal{U}(\mathcal{X})$ and is contained in the center of the algebra $\mathbb{D}(\mathcal{X})$ of $G_\mathbb{C}$-invariant differential operators on $\mathcal{X}$.

We now consider $\mathcal{X} = Z_\mathbb{C}$. We first apply the local structure theorem, [28, Theorem 4.2], to $Z_\mathbb{C}$. Let B be a Borel subgroup of $G_\mathbb{C}$ that is contained in $P_\mathbb{C}$. The local structure theorem then yields a parabolic subgroup R of $G_\mathbb{C}$ and a Levi-decomposition $R = L_R N_R$ so that $R \cap H_\mathbb{C} = L_R \cap H_\mathbb{C}$ is a normal subgroup of L_R and $L_R/(L_R \cap H_\mathbb{C})$ is a torus. By [28, Lemma 9.3] L_R may be chosen so that $A_\mathbb{C} \subseteq L_R$. Let now $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{m} \cap \mathfrak{l}_R$. Then $\mathfrak{j} := \mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Without loss of generality we may assume that $\mathfrak{j}_\mathbb{C} = \mathfrak{a}_\mathbb{C} \oplus \mathfrak{t}_\mathbb{C}$ is contained in the Lie algebra of B.

Let V be a finite dimensional representation of $G_\mathbb{C}$ which contains a vector v whose stabilizer is equal to $H_\mathbb{C}$. Such a representation exists since $Z_\mathbb{C}$ is quasi-affine. See [17, Lemma 12.7]. We embed $Z_\mathbb{C}$ in V via the map

$$gH_\mathbb{C} \mapsto g \cdot v.$$

Let $\overline{Z}_{\mathbb{C}}$ be the Zariski closure of $Z_{\mathbb{C}}$. Note that $\overline{Z}_{\mathbb{C}}$ is affine. Let $X_1 \in \mathfrak{j}_{\mathbb{C}}/(\mathfrak{j}_{\mathbb{C}} \cap \mathfrak{h}_{\mathbb{C}})$ be a cocharacter so that $\alpha(X_1) > 0$ for every $\alpha \in \Pi_{Z_{\mathbb{C}}}$, where $\Pi_{Z_{\mathbb{C}}} \subseteq (\mathfrak{j}_{\mathbb{C}}/\mathfrak{j}_{\mathbb{C}} \cap \mathfrak{h}_{\mathbb{C}})^*$ is a simple system of the spherical roots system of $Z_{\mathbb{C}}$ that consists of roots whose restrictions to $\mathcal{C}$ is non-positive. We decompose $\mathbb{C}[\overline{Z}_{\mathbb{C}}]$ as

$$\mathbb{C}[\overline{Z}_{\mathbb{C}}] = \bigoplus_{\nu} \mathbb{C}[\overline{Z}_{\mathbb{C}}]_{(\nu)},$$

where the sum ranges over all highest weights ν occurring in $\mathbb{C}[\overline{Z}_{\mathbb{C}}]$ considered as a representation of $G_{\mathbb{C}}$, and $\mathbb{C}[\overline{Z}_{\mathbb{C}}]_{(\nu)}$ is the sum of all irreducible subrepresentations of $\mathbb{C}[\overline{Z}_{\mathbb{C}}]$ with highest weight ν. We provide $\mathbb{C}[\overline{Z}_{\mathbb{C}}]$ with a filtration induced by X_1 by setting

$$\mathbb{C}[\overline{Z}_{\mathbb{C}}]^{(n)} = \bigoplus_{\nu(X_1) \leq n} \mathbb{C}[\overline{Z}_{\mathbb{C}}]_{(\nu)} \qquad (n \in \mathbb{N}).$$

With this filtration we define the ring

$$R = \bigoplus_{n=0}^{\infty} \mathbb{C}[\overline{Z}_{\mathbb{C}}]^{(n)} t^n \subseteq \mathbb{C}[\overline{Z}_{\mathbb{C}}][t],$$

and set $\tilde{\mathcal{Y}}_{\text{hor}} = \operatorname{spec}(R) \subseteq V \times \mathbb{C}$. Let $\tilde{\Delta}_{\text{hor}} : \tilde{\mathcal{Y}}_{\text{hor}} \to \mathbb{C}$ be the projection onto the second component. Now $\tilde{\Delta}_{\text{hor}}$ is the regular $G_{\mathbb{C}} \times \mathbb{C}^{\times}$-equivariant map corresponding to the inclusion homomorphism $\mathbb{C}[t] \hookrightarrow R$. Let $\mathcal{S}$ be the horospherical type of $Z_{\mathbb{C}}$, see [25, p. 5]. By [25, Satz 2.2] there exists a $G_{\mathbb{C}}$-stable Zariski open subset of $\tilde{\Delta}_{\text{hor}}^{-1}(\{0\})$ that is $G_{\mathbb{C}}$-equivariantly isomorphic to $\tilde{\mathcal{V}} \times G_{\mathbb{C}}/S$, where $\tilde{\mathcal{V}}$ is a complex algebraic variety on which $G_{\mathbb{C}}$ acts trivially and S is a subgroup of $G_{\mathbb{C}}$ in the horospherical type $\mathcal{S}$. Let $W_1 \subseteq \tilde{\Delta}_{\text{hor}}^{-1}(\{0\})$ be the complement of this Zariski open subset and let $W_2 \subseteq \tilde{\mathcal{Y}}_{\text{hor}}$ be the Zariski closure of all $G_{\mathbb{C}}$ orbits in $\tilde{\Delta}_{\text{hor}}^{-1}(\mathbb{C}^{\times})$ of dimension strictly smaller than $\dim(Z_{\mathbb{C}})$. Note that W_2 is the Zariski closure of the $G_{\mathbb{C}} \times \mathbb{C}^{\times}$-orbits through $\overline{Z}_{\mathbb{C}} \setminus Z_{\mathbb{C}}$. Therefore, $W_2 \cap \tilde{\Delta}_{\text{hor}}^{-1}(\{0\})$ has dimension strictly smaller than the dimension of $Z_{\mathbb{C}}$. As the dimension of $\tilde{\Delta}_{\text{hor}}^{-1}(\{0\})$ equals the dimension of $Z_{\mathbb{C}}$, it follows that $W_2 \cap \tilde{\Delta}_{\text{hor}}^{-1}(\{0\})$ is a proper Zariski closed subset of $\tilde{\Delta}_{\text{hor}}^{-1}(\{0\})$ with Zariski dense complement. We now define $\mathcal{Y}_{\text{hor}} := \tilde{\mathcal{Y}}_{\text{hor}} \setminus (W_1 \cup W_2)$ and

$$\Delta_{\text{hor}} := \tilde{\Delta}_{\text{hor}}|_{\mathcal{Y}_{\text{hor}}} : \mathcal{Y}_{\text{hor}} \to \mathbb{C}.$$

The latter construction is called the horospherical degeneration of $Z_{\mathbb{C}}$. By construction the fibers of Δ_{hor} satisfy

$$\Delta_{\text{hor}}^{-1}(\{(t)\}) \simeq \begin{cases} Z_{\mathbb{C}} & (t \neq 0) \\ \mathcal{V} \times G_{\mathbb{C}}/S & (t = 0), \end{cases}$$

where $\mathcal{V}$ is a non-empty Zariski open subset of $\tilde{\mathcal{V}}$. As in the proof of [26, Theorem 6.5] we obtain a canonical map

$$i_{\text{hor}} : \mathcal{U}(Z_{\mathbb{C}}) = \mathcal{U}(Z_{\mathbb{C}} \times \mathbb{C}) = \mathcal{U}(\mathcal{Y}_{\text{hor}}) \to \mathcal{U}\big(\Delta_{\text{hor}}^{-1}(\{0\})\big) = \mathcal{U}(G_{\mathbb{C}}/S). \qquad (8.3)$$

The equalities follow from [26, Lemma 3.5] and the canonical map

$$\mathcal{U}(\mathcal{Y}_{\mathrm{hor}}) \to \mathcal{U}\big(\Delta_{\mathrm{hor}}^{-1}(\{0\})\big)$$

is obtained by applying [26, Lemma 3.1] to the injection $\iota_{\mathrm{hor}} : \Delta_{\mathrm{hor}}^{-1}(\{0\}) \hookrightarrow \mathcal{Y}_{\mathrm{hor}}$ in the same way as in [26, Corollary 3.4]. (In the proof of [26, Corollary 3.4] the spaces X and Y are erroneously interchanged.) From the fact that ι_{hor} is $G_{\mathbb{C}}$-equivariant it follows that i_{hor} maps $\mathcal{Z}(Z_{\mathbb{C}})$ to $\mathcal{Z}(G_{\mathbb{C}}/S)$. The latter is isomorphic to $\mathbb{C}\big[(\mathfrak{j}/\mathfrak{j}\cap\mathfrak{h})^*_{\mathbb{C}}\big]$, see the top of page 272 in [26]. Let ρ_B be the half-sum of the roots of $\mathrm{Lie}(B)$ in $\mathfrak{j}$. After a ρ-shift, we obtain Knop's Harish-Chandra homomorphism $\gamma_{Z_{\mathbb{C}}} : \mathcal{Z}(Z_{\mathbb{C}}) \to \mathbb{C}[\rho_B + (\mathfrak{j}/\mathfrak{j}\cap\mathfrak{h})^*_{\mathbb{C}}]$. Let $W_{Z,\mathbb{C}}$ be the little Weyl group of $Z_{\mathbb{C}}$, see [28, (9.13)], and $W_{\mathbb{C}}$ the Weyl group of the root system of $\mathfrak{g}_{\mathbb{C}}$ in $\mathfrak{j}$. Finally, let $\gamma : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}[\mathfrak{j}^*_{\mathbb{C}}]^{W_{\mathbb{C}}}$ be the Harish-Chandra isomorphism. By [26, Theorem 6.5] the map $\gamma_{Z_{\mathbb{C}}}$ is an isomorphism onto $\mathbb{C}[\rho_B + (\mathfrak{j}/\mathfrak{j}\cap\mathfrak{h})^*_{\mathbb{C}}]^{W_{Z,\mathbb{C}}}$ and the diagram

$$\begin{tikzcd}
\mathcal{Z}(\mathfrak{g}) \arrow[r, "\gamma"] \arrow[d] & \mathbb{C}[\mathfrak{j}^*_{\mathbb{C}}]^{W_{\mathbb{C}}} \arrow[d] \\
\mathcal{Z}(Z_{\mathbb{C}}) \arrow[r, "\gamma_{Z_{\mathbb{C}}}"] & \mathbb{C}\big[\rho_B + (\mathfrak{j}/\mathfrak{j}\cap\mathfrak{h})^*_{\mathbb{C}}\big]^{W_{Z,\mathbb{C}}}
\end{tikzcd} \tag{8.4}$$

commutes. Here the right vertical arrow is the restriction map.

We now wish to compare Knop's Harish-Chandra homomorphism for $Z_{\mathbb{C}}$ to that for a degeneration of $Z_{\mathbb{C}}$. The degeneration of $Z_{\mathbb{C}}$ is obtained by a degeneration to the normal bundle, as in [28, Remark 12.2.3]. Let $\mathcal{F}$ be a face of $\overline{\mathcal{C}}$. In this article we only need to consider $\mathcal{F} = \overline{\mathcal{C}}$, but for reference in future articles we treat here the general case.

Let $X_2 \in \mathfrak{a}$ be an element contained in the coweight-lattice so that $-X_2$ is in the interior of the face $\mathcal{F}$. We consider the partial toroidal $G_{\mathbb{C}} \times \mathbb{C}^\times$-compactification $\mathcal{Y}_{\mathcal{F}}$ of $Z_{\mathbb{C}} \times \mathbb{C}^\times$ attached to the fan with only one non trivial cone $\mathbb{R}_+(X_2, 1)$. By functoriality of toroidal compactifications there exists a $G_{\mathbb{C}} \times \mathbb{C}^\times$ equivariant map $\Delta_{\mathcal{F}} : \mathcal{Y}_{\mathcal{F}} \to \mathbb{C}$ whose restriction to $Z_{\mathbb{C}} \times \mathbb{C}^\times$ equals the projection $Z_{\mathbb{C}} \times \mathbb{C}^\times \to \mathbb{C}^\times$, i.e., the diagram

$$\begin{tikzcd}
Z_{\mathbb{C}} \times \mathbb{C}^\times \arrow[r, hook] \arrow[d] & \mathcal{Y}_{\mathcal{F}} \arrow[d, "\Delta_{\mathcal{F}}"] \\
\mathbb{C}^\times \arrow[r, hook] & \mathbb{C}
\end{tikzcd}$$

commutes. For $t \in \mathbb{C}^\times$, let $a_t = \exp\big(\log(t)X_2\big) \in A$. It follows from the local structure theorem [28, Theorem 4.2] that the map

$$\mathbb{C}^\times \to \mathcal{Y}_{\mathcal{F}}, \quad t \mapsto (a_t \cdot H_{\mathbb{C}}, t)$$

extends to a regular map $s : \mathbb{C} \to \mathcal{Y}_{\mathcal{F}}$.

For $t \in \mathbb{C}$, let $z_t = s(t)$. Then for $t \in \mathbb{C}^\times$ the stabilizer H_t of z_t in $G_{\mathbb{C}}$ is equal to $a_t H_{\mathbb{C}} a_t^{-1}$ and $\Delta_{\mathcal{F}}^{-1}(\{t\}) = G_{\mathbb{C}} \cdot z_t \simeq Z_{\mathbb{C}}$. Let

$$Z_{\mathcal{F},\mathbb{C}} := \Delta_{\mathcal{F}}^{-1}(\{0\}).$$

From the properties of compactifications it follows that $Z_{\mathcal{F},\mathbb{C}}$ is the $G_{\mathbb{C}} \times \mathbb{C}^\times$-orbit through z_0 and the subgroup $\{(a_t, t) : t \in \mathbb{C}^\times\}$ stabilizes z_0. Therefore, $Z_{\mathcal{F},\mathbb{C}} = G_{\mathbb{C}} \cdot z_0$. Moreover, the stabilizer of z_0 in $G_\mathbb{C}$ has Lie algebra $\mathfrak{h}_{\mathcal{F},\mathbb{C}}$. This implies that the space $Z_\mathcal{F} := Z_{eH,\mathcal{F}}$ from Section 7.3 is a finite cover of the real form $G \cdot z_0$ of $Z_{\mathcal{F},\mathbb{C}}$.

Lemma 8.3. *The map $\Delta_\mathcal{F} : \mathcal{Y}_\mathcal{F} \to \mathbb{C}$ is smooth.*

Proof. Let $\mathcal{A}$ be the closure of $A_\mathbb{C} \cdot H_\mathbb{C} \times \mathbb{C}^\times$ in $\mathcal{Y}_\mathcal{F}$. Then $\mathcal{A}$ is the product of the toroidal compactification of $A_\mathbb{C}/(A_\mathbb{C} \cap H_\mathbb{C})$ and $\mathbb{C}$. The restriction of $\Delta_\mathcal{F}$ to $\mathcal{A}$ equals the projection onto the second factor, and hence is smooth. By the local structure theorem [28, Theorem 4.2] the natural map $\Phi : U_{Q,\mathbb{C}} \times L_{Q,\mathbb{C}}/(L_Q \cap H_\mathbb{C}) \times_{A_\mathbb{C}} \mathcal{A} \to \mathcal{Y}_\mathcal{F}$ is an isomorphism onto an open subset $\mathcal{Y}_\mathcal{F}^\circ$ of $\mathcal{Y}_\mathcal{F}$ that intersects with every $G_\mathbb{C}$-orbit. Since

$$\Delta_\mathcal{F} \circ \Phi(u,l,a) = \Delta_\mathcal{F}(a) \qquad \big(u \in U_{Q,\mathbb{C}}, l \in L_{Q,\mathbb{C}}/(L_{Q,\mathbb{C}} \cap H_\mathbb{C}, a \in \mathcal{A}\big)$$

it follows that $\Delta_\mathcal{F}$ is smooth on the open subset $\mathcal{Y}_\mathcal{F}^\circ$. The claim that $\Delta_\mathcal{F}$ is smooth now follows as $\mathcal{Y}_\mathcal{F} = G_\mathbb{C} \cdot \mathcal{Y}_\mathcal{F}^\circ$ and $\Delta_\mathcal{F}$ is $G_\mathbb{C}$-equivariant. □

We move on to relate the Harish-Chandra homomorphisms for $Z_\mathbb{C}$ and $Z_{\mathcal{F},\mathbb{C}}$. As in the proof of [26, Theorem 6.5] the inclusion $\iota_\mathcal{F} : Z_{\mathcal{F},\mathbb{C}} \hookrightarrow \mathcal{Y}_\mathcal{F}$ induces a canonical map

$$i_\mathcal{F} : \mathcal{U}(Z_\mathbb{C}) = \mathcal{U}(Z_\mathbb{C} \times \mathbb{C}) = \mathcal{U}(\mathcal{Y}_\mathcal{F}) \to \mathcal{U}(Z_{\mathcal{F},\mathbb{C}}).$$

The first two equalities follow from [26, Lemma 3.5] and canonical $\mathcal{U}(\mathcal{Y}_\mathcal{F}) \to \mathcal{U}(Z_{\mathcal{F},\mathbb{C}})$ is obtained by applying [26, Lemma 3.1] to the injection $\iota_\mathcal{F}$ in the same way as in [26, Corollary 3.4]. Since $\iota_\mathcal{F}$ is $G_\mathbb{C}$-equivariant the map $i_\mathcal{F}$ maps $\mathcal{Z}(Z_\mathbb{C})$ to $\mathcal{Z}(Z_{\mathcal{F},\mathbb{C}})$.

Lemma 8.4. *The diagram*

$$\begin{tikzcd}
\mathcal{Z}(Z_\mathbb{C}) \arrow[r, "\gamma_{Z_\mathbb{C}}"] \arrow[d, hook, "i_\mathcal{F}"] & \mathbb{C}\big[\rho_B + (\mathfrak{j}/\mathfrak{j} \cap \mathfrak{h})^*_\mathbb{C}\big]^{W_{Z,\mathbb{C}}} \arrow[d, hook] \\
\mathcal{Z}(Z_{\mathcal{F},\mathbb{C}}) \arrow[r, "\gamma_{Z_{\mathcal{F},\mathbb{C}}}"] & \mathbb{C}\big[\rho_B + (\mathfrak{j}/\mathfrak{j} \cap \mathfrak{h})^*_\mathbb{C}\big]^{W_{Z_{\mathcal{F},\mathbb{C}}}}
\end{tikzcd}$$

commutes.

Proof. The strategy of the proof is to construct a horospherical degeneration $\mathcal{Y} \to \mathbb{C}$ of the $G_\mathbb{C} \times \mathbb{C}^\times$-variety $\mathcal{Y}_\mathcal{F}$. This yields a map $\Delta : \mathcal{Y} \to \mathbb{C}^2$ whose fibers are isomorphic to $Z_\mathbb{C}$, $Z_{\mathcal{F},\mathbb{C}}$ and $\mathcal{V} \times G_\mathbb{C}/S$, where $G_\mathbb{C}/S$ is horospherical and $\mathcal{V}$ is a variety on which $G_\mathbb{C}$ acts trivially. From the various inclusions of these fibers into $\mathcal{Y}$ we then obtain canonical maps between the rings of completely regular invariant differential operators on these spaces as in (8.3).

Let $X_1 \in \mathfrak{j}_\mathbb{C}/(\mathfrak{j}_\mathbb{C} \cap \mathfrak{h}_\mathbb{C})$ be the cocharacter we used to define $\mathcal{Y}_{\mathrm{hor}}$. Recall that V is a finite dimensional representation of $G_\mathbb{C}$ and $v \in V$ is a vector whose stabilizer is equal to $H_\mathbb{C}$. The variety $\mathcal{Y}_\mathcal{F}$ embeds into $V \times \mathbb{C}$. To be more precise, let $V = V_1 \oplus \cdots \oplus V_n$, where the V_n are irreducible subrepresentations. Let ν_i be the lowest weight of V_i and let $v = \sum_{i=1}^n v_i$ be the decomposition of v with $v_i \in V_i$. Then $\mathcal{Y}_\mathcal{F}$ equals the set of $G_\mathbb{C}$ orbits through

$$\Big\{\big(\sum_{i=1}^n t^{-\nu_i(X_2)} a_t \cdot v_i, t\big) : t \in \mathbb{C}\Big\}.$$

Let $\overline{\mathcal{Y}}_{\mathcal{F}}$ be the Zariski closure of $\mathcal{Y}_{\mathcal{F}}$. We note that $\overline{\mathcal{Y}}_{\mathcal{F}}$ is affine and $\mathcal{Y}_{\mathcal{F}}$ is Zariski open in $\overline{\mathcal{Y}}_{\mathcal{F}}$. We provide $\mathbb{C}[\overline{\mathcal{Y}}_{\mathcal{F}}]$ with a filtration induced by the cocharacters X_1 by setting

$$\mathbb{C}[\overline{\mathcal{Y}}_{\mathcal{F}}]^{(n)} = \bigoplus_{\nu(X_1)\leq n} \mathbb{C}[\overline{\mathcal{Y}}_{\mathcal{F}}]_{(\nu)} \qquad (n \in \mathbb{N}),$$

where $\mathbb{C}[\overline{\mathcal{Y}}_{\mathcal{F}}]_{(\nu)}$ is the sum of all irreducible subrepresentations of $\mathbb{C}[\overline{\mathcal{Y}}_{\mathcal{F}}]$ with highest weight ν. We define $\overline{\mathcal{Y}} = \operatorname{spec}(R)$, where

$$R = \bigoplus_{n=0}^{\infty} \mathbb{C}[\overline{\mathcal{Y}}_{\mathcal{F}}]^{(n)} t^n \subseteq \mathbb{C}[\overline{\mathcal{Y}}_{\mathcal{F}}][t].$$

Recall the regular map $\Delta_{\mathcal{F}} : \mathcal{Y}_{\mathcal{F}} \to \mathbb{C}$. Let $\phi_{\mathcal{F}}$ be the corresponding homomorphism $\mathbb{C}[s] \to \mathbb{C}[\overline{\mathcal{Y}}_{\mathcal{F}}]$. We define $\overline{\Delta} : \overline{\mathcal{Y}} \to \mathbb{C}^2$ to be the regular map corresponding to the homomorphism $\mathbb{C}[s,t] \to \mathbb{C}[\overline{\mathcal{Y}}_{\mathcal{F}}][t]$ that restricts to $\phi_{\mathcal{F}} : \mathbb{C}[s] \to \mathbb{C}[\overline{\mathcal{Y}}_{\mathcal{F}}] \hookrightarrow \mathbb{C}[\overline{\mathcal{Y}}_{\mathcal{F}}][t]$ on $\mathbb{C}[s] \subseteq \mathbb{C}[s,t]$ and to the inclusion homomorphism $\mathbb{C}[t] \hookrightarrow \mathbb{C}[\overline{\mathcal{Y}}_{\mathcal{F}}][t]$ on $\mathbb{C}[t] \subseteq \mathbb{C}[s,t]$. Note that $\overline{\Delta}$ is $G_{\mathbb{C}} \times (\mathbb{C}^{\times})^2$-equivariant. Let W_1 be the Zariski closure of all $G_{\mathbb{C}}$-orbits in $\overline{\Delta}^{-1}(\{\mathbb{C} \times \mathbb{C}^{\times}\})$ of dimension strictly smaller than $\dim(Z_{\mathbb{C}})$. We note that the complement of W_1 in $\tilde{\Delta}^{-1}(\mathbb{C} \times \{0\})$ is Zariski dense. Let $\mathcal{S}$ be the horospherical type of $Z_{\mathbb{C}}$, see [25, p. 5]. In view of [25, Satz 2.5] the varieties $Z_{\mathcal{F},\mathbb{C}}$ and $Z_{\mathbb{C}}$ have the same horospherical type. By [25, Satz 2.2] there exists a $G_{\mathbb{C}} \times \mathbb{C}^{\times}$-stable Zariski open subset of $\tilde{\Delta}^{-1}(\mathbb{C} \times \{0\})$ that is $G_{\mathbb{C}} \times \mathbb{C}^{\times}$-equivariantly isomorphic to $\tilde{\mathcal{V}} \times G_{\mathbb{C}}/S \times \mathbb{C}$, where $\tilde{\mathcal{V}}$ is a complex algebraic variety on which $G_{\mathbb{C}} \times \mathbb{C}^{\times}$ acts trivially and S is a subgroup of $G_{\mathbb{C}}$ in the horospherical type $\mathcal{S}$. Let W_2 be the complement of this Zariski open subset of $\tilde{\Delta}^{-1}(\mathbb{C} \times \{0\})$. Then we define $\mathcal{Y} := \overline{\mathcal{Y}} \setminus \big(W_1 \cup W_2\big)$ and $\Delta := \overline{\Delta}|_{\mathcal{Y}}$. The fibers of Δ satisfy by construction

$$\Delta^{-1}(\{(s,t)\}) \simeq \begin{cases} Z_{\mathbb{C}} & (s, t \neq 0) \\ Z_{\mathcal{F},\mathbb{C}} & (s = 0, t \neq 0) \\ \mathcal{V} \times G_{\mathbb{C}}/S & (s \in \mathbb{C}, t = 0), \end{cases}$$

where $\mathcal{V}$ is a non-empty Zariski open subset of $\tilde{\mathcal{V}}$.

We now consider the inclusions

$$\begin{aligned} &\iota_1 : \mathcal{V} \times G_{\mathbb{C}}/S = \Delta^{-1}(\{(0,0)\}) \hookrightarrow \Delta^{-1}(\{0\} \times \mathbb{C}), \\ &\iota_2 : \Delta^{-1}(\{0\} \times \mathbb{C}) \hookrightarrow \mathcal{Y}, \\ &\iota_{\mathrm{hor}} = \iota_2 \circ \iota_1 : \mathcal{V} \times G_{\mathbb{C}}/S = \Delta^{-1}(\{(0,0)\}) \hookrightarrow \mathcal{Y}. \end{aligned}$$

As in the proof of [26, Theorem 6.5] we may apply [26, Lemma 3.1] to these inclusions and use this in combination with [26, Lemma 3.5] to obtain canonical maps

$$i_1 : \mathcal{U}(Z_{\mathcal{F},\mathbb{C}}) \to \mathcal{U}(G_{\mathbb{C}}/S), \quad i_2 : \mathcal{U}(Z_{\mathbb{C}}) \to \mathcal{U}(Z_{\mathcal{F},\mathbb{C}}), \quad i_{\mathrm{hor}} : \mathcal{U}(Z_{\mathbb{C}}) \to \mathcal{U}(G_{\mathbb{C}}/S).$$

The maps $\gamma_{Z_{\mathbb{C}}}$ and $\gamma_{Z_{\mathcal{F},\mathbb{C}}}$ are obtained from i_{hor} and i_1 by restricting them to $\mathcal{Z}(Z_{\mathbb{C}})$ and $\mathcal{Z}(Z_{\mathcal{F},\mathbb{C}})$, respectively, and applying [26, Lemma 6.4.]. As $\iota_2 \circ \iota_1 = \iota_{\mathrm{hor}}$, the uniqueness of the maps obtained from [26, Lemma 3.1] implies that $i_1 \circ i_2 = i_{\mathrm{hor}}$. Moreover, it follows from [26, Lemma 3.5] that $i_2 = i_{\mathcal{F}}$. This proves the lemma. □

We now give an alternative description of the map $i_{\mathcal{F}}$. The right-action of G induces a natural isomorphism

$$\mathbb{D}(Z) \simeq \mathcal{U}(\mathfrak{g})_H/\mathcal{U}(\mathfrak{g})\mathfrak{h},$$

where

$$\mathcal{U}(\mathfrak{g})_H := \big\{u \in \mathcal{U}(\mathfrak{g}) : \mathrm{Ad}(h)u - u \in \mathcal{U}(\mathfrak{g})\mathfrak{h} \text{ for all } h \in H\big\}.$$

Likewise, we have

$$\mathbb{D}(Z_{\mathbb{C}}) \simeq \mathcal{U}(\mathfrak{g})_{H_{\mathbb{C}}}/\mathcal{U}(\mathfrak{g})\mathfrak{h},$$

where

$$\mathcal{U}(\mathfrak{g})_{H_{\mathbb{C}}} := \big\{u \in \mathcal{U}(\mathfrak{g}) : \mathrm{Ad}(h)u - u \in \mathcal{U}(\mathfrak{g})\mathfrak{h} \text{ for all } h \in H_{\mathbb{C}}\big\}.$$

Clearly, $\mathbb{D}(Z_{\mathbb{C}}) \subseteq \mathbb{D}(Z)$. Let $\mathfrak{b} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_Q$ and $\mathfrak{b}_H = (\mathfrak{m} \cap \mathfrak{h}) \oplus \mathfrak{a}_{\mathfrak{h}}$. Then $\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ is a two-sided ideal of $\mathcal{U}(\mathfrak{b})$. From the Poincaré-Birkhoff-Witt theorem we obtain the isomorphism

$$\mathcal{U}(\mathfrak{b})_H/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H \simeq \mathcal{U}(\mathfrak{g})_H/\mathcal{U}(\mathfrak{g})\mathfrak{h} \simeq \mathbb{D}(Z), \tag{8.5}$$

where $\mathcal{U}(\mathfrak{b})_H = \mathcal{U}(\mathfrak{b}) \cap \mathcal{U}(\mathfrak{g})_H$. We thus may view $\mathbb{D}(Z)$ as a subring of $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$. By the same reasoning we may view $\mathbb{D}(Z_{\mathcal{F}})$ as a subring of $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$. We recall from [18, Lemma 5.2] that the limit

$$\lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)D$$

exists (in $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$) for every $D \in \mathbb{D}(Z)$ and defines a G-invariant differential operator on $Z_{\mathcal{F}}$. The limit does not depend on the choice of X. Moreover, the map

$$\delta_{\mathcal{F}} : \mathbb{D}(Z) \to \mathbb{D}(Z_{\mathcal{F}}); \quad D \mapsto \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)D$$

is an injective algebra morphism. Since the complexification of $Z_{\mathcal{F}}$ is a finite cover of $Z_{\mathcal{F},\mathbb{C}}$, we may view $\mathbb{D}(Z_{\mathcal{F},\mathbb{C}})$ as a subalgebra of $\mathbb{D}(Z_{\mathcal{F}})$. The following lemma is due to Delorme and Beuzart-Plessis and was communicated to us by Delorme.

Lemma 8.5. *The image of $\mathcal{Z}(Z_{\mathbb{C}})$ under the map $\delta_{\mathcal{F}}$ is contained in $\mathcal{Z}(Z_{\mathcal{F},\mathbb{C}})$. Moreover,*

$$i_{\mathcal{F}}(u) = \delta_{\mathcal{F}}(u) \qquad \big(u \in \mathcal{Z}(Z_{\mathcal{F},\mathbb{C}})\big).$$

Proof. We claim that if $\mathcal{X}$ is a smooth complex $G_{\mathbb{C}}$-variety and $V \subseteq \mathcal{X}$ is a Zariski open and pseudo-free subvariety, then

$$\big(\overline{\mathcal{U}}_{\mathcal{X}}\big)\big|_V = \mathcal{U}_V. \tag{8.6}$$

To see this, let $\tilde{\mathcal{X}}$ be pseudo-free and $\phi : \tilde{\mathcal{X}} \to \mathcal{X}$ an equivariant, birational, proper morphism. Then $\overline{\mathcal{U}}_{\mathcal{X}} := \phi_*\mathcal{U}_{\tilde{\mathcal{X}}}$. The variety $\phi^{-1}(V)$ is an open in the pseudo-free variety $\tilde{\mathcal{X}}$ and therefore is pseudo-free. Therefore,

$$\big(\overline{\mathcal{U}}_{\mathcal{X}}\big)\big|_V = \big(\phi_*\mathcal{U}_{\tilde{X}}\big)\big|_V = \phi_*\mathcal{U}_{\phi^{-1}(v)} = \overline{\mathcal{U}}_V$$

The claim now follows as V is pseudo-free, and hence $\overline{\mathcal{U}}_V = \mathcal{U}_V$.

Let $\overline{\mathcal{Y}}_{\mathcal{F}}$ be any smooth $G_{\mathbb{C}}$-equivariant completion of $\mathcal{Y}_{\mathcal{F}}$. By [26, Lemma 2.3] the variety $\mathcal{Y}_{\mathcal{F}}$ is pseudo-free as all $G_{\mathbb{C}}$-orbits in $\mathcal{Y}_{\mathcal{F}}$ have the same dimension. We may thus apply (8.6) to $\mathcal{X} = \overline{\mathcal{Y}}_{\mathcal{F}}$ and $V = \mathcal{Y}_{\mathcal{F}}$. Thus we obtain

$$(\overline{\mathcal{U}}_{\overline{\mathcal{Y}}_{\mathcal{F}}})\big|_{\mathcal{Y}_{\mathcal{F}}} = \mathcal{U}_{\mathcal{Y}_{\mathcal{F}}}$$

Therefore,

$$\mathcal{U}(\mathcal{Y}_{\mathcal{F}}) = H^0(\overline{\mathcal{Y}}_{\mathcal{F}}, \overline{\mathcal{U}}_{\overline{\mathcal{Y}}_{\mathcal{F}}}) \subseteq \mathcal{U}_{\mathcal{Y}_{\mathcal{F}}}(\mathcal{Y}_{\mathcal{F}}).$$

In particular, every differential operator in $\mathcal{U}(\mathcal{Y}_{\mathcal{F}})$ restricts to a differential operator on any $G_{\mathbb{C}}$-orbit in $\mathcal{Y}_{\mathcal{F}}$.

Let u be an element in $\mathcal{Z}(Z_{\mathbb{C}})$. By [26, Lemma 3.5] the latter algebra is isomorphic to $\mathcal{U}(\mathcal{Y}_{\mathcal{F}})$. Let $v \in \mathcal{U}(\mathcal{Y}_{\mathcal{F}})$ be the image of u. Then $i_{\mathcal{F}}(u)$ is given by restricting v to $Z_{\mathcal{F},\mathbb{C}} \subseteq \mathcal{Y}_{\mathcal{F}}$. Let f be a regular function on an open affine set of $\mathcal{Y}_{\mathcal{F}}$ containing z_0. Now

$$(vf)(z_0) = \lim_{t \to 0}(vf)(z_t).$$

If $t \neq 0$, then $(vf)(z_t) = ((u \otimes 1)f)(z_t)$. In view of (8.5) there exists an element $w \in \mathcal{U}(\mathfrak{b})$ so that u is given by the right action of w. Hence

$$(vf)(z_{e^s}) = (L_{\mathrm{Ad}(\exp(sX))w} f)(z_{e^s})$$

Taking the limit for $s \to -\infty$ and using the definition of $\delta_{\mathcal{F}}$, we obtain

$$(vf)(z_0) = (L_{\delta_{\mathcal{F}}(u)} f)(z_0).$$

Using $G_{\mathbb{C}}$-invariance we deduce from this that the restriction of v to $\mathcal{Y}_{\mathcal{F}}$ is given by $L_{\delta_{\mathcal{F}}(v)}$. □

Proposition 8.6. *The diagram*

$$\begin{tikzcd}
\mathcal{Z}(\mathfrak{g}) \arrow[r, "\gamma"] \arrow[d] & \mathbb{C}[\mathfrak{j}_{\mathbb{C}}^*]^{W_{\mathbb{C}}} \arrow[d] \\
\mathcal{Z}(Z_{\mathbb{C}}) \arrow[r, "\gamma_{Z_{\mathbb{C}}}"] \arrow[d, hook, "\delta_{\mathcal{F}}"] & \mathbb{C}\big[\rho_B + (\mathfrak{j}/\mathfrak{j} \cap \mathfrak{h})_{\mathbb{C}}^*\big]^{W_{Z,\mathbb{C}}} \arrow[d, hook] \\
\mathcal{Z}(Z_{\mathcal{F},\mathbb{C}}) \arrow[r, "\gamma_{Z_{\mathcal{F},\mathbb{C}}}"] & \mathbb{C}\big[\rho_B + (\mathfrak{j}/\mathfrak{j} \cap \mathfrak{h})_{\mathbb{C}}^*\big]^{W_{Z_{\mathcal{F},\mathbb{C}}}}
\end{tikzcd}$$

commutes.

Proof. The assertion is a direct corollary of (8.4), Lemma 8.4 and Lemma 8.5. □

We return our attention to the face $\mathcal{F} = \overline{\mathcal{C}}$. Since the corresponding real spherical space is denoted by $Z_\emptyset$, we change notation and write $\delta_\emptyset$ instead of $\delta_{\overline{\mathcal{C}}}$.

For every $D \in \mathbb{D}(Z)$ the $\mathfrak{a}$-weights that occur in $D - \delta_\emptyset(D)$, considered as an element of $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$, are strictly negative on $\mathcal{C}$. When applied to $Z_\emptyset$ this leads to

$$\mathbb{D}(Z_\emptyset) \simeq S(\mathfrak{a})/S(\mathfrak{a})\mathfrak{a}_{\mathfrak{h}} \otimes \mathcal{U}(\mathfrak{m})_H/\mathcal{U}(\mathfrak{m})(\mathfrak{m} \cap \mathfrak{h}),$$

where $\mathcal{U}(\mathfrak{m})_H := \mathcal{U}(\mathfrak{m}) \cap \mathcal{U}(\mathfrak{g})_H$. We note that $\mathcal{U}(\mathfrak{m})_H/\mathcal{U}(\mathfrak{m})(\mathfrak{m} \cap \mathfrak{h})$ may be identified with the ring $\mathbb{D}\big(M/(M \cap H)\big)$ of M-invariant differential operators on $M/(M \cap H)$. In particular,

$$\mathcal{Z}(Z_{\emptyset,\mathbb{C}}) \subseteq S(\mathfrak{a})/S(\mathfrak{a})\mathfrak{a}_{\mathfrak{h}} \otimes \mathbb{D}\big(M/(M \cap H)\big).$$

The little Weyl group $W_{Z_{\emptyset,\mathbb{C}}}$ of $Z_{\emptyset,\mathbb{C}}$ acts trivially on the subspace $(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$ of $(\mathfrak{j}/\mathfrak{j} \cap \mathfrak{h})^*_{\mathbb{C}}$ and is isomorphic to the little Weyl group $W_{Z,M}$ of $M_{\mathbb{C}}/(M_{\mathbb{C}} \cap H_{\mathbb{C}})$.

Using that $\gamma_{Z_{\mathbb{C}}}$ is an isomorphism, we see that the set of characters of $\mathcal{Z}(Z_{\mathbb{C}})$ is in bijection with $\big(\rho_B + (\mathfrak{j}/\mathfrak{j} \cap \mathfrak{h})^*_{\mathbb{C}}\big)/W_{Z,\mathbb{C}}$. Likewise, the set of characters of $\mathcal{Z}(Z_{\emptyset,\mathbb{C}})$ is in bijection with $\big(\rho_B + (\mathfrak{j}/\mathfrak{j} \cap \mathfrak{h})^*_{\mathbb{C}}\big)/W_{Z_{\emptyset},\mathbb{C}}$.

Lemma 8.7. *Let ξ be a finite dimensional unitary representation of M_Q and let $\lambda \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_{\mathbb{C}}$ with $\operatorname{Im}\lambda \notin \mathcal{S}$. Let further $v \in \mathfrak{N}$. The image of $\mu^\circ_\emptyset(\xi : \lambda) \circ \iota_v$ is stable under the action of $\mathbb{D}(Z_\emptyset)$. Moreover, if the $\mathcal{Z}(Z_{\emptyset,\mathbb{C}})$-character $W_{Z_\emptyset,\mathbb{C}} \cdot \nu \in \big(\rho_B + (\mathfrak{j}/\mathfrak{j} \cap \mathfrak{h})^*_{\mathbb{C}}\big)/W_{Z_\emptyset,\mathbb{C}}$ occurs in the spectrum of the action of $\mathcal{Z}(Z_{\emptyset,\mathbb{C}})$ on the image of $\mu^\circ_\emptyset(\xi : \lambda) \circ \iota_v$, then*

$$\operatorname{Im}\nu\big|_{\mathfrak{a}} = -\operatorname{Ad}(v^{-1})\lambda,$$
$$\nu\big|_{\mathfrak{t}} \in i\mathfrak{t}^*/W_{Z,M}.$$

Proof. Let $X \in \mathfrak{a}$. The corresponding invariant differential operator $D_X = R^\vee(X)$ acts in view of (6.6) on the image of $\mu^\circ_\emptyset(\xi : \lambda) \circ \iota_v$ by the scalar $-\operatorname{Ad}(v^{-1})\lambda(X) + \rho_Q(X)$. Since the operators D_X with $X \in \mathfrak{a}$ are contained in the center of $\mathbb{D}(Z_\emptyset)$, it follows that action of $\mathbb{D}(Z_\emptyset)$ preserves the image of $\mu^\circ_\emptyset(\xi : \lambda) \circ \iota_v$ if λ is sufficiently regular. By meromorphic continuation the same then holds for all λ.

Assume that the $\mathcal{Z}(Z_{\emptyset,\mathbb{C}})$-character $W_{Z_\emptyset,\mathbb{C}} \cdot \nu \in \big(\rho_B + (\mathfrak{j}/\mathfrak{j} \cap \mathfrak{h})^*_{\mathbb{C}}\big)/W_{Z_\emptyset,\mathbb{C}}$ occurs in the spectrum of the action of $\mathcal{Z}(Z_{\emptyset,\mathbb{C}})$ on the image of $\mu^\circ_\emptyset(\xi : \lambda) \circ \iota_v$. Since

$$\operatorname{Im}\big(\gamma_{Z_\emptyset}(D_X)\big)(\nu) = \operatorname{Im}\nu(X)$$

it follows that $\operatorname{Im}\nu|_{\mathfrak{a}} = -\operatorname{Ad}(v^{-1})\lambda$.

From the explicit formula (6.8) for $\mu^\circ(\xi : \lambda) \circ \iota_v$ for sufficiently anti-dominant λ and by using meromorphic continuation one easily sees that $\mathcal{Z}(\mathfrak{m}) \subseteq \mathbb{D}\big(M/(M \cap H)\big)$ acts on the image of $\mu^\circ \circ \iota_v$ by the infinitesimal character of the restriction $(v^{-1} \cdot \xi)|_M$ of $v^{-1} \cdot \xi$ to M. Since this infinitesimal character is real, it follows that the restriction of ν to $\mathfrak{t}$ is contained $i\mathfrak{t}^*/W_{Z,M}$. □

The final ingredient for the proof of Proposition 8.1 is a relation between the constant term and the map $\delta_\emptyset$.

Lemma 8.8. *Let $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^* \setminus i\mathcal{S}$ and let ξ be a finite dimensional unitary representation of M_Q. Then*

$$\operatorname{CT}_{\mathcal{O}}(\xi : \lambda) \circ D = \delta_\emptyset(D) \circ \operatorname{CT}_{\mathcal{O}}(\xi : \lambda) \qquad \big(D \in \mathbb{D}(Z), \mathcal{O} \in (P\backslash Z)_{\text{open}}\big).$$

Proof. After replacing H by $x_{\mathcal{O}} H x_{\mathcal{O}}^{-1}$ we may assume that $\mathcal{O} = PH$. Let $D \in \mathbb{D}(Z)$ and $\mu \in \mathcal{D}'(\overline{Q} : \xi : \lambda)^H$. The $\mathfrak{a}$-weights that occur in $D - \delta_\emptyset(D)$, considered as an element of $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$, are strictly negative on $\mathcal{C}$. Therefore, for a fixed $X \in \mathcal{C}$ there exists an $\epsilon > 0$ and a $u \in \mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ whose weights are non-positive on $\mathcal{C}$ so that

$$\mathrm{Ad}\big(\exp(tX)\big)\Big(D - \delta_\emptyset(D)\Big) = e^{-\epsilon t}\,\mathrm{Ad}\big(\exp(tX)\big)u \qquad (t > 0).$$

It follows from Theorem 7.2 that

$$\lim_{t\to\infty} e^{-t\rho_Q(X)-\epsilon t} R^\vee\big(\exp(tX)\big)\mu = 0.$$

Since $\mathrm{Ad}\big(\exp(tX)\big)u$ converges for $t \to \infty$, it follows that

$$\lim_{t\to\infty} e^{-t\rho_Q(X)-\epsilon t} R^\vee\big(\exp(tX)\big)R^\vee(u)\mu = 0. \tag{8.7}$$

Using that X centralizes $\delta_\emptyset(D)$, we obtain

$$\begin{aligned} &e^{-t\rho_Q(X)}\Big(R^\vee\big(\exp(tX)\big)D\mu - R^\vee\big(\exp(tX)\big)\delta_\emptyset(D)\mu_\emptyset\Big)\\ &\quad= e^{-t\rho_Q(X)-\epsilon t}R^\vee\big(\exp(tX)\big)R^\vee(u)\mu\\ &\qquad+ e^{-t\rho_Q(X)}\delta_\emptyset(D)\Big(R^\vee\big(\exp(tX)\big)\mu - R^\vee\big(\exp(tX)\big)\mu_\emptyset\Big). \end{aligned}$$

In view of (8.7) and (7.1) in Theorem 7.1 the right-hand side converges to 0 for $t \to \infty$. By [18, Lemma 6.5] this identifies $\delta_\emptyset(D)\mathrm{CT}_\mathcal{O}(\xi : \lambda)(\mu)$ as the constant term of $D\mu$. $\square$

Recall the maps pr_v from (8.5). From Corollary 6.1 it is easily seen that for every $v \in \mathfrak{N}$ there exists an element $u \in S(\mathfrak{a})/S(\mathfrak{a})\mathfrak{a}_\mathfrak{h} \subseteq \mathbb{D}(Z_\emptyset)$ so that the diagram

$$\begin{tikzcd} \mathcal{D}'(\overline{Q} : \xi : \lambda)^{H_\emptyset} \arrow[r, "R^\vee(u)"] & \mathcal{D}'(\overline{Q} : \xi : \lambda)^{H_\emptyset} \\ V_\emptyset^*(\xi) \arrow[u, "\mu_\emptyset^\circ(\xi:\lambda)"] \arrow[r, "\mathrm{pr}_v"] & (V_\xi^*)^{M_Q\cap vHv^{-1}} \arrow[u, "\mu_\emptyset^\circ(\xi:\lambda)\circ\iota_v"'] \end{tikzcd}$$

commutes. Note that $R^\vee(u)$ is contained in the center of $\mathbb{D}(Z_\emptyset)$. For $v \in \mathfrak{N}$ we define a map $r_{\emptyset,v}(\xi : \lambda) : \mathbb{D}(Z_\emptyset) \to \mathrm{End}\big(V_{\emptyset,v}^*(\xi)\big)$ similar to (8.2) by requiring that the identity

$$D\mu_\emptyset^\circ(\xi : \lambda)(\eta_\mathcal{O}) = \mu_\emptyset^\circ(\xi : \lambda)\Big(\big(r_{\emptyset,v}(\xi : \lambda)(D)\eta\big)_\mathcal{O}\Big)$$

holds for every $D \in \mathbb{D}(Z_\emptyset)$, $\eta \in V_{\emptyset,v}^*(\xi)$ and $\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}$. As before $V_{\emptyset,v}^*(\xi)$ is considered here to be a subspace of $\bigoplus_{\mathcal{O}\in(P\backslash Z)_{\mathrm{open}}} V_\emptyset^*(\xi)$. Now Lemma 8.8 has the following immediate corollary.

Corollary 8.9. *Let ξ be a finite dimensional unitary representation of M_Q and $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \setminus i\mathcal{S}$. Then for every $D \in \mathbb{D}(Z)$ and $v \in \mathfrak{N}$*

$$\Gamma_v(\xi : \lambda) \circ r(\xi : \lambda)(D) = r_{\emptyset,v}(\xi : \lambda)\big(\delta_\emptyset(D)\big) \circ \Gamma_v(\xi : \lambda).$$

Proof of Proposition 8.1. We only consider $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ whose stabilizer in $\mathcal{N}$ is equal to $\mathcal{Z}$ and for which the implication for $v \in N_{G_\mathbb{C}}(\mathfrak{j}/\mathfrak{j}\cap\mathfrak{h})$

$$\mathrm{Ad}^*(v)\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \Rightarrow v \in N_{G_\mathbb{C}}\big((\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\big) \cap N_{G_\mathbb{C}}\big((\mathfrak{t}/\mathfrak{t}\cap\mathfrak{h})^*\big)$$

holds. It suffices to prove the proposition only for these λ as all elements $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ outside of a finite union of proper subspaces have these properties.

We claim that the $W_{Z,\mathbb{C}}$-orbit $W_{Z,\mathbb{C}}\cdot\lambda$ does not contain the points $\mathrm{Ad}^*(v_w^{-1})\lambda$ for $w \neq e\mathcal{W} \in \mathcal{N}/\mathcal{W}$. To prove the claim we assume that there exists a $v \in W_{Z,\mathbb{C}}$ so that $v\cdot\lambda = \mathrm{Ad}^*(v_w^{-1})\lambda$ for some $w \in \mathcal{N}/\mathcal{W}$. We will prove the claim by showing that $w = e\mathcal{W}$.

The assumption on λ guarantees that v normalizes $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$. We write $N_{W_{Z,\mathbb{C}}}(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})$ and $Z_{W_{Z,\mathbb{C}}}(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})$ for the normalizer and centralizer, respectively, of $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$ in $W_{Z,\mathbb{C}}$. By [28, Theorem 9.5] the little Weyl group W_Z of Z is related to the little Weyl group $W_{Z,\mathbb{C}}$ of $Z_\mathbb{C}$ by

$$W_Z = N_{W_{Z,\mathbb{C}}}(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})/Z_{W_{Z,\mathbb{C}}}(\mathfrak{a}/\mathfrak{a}_\mathfrak{h}).$$

This identity is to be considered as an identity of finite reflection groups on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$. It follows that $v\cdot\lambda \in W_Z\cdot\lambda = \mathrm{Ad}^*(\mathcal{W})\lambda$. Since the stabilizer of λ in $\mathcal{N}$ is by assumption equal to $\mathcal{Z}$, it follows that $v_w \in \mathcal{W}$, and hence $w = e\mathcal{W}$. This proves the claim.

Let $v \in \mathcal{N}$. After replacing λ by $-\mathrm{Ad}^*(v^{-1})\lambda$ we may conclude from the claim that the $W_{Z,\mathbb{C}}$-orbit through $-\mathrm{Ad}^*(v^{-1})\lambda$ is for every $v' \in \mathcal{N}\setminus v\mathcal{W}$ disjunct from the $W_{Z,\mathbb{C}}$-orbit through $-\mathrm{Ad}^*(v'^{-1})\lambda$.

In view of Lemma 8.7 there exist $\nu_1,\dots,\nu_r \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \oplus i(\mathfrak{t}/\mathfrak{t}\cap\mathfrak{h})^*$ so that the $\mathcal{Z}(Z_{\emptyset,\mathbb{C}})$-characters occurring in $\mathcal{D}'(\overline{Q}:\xi:\lambda)^{H_\emptyset}$ are given by

$$W_{Z,\mathbb{C}}\cdot\big(-\mathrm{Ad}(v_w^{-1})\lambda+\nu_j\big) \qquad \big(w\in\mathcal{N}/\mathcal{W}, 1\leq j\leq r\big).$$

The real subspaces $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\oplus i(\mathfrak{t}/\mathfrak{t}\cap\mathfrak{h})^*$ and $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\oplus(\mathfrak{t}/\mathfrak{t}\cap\mathfrak{h})^*$ of $(\mathfrak{j}/\mathfrak{j}\cap\mathfrak{h})^*_\mathbb{C}$ are stable under the action of $N_{G_\mathbb{C}}(\mathfrak{j}/\mathfrak{j}\cap\mathfrak{h})$. Therefore, the $W_{Z,\mathbb{C}}$-orbits through $-\mathrm{Ad}^*(v^{-1})\lambda+\nu_j$ for $1\leq j\leq r$ are disjunct from the $W_{Z,\mathbb{C}}$-orbits through $-\mathrm{Ad}^*(v'^{-1})\lambda+\nu_j$ for $1\leq j\leq r$ and $v'\in\mathcal{N}\setminus v\mathcal{W}$.

Let now $w\in\mathcal{N}/\mathcal{W}$. It follows that there exists a polynomial

$$p_w \in \mathbb{C}\big[\rho_B+(\mathfrak{j}/\mathfrak{j}\cap\mathfrak{h})^*_\mathbb{C}\big]^{W_{Z,\mathbb{C}}}$$

so that

$$\begin{aligned} p_w(-\mathrm{Ad}^*(v^{-1})\lambda+\nu_j) &= 1, &&\big(1\leq j\leq r, v\in v_w\mathcal{W}\big)\\ p_w\big(-\mathrm{Ad}^*(v^{-1})\lambda+\nu_j\big) &= 0 &&\big(1\leq j\leq r, v\in\mathcal{N}\setminus v_w\mathcal{W}\big). \end{aligned}$$

Let $D_w := \gamma_Z^{-1}\big(p_w(\lambda)\big)$. We claim that the differential operator D has the desired properties.

To prove the claim, let $v\in\mathfrak{N}$. Proposition 8.6, Corollary 8.9 and the construction of D guarantee that

$$\begin{aligned} &\Gamma_v(\xi:\lambda)\circ r(\xi:\lambda)(D_w)\\ &\qquad = r_{\emptyset,v}(\xi:\lambda)\big(\delta(D_w)\big)\circ\Gamma_v(\xi:\lambda) = \begin{cases}\Gamma_v(\xi:\lambda) & (v\mathcal{W}=w),\\ 0 & (v\mathcal{W}\neq w).\end{cases} \end{aligned} \tag{8.8}$$

We now substitute $v = v_w$ with $w \in \mathcal{N}/\mathcal{W}$ and apply Corollary 8.3. We then have for $\eta \in V^*(\xi)$

$$\Big(r(\xi : \lambda)(D_w)\eta\Big)_{\mathcal{O}} = \begin{cases} \eta_{\mathcal{O}} & \big(\mathcal{O} \in w \cdot (P\backslash Z)_{\mathrm{open}}\big), \\ 0 & \big(\mathcal{O} \notin w \cdot (P\backslash Z)_{\mathrm{open}}\big). \end{cases}$$

The proposition now follows by applying $\mu^\circ(\xi : \lambda)$ to both sides and using the identity $\mu^\circ(\xi : \lambda) \circ r(\xi : \lambda)(D_w) = D_w \circ \mu^\circ(\xi : \lambda)$. $\square$

8.6 Plancherel decomposition of $L^2_{\mathrm{mc}}(Z)$

We now come to the main theorem of this article.

For a finite dimensional unitary representation ξ of M_Q and $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^* \setminus i\mathcal{S}$ we define the Fourier transform

$$\mathscr{F}f(\xi : \lambda) \in \mathrm{Hom}_{\mathbb{C}}\big(V^*(\xi^\vee), C^\infty(\overline{Q} : \xi : \lambda)\big) = V^*(\xi) \otimes C^\infty(\overline{Q} : \xi : \lambda)$$

of a function $f \in \mathcal{D}(Z)$ by

$$\mathscr{F}f(\xi : \lambda)\eta = \int_Z f(gH) R^\vee(g)\Big(\mu^\circ(\xi^\vee : -\lambda)\eta\Big)\, dgH.$$

Let $\langle \cdot, \cdot \rangle_{\xi^\vee}$ be the inner product on $V^*(\xi^\vee)$ induced by the inner product on $V_{\xi^\vee}$, and let $\langle \cdot, \cdot \rangle_{\xi,\lambda}$ be the inner product on $V^*(\xi) \otimes \mathrm{Ind}_{\overline{Q}}^G(\xi \otimes \lambda \otimes \mathbf{1})$ induced by the inner products $\langle \cdot, \cdot \rangle_\xi$ and $\langle \cdot, \cdot \rangle_{\overline{Q},\xi,\lambda}$ on $V^*(\xi)$ and $\mathrm{Ind}_{\overline{Q}}^G(\xi \otimes \lambda \otimes \mathbf{1})$, respectively. We then have the following description of the Plancherel decomposition of $L^2_{\mathrm{mc}}(Z)$.

Theorem 8.1. *The Fourier transform $f \mapsto \mathscr{F}f$ extends to a continuous linear operator*

$$L^2(Z) \to \widehat{\bigoplus_{\xi \in \widehat{M}_{Q,\mathrm{fu}}}} \int_{i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_+}^{\oplus} V^*(\xi) \otimes \mathrm{Ind}_{\overline{Q}}^G(\xi \otimes \lambda \otimes \mathbf{1})\, d\lambda.$$

Moreover, for every $f_1, f_2 \in L^2_{\mathrm{mc}}(Z)$ we have

$$\int_Z f_1(z)\overline{f_2(z)}\, dz = \sum_{[\xi] \in \widehat{M}_{Q,\mathrm{fu}}} \dim(V_\xi) \int_{i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*_+} \Big\langle \mathscr{F}f_1(\xi : \lambda), \mathscr{F}f_2(\xi : \lambda)\Big\rangle_{\xi,\lambda}\, d\lambda.$$

Proof. By Corollary 8.2 the multiplicity space $\mathcal{M}_{\xi,\lambda}$ is isomorphic to $V^*(\xi)$ for all $\xi \in \widehat{M}_{Q,\mathrm{fu}}$ and almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$. In view of (8.1) it therefore suffices to show that for almost all $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ we have the equality

$$\langle \cdot, \cdot \rangle_{\mathrm{Pl},\xi,\lambda} = \dim(V_\xi)\langle \cdot, \cdot \rangle_\xi \tag{8.1}$$

of inner products on $V^*(\xi)$. To prove this identity we will use Theorem 8.1, Corollary 8.2 and the Maaß-Selberg relations from [17, Theorem 9.6].

We fix a finite dimensional unitary representation ξ of M_Q. It follows from Theorem 8.1 that the multiplicity space for the representation $\mathrm{Ind}_{\overline{Q}}^G\big(\xi \otimes \lambda \otimes \mathbf{1}\big)$ in the Plancherel decomposition of $L^2(Z_\emptyset)$ is for almost all $\lambda \in i(\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ equal to $V^*_\emptyset(\xi)$. Moreover, the inner product on the multiplicity spaces induced by the Plancherel decomposition is the inner product $\dim(V_\xi)\langle\cdot,\cdot\rangle_{\emptyset,\xi}$, where $\langle\cdot,\cdot\rangle_{\emptyset,\xi}$ is the inner product on $V^*_\emptyset(\xi)$ induced from the natural M_Q-invariant inner product on V_ξ. We recall the spaces $V^*_{\emptyset,v_w}(\xi)$ from (8.4). We view these spaces as subspaces of $V^*_\emptyset(\xi)$. The inner product $\langle\cdot,\cdot\rangle_{v_w,\xi}$ on $V^*_{\emptyset,v_w}(\xi)$ obtained by restriction equals the inner-product induced by the natural inner product on V_ξ.

By Corollary 8.3 the kernel of $\Gamma_{v_w}(\xi : \lambda)$ equals the direct sum of the subspaces $V^*_{w'}(\xi)$ with $w' \in \mathcal{N}/\mathcal{W}$, $w' \neq w$. In view of Corollary 8.2 the latter equals the orthocomplement of $V^*_w(\xi)$ with respect to $\langle\cdot,\cdot\rangle_{\mathrm{Pl},\xi,\lambda}$. Let $\Gamma_{v_w}(\xi : \lambda)^\dagger$ be the dual map of $\Gamma_{v_w}(\xi : \lambda)$ with respect to the inner products $\dim(V_\xi)\langle\cdot,\cdot,\rangle_{v_w,\xi}$ and $\langle\cdot,\cdot\rangle_{\mathrm{Pl},\xi,\lambda}$ on $V^*_{\emptyset,v_w}(\xi)$ and $V^*(\xi)$, respectively. It then follows that the image of $\Gamma_{v_w}(\xi : \lambda)^\dagger$ is equal to $V^*_w(\xi)$. Moreover, by the Maaß-Selberg relations from [17, Theorem 9.6] the map $\Gamma_{v_w}(\xi : \lambda)$ is a partial isometry for every $w \in \mathcal{N}/\mathcal{W}$ and almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$, i.e., $\Gamma_{v_w}(\xi : \lambda)^\dagger$ is for almost all $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ a unitary map onto its image $V^*_w(\xi)$. Since $\Gamma_{v_w}(\xi : \lambda)$ is essentially the identity map, the restrictions of $\dim(\xi)\langle\cdot,\cdot\rangle_\xi$ and $\langle\cdot,\cdot\rangle_{\mathrm{Pl},\xi,\lambda}$ to the subspaces $V^*_w(\xi)$ with $w \in \mathcal{N}/\mathcal{W}$ coincide for almost every $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. This proves the identity (8.1) as the decomposition (8.1) is orthogonal with respect to both $\dim(\xi)\langle\cdot,\cdot\rangle_\xi$ and $\langle\cdot,\cdot\rangle_{\mathrm{Pl},\xi,\lambda}$ by Corollary 8.2. ☐

8.7 Corollaries I: regularity of the families of distributions

In this and the next section we record two corollaries of Theorem 8.1. The first corollary is the regularity of the families of distributions we constructed in Section 6 on the imaginary axis.

Corollary 8.1. *Let ξ be a finite dimensional unitary representation of M_Q. For every $\eta \in V^*(\xi)$ the family of distributions $\lambda \mapsto \mu^\circ(\xi : \lambda)\eta$ is holomorphic on a neighborhood of $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$.*

Proof. Let $\eta \in V^*(\xi)$. By Theorem 6.1 the family $\lambda \mapsto \mu^\circ(\xi : \lambda)\eta$ is meromorphic. It therefore suffices to proof that the family does not have any singularities on $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$.

We aim for a contradiction and assume that $\lambda \mapsto \mu^\circ(\xi : \lambda)\eta$ has a singularity on $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$. The poles of the family lie in view of Theorem 6.1 on a locally finite union of complex affine hyperplanes of the form

$$\{\lambda \in (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C} : \lambda(X) = a\} \quad \text{for some } X \in \mathfrak{a} \text{ and } a \in \mathbb{R}.$$

Since the singular set of a meromorphic function on $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ is a union of complex analytic submanifolds of $\mathbb{C}$-codimension 1, it follows that there exists a subspace $\mathcal{H}$ of $i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ of codimension 1 so that $\lambda \mapsto \mu^\circ(\xi : \lambda)$ is singular on $\mathcal{H}$. For every $f \in \mathcal{D}(Z)$ the assignment $\lambda \mapsto \mathscr{F}f(\xi^\vee : \lambda)\eta$ defines a meromorphic function on $(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*_\mathbb{C}$ and there exist functions $f \in \mathcal{D}(Z)$ so that $\mathscr{F}f(\xi^\vee : \lambda)\eta$ is singular on $\mathcal{H}$. Let f be such a function and let $\omega \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*$ be transversal to $\mathcal{H}$. Then there exists a $\mu \in \mathcal{H}$ and $n \in \mathbb{N}$ with $n \geq 1$ so that

$$t \mapsto t^n \mathscr{F}f(\xi^\vee : \mu + t\omega)\eta$$

extends to a continuous function on a neighborhood of 0 that is non-zero in 0. Therefore, there exists an $\epsilon > 0$ and an open neighborhood Ω of μ in $\mathcal{H}$ so that

$$\|\mathscr{F}f(\xi^\vee : \lambda + t\omega)\eta\|_{\xi,\lambda} \geq c|t|^{-n} \qquad \big(-\epsilon < t < \epsilon, t \neq 0, \lambda \in \Omega\big).$$

It follows that $\lambda \mapsto \|\mathscr{F}f(\xi^\vee : \lambda)\eta\|_{\xi,\lambda}$ is not square integrable. This is however in contradiction with Theorem 8.1. □

8.8 Corollaries II: refined Maaß-Selberg relations

The second corollary of Theorem 8.1 is a refinement of the Maaß-Selberg relations. The Maaß-Selberg relations from [17, Theorem 9.6] state that each of the maps

$$\Gamma_v(\xi : \lambda) : V^*(\xi) \to V^*_{\emptyset,v}(\xi) \qquad \big(v \in \mathfrak{N}\big).$$

is a partial isometry, i.e., its Hermitian dual of $\Gamma_v(\xi : \lambda)$ with respect to the inner products on $V^*(\xi)$ and $V^*_{\emptyset,v}(\xi)$, induced by the Plancherel decompositions Theorems 8.1 and 8.1, is a unitary isometry. In view of the Theorems 8.1 and 8.1 these inner products are up to factor $\dim(V_\xi)$ equal to the inner products induced by the inner product on V_ξ. We can now refine the Maaß-Selberg relations from [17] for the most continuous part of $L^2(Z)$ as follows.

Corollary 8.1. *Let ξ be a finite dimensional unitary representation of M_Q, $\lambda \in i(\mathfrak{a}/\mathfrak{a}^*)$ and $v \in \mathfrak{N}$. Then*

$$\Gamma_v(\xi : \lambda)\big|_{V^*_w(\xi)} = 0 \qquad \big(w \in \mathcal{N}/\mathcal{W}, v\mathcal{W} \neq w\big).$$

Moreover, if $w = v\mathcal{W}$, then

$$\Gamma_v(\xi : \lambda)\big|_{V^*_w(\xi)} : V^*_w(\xi) \to V^*_{\emptyset,v}(\xi)$$

is a unitary map.

Proof. The assertions follow Proposition 8.1, (8.8) and the Maaß-Selberg relations from [17, Theorem 9.6]. □

The Maaß-Selberg relations from Corollary 8.1 are reflected in the symmetries of the combined constant term map

$$\Gamma(\xi : \lambda) : V^*(\xi) \to \bigoplus_{\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}} V^*_\emptyset(\xi)$$

given by

$$\Big(\Gamma(\xi : \lambda)\eta\Big)_{\mathcal{O}} = \Gamma_{\mathcal{O}}(\xi : \lambda)\eta \qquad \big(\eta \in V^*(\xi), \mathcal{O} \in (P\backslash Z)_{\mathrm{open}}\big).$$

Note that $\Gamma(\xi : \lambda)$ decomposes according to the decomposition $V^*_\emptyset(\xi) = \bigoplus_{v\in\mathfrak{N}} V^*_{\emptyset,v}(\xi)$ as

$$\Big(\Gamma(\xi : \lambda)\eta\Big)_v = \Gamma_v(\xi : \lambda)\eta \in V^*_{\emptyset,v}(\xi) \qquad \big(\eta \in V^*(\xi), v \in \mathfrak{N}\big).$$

To describe the symmetries we first introduce an action of $\mathcal{N}/\mathcal{Z}$ on $\mathfrak{N}$. For $w \in \mathcal{N}/\mathcal{Z}$ and $v \in \mathfrak{N} \subseteq \mathcal{N}$ we define $w \cdot v$ to be the element in $\mathfrak{N}$ determined by the identity in $\mathcal{N}/\mathcal{Z}$

$$(w \cdot v)\mathcal{Z} = vw^{-1},$$

i.e., $w \cdot v$ is the representative of $vw^{-1} \in \mathcal{N}/\mathcal{Z}$ in $\mathfrak{N}$.

For $w \in \mathcal{N}/\mathcal{Z}$ we now define the scattering operator

$$\mathcal{S}_w(\xi : \lambda) := \sum_{v \in \mathfrak{N}} \Gamma_{w\cdot v}(\xi : \lambda) \circ \Gamma_v(\xi : \lambda)^\dagger : \bigoplus_{\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}} V_\emptyset^*(\xi) \to \bigoplus_{\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}} V_\emptyset^*(\xi).$$

We then have the following immediate corollary of Corollary 8.1.

Corollary 8.2. *Let ξ be a finite dimensional unitary representation of M_Q and $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \setminus i\mathcal{S}$. The following hold true.*

(i) $\mathcal{S}_w(\xi : \lambda) = 0$ for every $w \in (\mathcal{N}/\mathcal{Z}) \setminus W_Z$.

(ii) The assignment $W_Z \ni w \mapsto \mathcal{S}_w(\xi : \lambda)$ defines a unitary representation of W_Z on

$$\bigoplus_{\mathcal{O} \in (P\backslash Z)_{\mathrm{open}}} V_\emptyset^*(\xi).$$

(iii) For every $w \in W_Z$ we have $\mathcal{S}_w(\xi : \lambda) \circ \Gamma(\xi : \lambda) = \Gamma(\xi : \lambda)$.

Remark 8.3. In [15] scattering operators were defined under the restricting assumption that G is split, but for all boundary degenerations, not just for the horospherical boundary degeneration $Z_\emptyset$ as we do here.

We finish this section with a description of the scattering operators in terms of the action of standard intertwining operators on $\mathcal{D}'(\overline{Q} : \xi : \lambda)^H$, or rather in terms of the induced action on the parameter spaces $V^*(\xi)$. We first define for $v \in \mathcal{N}$ the normalized $\mathcal{B}$-matrix

$$\mathcal{B}_v^\circ(\xi : \lambda) : V^*(\xi) \to V^*(v \cdot \xi)$$

by

$$\mathcal{B}_v^\circ(\xi : \lambda) := \frac{1}{\gamma(v^{-1}\overline{Q}v : \overline{Q} : \xi : \lambda)} \beta(v \cdot \xi : \mathrm{Ad}^*(v)\lambda) \circ \mathcal{B}_v(Q : \xi : \lambda) \circ \beta(\xi : \lambda)^{-1}.$$

The normalized $\mathcal{B}$-matrices are characterized by the fact that the diagram

$$\begin{tikzcd}
\mathcal{D}'(\overline{Q} : \xi : \lambda)^H \arrow[r, "\mathcal{I}_v^\circ(\xi:\lambda)"] & \mathcal{D}'(\overline{Q} : v \cdot \xi : \mathrm{Ad}^*(v)\lambda)^H \\
V^*(\xi) \arrow[u, "\mu^\circ(\xi:\lambda)"] \arrow[r, "\mathcal{B}_v^\circ(\xi:\lambda)"] & V^*(v \cdot \xi) \arrow[u, "\mu^\circ\left(v\cdot\xi:\mathrm{Ad}^*(v)\lambda\right)"']
\end{tikzcd}$$

commutes. If $v_1, v_2 \in \mathcal{N}$ then the identity

$$\mathcal{I}^\circ_{v_1}\big(v_2 \cdot \xi : \mathrm{Ad}^*(v_2)\lambda\big) \circ \mathcal{I}^\circ_{v_2}(\xi : \lambda) = \mathcal{I}^\circ_{v_1 v_2}(\xi : \lambda)$$

implies

$$\mathcal{B}^\circ_{v_1}\big(v_2 \cdot \xi : \mathrm{Ad}^*(v_2)\lambda\big) \circ \mathcal{B}^\circ_{v_2}(\xi : \lambda) = \mathcal{B}^\circ_{v_1 v_2}(\xi : \lambda). \tag{8.1}$$

Each normalized intertwining operator $\mathcal{I}^\circ_v(\xi : \lambda)$ is unitary and hence is a unitary map between the multiplicity spaces $\mathcal{M}_{\xi,\lambda}$ and $\mathcal{M}_{v\cdot\xi,\mathrm{Ad}^*(v)\lambda}$. Furthermore, $\mu^\circ(\xi : \lambda)$ is in view of Theorem 8.1 a unitary map from $V^*(\xi)$ (equipped with $\dim(V_\xi)$ times the inner product induced by the one on V_ξ) to $\mathcal{M}_{\xi,\lambda}$. Therefore, the normalized $\mathcal{B}$-matrices $\mathcal{B}^\circ_v(\xi : \lambda)$ are unitary.

For $v \in \mathfrak{N}$ we define the map

$$j_v(\xi) : V^*(v^{-1} \cdot \xi) \to V^*_{\emptyset,v}(\xi) \subseteq \bigoplus_{\mathcal{O}\in(P\backslash Z)_{\mathrm{open}}} V^*_\emptyset(\xi)$$

by

$$\big(j_v(\xi)\eta\big)_{\mathcal{O}} := \eta_{\mathcal{O},e} \qquad \big(\eta \in V^*(v^{-1} \cdot \xi), \mathcal{O} \in (P\backslash Z)_{\mathrm{open}}\big).$$

The dual map

$$j_v(\xi)^\dagger : \bigoplus_{\mathcal{O}\in(P\backslash Z)_{\mathrm{open}}} V^*_\emptyset(\xi) \to V^*_{e\mathcal{W}}(v^{-1} \cdot \xi) \subseteq V^*(v^{-1} \cdot \xi)$$

is given by

$$j_v(\xi)^\dagger\big|_{V^*_{\emptyset,v'}(\xi)} = 0 \qquad (v' \in \mathfrak{N}, v' \neq v)$$

and

$$\big(j_v(\xi)^\dagger\eta\big)_{\mathcal{O}} = \eta_{\mathcal{O}} \qquad \big(\eta \in V^*_{\emptyset,v}(\xi), \mathcal{O} \in (P\backslash Z)_{\mathrm{open}}\big).$$

Now the scattering maps are given by the following.

Corollary 8.4. *Let ξ be a finite dimensional unitary representation of M_Q and let $\lambda \in i(\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^* \setminus i\mathcal{S}$. Then*

$$\mathcal{S}_w(\xi : \lambda) = \sum_{v\in\mathfrak{N}} j_{w\cdot v}(\xi) \circ \mathcal{B}^\circ_{(w\cdot v)^{-1}v}(\xi : \lambda) \circ j_v(\xi)^\dagger \qquad (w \in W_Z).$$

Proof. Let $w \in \mathcal{N}/\mathcal{W}$ and let $v \in \mathfrak{N}$ be so that $v\mathcal{W} = w$. In view of Proposition 8.1 we have

$$\Gamma_v(\xi : \lambda) = \Gamma_e\big(v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) \circ \mathcal{B}^\circ_{v^{-1}}(\xi : \lambda) \qquad (v \in \mathfrak{N}).$$

Moreover, by Corollary 8.3

$$\Gamma_e\big(v^{-1} \cdot \xi : \mathrm{Ad}^*(v^{-1})\lambda\big) = j_v(\xi),$$

and hence

$$\Gamma_v(\xi : \lambda) = j_v(\xi) \circ \mathcal{B}^\circ_{v^{-1}}(\xi : \lambda) \qquad (v \in \mathfrak{N}).$$

The assertion now follows from the unitarity of the normalized $\mathcal{B}$-matrices and (8.1). □

Appendices

Appendix A: Wavefront spaces

The space Z is called wavefront if the compression cone of Z is given by

$$\mathcal{C} = \mathfrak{a}^- + \mathfrak{a}_\mathfrak{h}.$$

A main class of examples of wavefront spaces is the class of reductive symmetric spaces, i.e., the spaces $Z = G/H$ with H an open subgroup of the fixed point subgroup of an involutive automorphism of G.

For the most continuous part of $L^2(Z)$ the P-orbits $\mathcal{O}$ of maximal rank with $\mathfrak{a}_\mathcal{O} = \mathfrak{a}_\mathfrak{h}$ are of relevance. The open P-orbits satisfy this condition. In view of the following proposition, the open P-orbits are all orbits with $\mathfrak{a}_\mathcal{O} = \mathfrak{a}_\mathfrak{h}$ in case Z is wavefront. We recall the groups $\mathcal{N}$, $\mathcal{Z}$ and $\mathcal{W}$ from (3.1), (3.2) and (3.3), respectively.

Proposition A.1. *Assume that Z is wavefront. Then $\mathcal{W} = \mathcal{N}$. In particular, the little Weyl group is equal to*

$$W_Z = \mathcal{N}/\mathcal{Z}.$$

Theorem 3.3 and Proposition A.1 have the following corollary.

Corollary A.2. *Assume that Z is wavefront and let $\mathcal{O} \in (P\backslash Z)_{\max}$. Then $\mathcal{O}$ is open if and only if $\mathfrak{a}_\mathcal{O} = \mathfrak{a}_\mathfrak{h}$.*

Proof of Proposition A.1. As $\mathcal{W}$ is a subgroup of $\mathcal{N}$, we only have to prove the inclusion $\mathcal{N} \subseteq \mathcal{W}$. Let $w \in \mathcal{N}$. From the fact that $\overline{\mathcal{C}}/\mathfrak{a}_\mathfrak{h}$ is a fundamental domain for action of the little Weyl group on $\mathfrak{a}/\mathfrak{a}_\mathfrak{h}$ it follows that there exists a $v \in \mathcal{W}$ so that $\mathrm{Ad}(v^{-1}w^{-1})\mathfrak{a}^- \cap \mathcal{C} \neq \emptyset$. After replacing w by wv we may thus assume that $\mathrm{Ad}(w)\mathcal{C} \cap \mathfrak{a}^- \neq \emptyset$. We may further adjust w by multiplying it from the right by an element from $\mathcal{Z}$ and assume that (3.3) holds. It now suffices to prove that $w \in MA$. As the stabilizer of ρ_P in $N_G(\mathfrak{a})$ is equal to MA, it is thus enough to show that w stabilizes ρ_P.

Since

$$\big(\mathrm{Ad}(w)\mathfrak{a}^+ + \mathfrak{a}_\mathfrak{h}\big) \cap \mathfrak{a}^+ = -\big(\mathrm{Ad}(w)\mathcal{C} \cap \mathfrak{a}^-\big)$$

is open and nonempty, its dual cone

$$\Big(\big(\sum_{\alpha \in \Sigma(wPw^{-1})} \mathbb{R}_{\geq 0}\alpha\big) \cap (\mathfrak{a}/\mathfrak{a}_\mathfrak{h})^*\Big) + \sum_{\alpha \in \Sigma(P)} \mathbb{R}_{\geq 0}\alpha \tag{A.1}$$

is proper. Note that

$$\rho_P - \mathrm{Ad}^*(w)\rho_P = \sum_{\alpha \in \Sigma(P) \cap -\Sigma(wPw^{-1})} \dim(\mathfrak{g}_\alpha)\alpha.$$

Since w normalizes $\mathfrak{a}_\mathfrak{h}$, it follows from Remark 3.2 that w normalizes $\mathfrak{l}_Q$. In view of (3.3) we have

$$\Sigma(P) \cap -\Sigma(wPw^{-1}) = \Sigma(P) \cap -\Sigma(wQw^{-1}) = \Sigma(Q) \cap -\Sigma(wQw^{-1}),$$

and hence

$$\rho_P - \mathrm{Ad}^*(w)\rho_P = \rho_Q - \mathrm{Ad}^*(w)\rho_Q.$$

As Z is unimodular, we have

$$\rho_Q(X) = \mathrm{tr}\,\mathrm{ad}(X)\big|_{\mathfrak{n}_Q} = -\,\mathrm{tr}\,\mathrm{ad}(X)\big|_{\mathfrak{m}+\mathfrak{a}+\mathfrak{h}} = -\,\mathrm{tr}\,\mathrm{ad}(X)\big|_{\mathfrak{h}} = 0 \qquad (X \in \mathfrak{a}_{\mathfrak{h}}).$$

Therefore, $\rho_Q \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$ and, as w normalizes $\mathfrak{a}_{\mathfrak{h}}$, also $\mathrm{Ad}(w)^*\rho_Q \in (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$. It follows that

$$\mathrm{Ad}^*(w)\rho_P - \rho_P \in \Big(\sum_{\alpha\in\Sigma(wPw^{-1})} \mathbb{R}_{\geq 0}\alpha\Big) \cap (\mathfrak{a}/\mathfrak{a}_{\mathfrak{h}})^*$$

and

$$\rho_P - \mathrm{Ad}^*(w)\rho_P \in \sum_{\alpha\in\Sigma(P)} \mathbb{R}_{\geq 0}\alpha.$$

Since (A.1) is proper and contains both $\mathrm{Ad}^*(w)\rho_P - \rho_P$ and $\rho_P - \mathrm{Ad}^*(w)\rho_P$, we conclude that $\rho_P = \mathrm{Ad}^*(w)\rho_P$. This proves the claim. □

Appendix B: Intertwining operators

Let S be a parabolic subgroup of G with Langlands decomposition $S = M_SA_SN_S$. Further, let ξ be a representation of M_S on a Hilbert space V_ξ and $\lambda \in \mathfrak{a}_{S,\mathbb{C}}^*$. In this appendix we are concerned with a description of the action of standard intertwining operators on $\mathcal{D}'(S : \xi : \lambda)$. In the course of this appendix we will prove all assertions in Proposition 4.1. We begin by introducing some spaces of functions.

Recall the map $a_S : G \to A_S$ which is given by

$$x \in N_S a_S(x) M_S K \qquad (x \in G).$$

We write $\mathcal{L}_{S,\xi,\lambda}$ for the space of equivalence classes of measurable functions $\phi : G \to V_\xi$ such that

$$x \mapsto a_S^{-\mathrm{Re}\,\lambda+\rho_S}(x)\|\phi(x)\|_\xi$$

is integrable. Here two functions are equivalent if and only if only differ on a set of measure 0. We endow $\mathcal{L}_{S,\xi,\lambda}$ with the norm

$$\phi \mapsto \int_G \|a_S^{-\lambda+\rho_S}(x)\phi(x)\|_\xi \, dx < \infty.$$

With this norm $\mathcal{L}_{S,\xi,\lambda}$ is a Banach space.

Lemma B.1. *For every compact subset C of G there exists a constant $c > 0$ such that for every $g \in C$ and $x \in G$*

$$c^{-1}|a_S^{-\lambda+\rho_S}(xg)| \leq |a_S^{-\lambda+\rho_S}(x)| \leq c|a_S^{-\lambda+\rho_S}(xg)|.$$

Proof. Let C be a compact subset of G, let $g \in C$ and let $x \in G$. Then

$$a_S(xg) \in a_S(x)a_S(Kg).$$

Since KC is a compact subset of G and $a_S^{\pm\lambda\pm\rho_S}$ is continuous, there exists a constant $c > 0$ so that

$$c \geq |a_S^{\mp\lambda\pm\rho_S}(y)| \qquad (y \in KC).$$

With this constant c the desired inequalities hold. □

It follows from Lemma B.1 that $\mathcal{L}_{S,\xi,\lambda}$ is invariant under right translations by elements of G. We write R for the right-regular representation of G on $\mathcal{L}_{S,\xi,\lambda}$.

Proposition B.2. *The representation $\big(R, \mathcal{L}_{S,\xi,\lambda}\big)$ is a continuous Banach representation.*

Proof. The proof is the same as the proof for [38, Proposition 2.9]. □

Let $\mathcal{V}_{S,\xi,\lambda}$ be the space of smooth V_ξ-valued functions on G that represent a smooth vector for R in $\mathcal{L}_{S,\xi,\lambda}$. The local Sobolev lemma ensures that every smooth vector in $\mathcal{L}_{S,\xi,\lambda}$ can indeed be represented by a smooth V_ξ-valued function. See also [43, Theorem 5.1]. We endow $\mathcal{V}_{S,\xi,\lambda}$ with the unique Fréchet topology so that the natural bijection $\mathcal{V}_{S,\xi,\lambda} \to \mathcal{L}^\infty_{S,\xi,\lambda}$ is a topological isomorphism. Note that

$$\mathcal{V}_{S,\xi,\lambda} = \Big\{\phi \in \mathcal{E}(G, V_\xi) : \int_G \|a_S^{-\lambda+\rho_S}(x)\, R(u)\phi(x)\|_\xi \, dx < \infty \text{ for every } u \in \mathcal{U}(\mathfrak{g})\Big\}.$$

Lemma B.3. *For $\phi \in \mathcal{V}_{S,\xi,\lambda}$ the function*

$$G \to V_\xi; \quad x \mapsto \int_{M_S}\int_{A_S}\int_{N_S} a^{-\lambda+\rho_S}\xi(m^{-1})\phi(manx)\, dn\, da\, dm$$

is defined by absolutely convergent integrals and forms an element of $C^\infty(S : \xi : \lambda)$. Moreover, the map

$$\mathcal{V}_{S,\xi,\lambda} \to C^\infty(S : \xi : \lambda)$$

thus obtained is G-equivariant and continuous.

Proof. Let $\phi \in \mathcal{V}_{S,\xi,\lambda}$. By Fubini's theorem the integral

$$\mathcal{T}\phi(x) := \int_{M_S}\int_{A_S}\int_{N_S} a^{-\lambda+\rho_S}\xi(m^{-1})\phi(manx)\, dn\, da\, dm \tag{B.1}$$

is absolutely convergent for almost every $x \in K$ and the function $\mathcal{T}\phi : K \to V_\xi$ thus obtained is integrable. Since $L(man)\mathcal{T}\phi = a^{-\lambda-\rho_S}\xi(m^{-1})\mathcal{T}\phi$, it follows that the integral $\mathcal{T}\phi(x)$ is absolutely convergent for almost every $x \in G$ and the function $\mathcal{T}\phi : G \to V_\xi$ thus obtained is locally integrable.

We claim that the integral (B.1) is in fact absolutely convergent for every $x \in G$ and that $\mathcal{T}\phi$ is a smooth function for every $\phi \in \mathcal{V}_{S,\xi,\lambda}$. From [19, Théorème 3.3] it follows that

$$\mathcal{V}_{S,\xi,\lambda} = \operatorname{span}\big\{\pi(f)\phi : f \in \mathcal{D}(G), \phi \in \mathcal{V}_{S,\xi,\lambda}\big\}. \tag{B.2}$$

Therefore, to prove the claim it suffices to show that for every $f \in \mathcal{D}(G)$ and $\phi \in \mathcal{V}_{S,\xi,\lambda}$ the function $\mathcal{T}\big(\pi(f)\phi\big)$ is smooth. Let $f \in \mathcal{D}(G)$ and $\phi \in \mathcal{V}_{S,\xi,\lambda}$. Then, by Fubini's theorem

$$\begin{aligned}
\mathcal{T}(\pi(f)\phi)(x) &= \int_{M_S}\int_{A_S}\int_{N_S} a^{-\lambda+\rho_S}\xi(m^{-1})\int_G f(y)\phi(manxy)\,dy\,dn\,da\,dm \\
&= \int_G f(x^{-1}y)\Big(\int_{M_S}\int_{A_S}\int_{N_S} a^{-\lambda+\rho_S}\xi(m^{-1})\phi(many)\,dn\,da\,dm\Big)\,dy \\
&= \int_G f(x^{-1}y)\mathcal{T}\phi(y)\,dy.
\end{aligned}$$

Since $\mathcal{T}\phi$ is locally integrable, the last expression defines a smooth function in $x \in G$. This proves the claim. Note that it follows from the claim that $\mathcal{T}\phi \in C^\infty(S:\xi:\lambda)$ for every $\phi \in \mathcal{V}_{S,\xi,\lambda}$. This proves the first statement in the lemma.

The equivariance of $\mathcal{T}$ is clear. It thus remains to prove the continuity. Let $L^1(K:\xi)$ be the space of integrable functions $\phi : K \to V_\xi$ that satisfy

$$\phi(mk) = \xi(m)\phi(k) \qquad (m \in M \cap K, k \in K).$$

As stated above, the restriction of $\mathcal{T}\phi$ to K is integrable for every $\phi \in \mathcal{V}_{S,\xi,\lambda}$. Moreover,

$$\int_K \|\mathcal{T}\phi(k)\|_\xi\,dk \leq \int_G \|a_S^{-\lambda+\rho_S}(x)\phi(x)\|_\xi\,dx.$$

Therefore $\mathcal{T}$ defines a continuous map $\mathcal{V}_{S,\xi,\lambda} \to L^1(K:\xi)$ which intertwines $\pi\big|_K$ and the right regular representation of K on $L^1(K:\xi)$. Since $\big(\pi\big|_K, \mathcal{V}_{S,\xi,\lambda}\big)$ is a smooth representation, $\mathcal{T}$ in fact defines a continuous map $\mathcal{V}_{S,\xi,\lambda} \to L^1(K:\xi)^\infty$. From the local Sobolev lemma it follows that there is a natural identification between $L^1(K:\xi)^\infty$ and the space $C^\infty(K:\xi)$ consisting of all smooth functions $f : K \to V_\xi$ such that

$$f(mk) = \xi(m)f(k) \qquad (m \in M \cap K, k \in K).$$

This identification is a topological isomorphism. See also [43, Theorem 5.1]. Finally, the restriction map $\phi \mapsto \phi\big|_K$ is a K-equivariant topological isomorphism between the spaces $C^\infty(S:\xi:\lambda)$ and $C^\infty(K:\xi)$. This proves the second claim in the lemma. $\square$

It follows from Lemma B.3 that for every $\eta \in C^\infty(S:\xi:\lambda)'$ the right-hand side of (4.2) defines a continuous linear functional on $\mathcal{V}_{S,\xi,\lambda}$. We thus conclude that every $\mu \in \mathcal{D}'(S:\xi:\lambda)$ extends to a continuous linear function on $\mathcal{V}_{S,\xi,\lambda}$. In fact, the injection

$$\big\{\mu \in \mathcal{V}'_{S,\xi,\lambda} : \mu \text{ satisfies } (4.1)\big\} \hookrightarrow \mathcal{D}'(S:\xi:\lambda); \quad \mu \mapsto \mu\big|_{\mathcal{D}(G,V_\xi)} \tag{B.3}$$

is a bijection.

Now let S_1 and S_2 be parabolic subgroups such that $A_{S_1} = A_{S_2} \subseteq A$. We identify $\mathfrak{a}^*_{S,\mathbb{C}}$ by the subspace of $\mathfrak{a}^*_\mathbb{C}$ of elements that vanish on $\mathfrak{a} \cap \mathfrak{m}_S$.

Proposition B.4. *Let $\lambda \in \mathfrak{a}^*_{S,\mathbb{C}}$ satisfy*

$$\langle \mathrm{Re}\,\lambda, \alpha\rangle > 0 \qquad \big(\alpha \in \Sigma(\mathfrak{a} : S_2) \cap -\Sigma(\mathfrak{a} : S_1)\big). \tag{B.4}$$

For every $\phi \in \mathcal{L}_{S_2,\xi,\lambda}$ and almost every $x \in G$ the integral

$$\int_{N_{S_2}\cap \overline{N}_{S_1}} \phi(nx)\, dn$$

is absolutely convergent and the function $\int_{N_{S_2}\cap \overline{N}_{S_1}} \phi(n\,\cdot\,)\, dn$ thus obtained represents an element of $\mathcal{L}_{S_1,\xi,\lambda}$. Moreover, the map

$$\mathcal{L}_{S_2,\xi,\lambda} \to \mathcal{L}_{S_1,\xi,\lambda}; \quad \phi \mapsto \int_{N_{S_2}\cap \overline{N}_{S_1}} \phi(n\,\cdot\,)\, dn$$

is continuous.

Proof. Let $\phi \in \mathcal{L}_{S_2,\xi,\lambda}$. In view of Fubini's theorem, it suffices to show that

$$\int_G \int_{N_{S_2}\cap \overline{N}_{S_1}} \Big\| a_{S_1}^{-\lambda+\rho_{S_1}}(x)\phi(nx)\Big\|_\xi \, dn\, dx \leq c \int_G \|a_{S_2}^{-\lambda+\rho_{S_2}}(x)\phi(x)\|_\xi \, dx \tag{B.5}$$

for some $c > 0$.

Using the invariance of the Haar measure on G, we obtain

$$\begin{aligned}
\int_G \int_{N_{S_2}\cap \overline{N}_{S_1}} \Big\| a_{S_1}^{-\lambda+\rho_{S_1}}(x)\phi(nx)\Big\|_\xi \, dn\, dx &= \int_{N_{S_2}\cap \overline{N}_{S_1}} \int_G \Big\| a_{S_1}^{-\lambda+\rho_{S_1}}(x)\phi(nx)\Big\|_\xi \, dx\, dn\\
&= \int_{N_{S_2}\cap \overline{N}_{S_1}} \int_G \Big\| a_{S_1}^{-\lambda+\rho_{S_1}}(nx)\phi(x)\Big\|_\xi \, dx\, dn\\
&= \int_G \int_{N_{S_2}\cap \overline{N}_{S_1}} \Big| a_{S_1}^{-\lambda+\rho_{S_1}}(nx)\Big| \, dn\, \|\phi(x)\|_\xi \, dx.
\end{aligned}$$

Note that

$$\int_{N_{S_2}\cap \overline{N}_{S_1}} \Big| a_{S_1}^{-\lambda+\rho_{S_1}}(nx)\Big| \, dn = c(S_2 : S_1 : -\mathrm{Re}\,\lambda) a_{S_2}^{-\mathrm{Re}\,\lambda+\rho_{S_2}}(x) \qquad (x \in G)$$

where $c(S_2 : S_1 : \,\cdot\,)$ is the partial c-function which is given by the absolutely convergent integral

$$c(S_2 : S_1 : \nu) = \int_{N_{S_2}\cap \overline{N}_{S_1}} a_{S_1}^{\nu+\rho_{S_1}}(n)\, dn$$

in case $\lambda = -\nu \in \mathfrak{a}^*_{\mathbb{C}}$ satisfies (B.4). Hence (B.5) holds with $c = c(S_2 : S_1 : -\mathrm{Re}\,\lambda)$. This proves the proposition. $\square$

Corollary B.5. *Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ satisfy (B.4). For every $\phi \in \mathcal{V}_{S_2,\xi,\lambda}$ and every $x \in G$ the integral*

$$\int_{N_{S_2}\cap \overline{N}_{S_1}} \phi(nx)\, dx \tag{B.6}$$

is absolutely convergent and the function $\int_{N_{S_2}\cap \overline{N}_{S_1}} \phi(n\,\cdot\,)\,dx$ *thus obtained is an element of* $\mathcal{V}_{S_1,\xi,\lambda}$. *Moreover, the map*

$$\mathcal{V}_{S_2,\xi,\lambda}\to\mathcal{V}_{S_1,\xi,\lambda};\quad \phi\mapsto\int_{N_{S_2}\cap \overline{N}_{S_1}}\phi(n\,\cdot\,)\,dn \tag{B.7}$$

is continuous.

Proof. It follows from Proposition B.4 that $\phi\mapsto\int_{N_{S_2}\cap \overline{N}_{S_1}}\phi(n\,\cdot\,)\,dn$ defines a continuous map between the spaces of smooth vectors in $\mathcal{L}_{S_2,\xi,\lambda}$ and $\mathcal{L}_{S_1,\xi,\lambda}$ respectively. It therefore suffices to show that for every $\phi\in\mathcal{V}_{S_2,\xi,\lambda}$ and $x\in G$ the integral (B.6) is absolutely convergent and the function $\int_{N_{S_2}\cap \overline{N}_{S_1}}\phi(n\,\cdot\,)\,dn$ is smooth. In view of (B.2) it suffices to do this for ϕ of the form $\phi=\pi(f)\psi$ with $f\in\mathcal{D}(G)$ and $\psi\in\mathcal{V}_{S_2,\xi,\lambda}$.

Let $f\in\mathcal{D}(G)$, $\phi\in\mathcal{V}_{S_2,\xi,\lambda}$ and $x\in G$. It follows from Proposition B.4 that the integral

$$\int_G f(x^{-1}y)\int_{N_{S_2}\cap \overline{N}_{S_1}}\psi(ny)\,dn\,dy$$

is absolutely convergent. Moreover, it depends smoothly on x. By Fubini's theorem this integral is equal to

$$\int_{N_{S_2}\cap \overline{N}_{S_1}}\big(\pi(f)\psi\big)(nx)\,dn.$$

This proves the corollary. $\square$

We define

$$\mathcal{A}(S_2:S_1:\xi:\lambda):=\theta^{S_2}_{\xi,\lambda}\circ A(S_1:S_2:\xi:\lambda)^*\circ\omega^{S_1}_{\xi,\lambda}$$

The following diagram commutes.

$$\begin{tikzcd}
\mathcal{D}'(S_1:\xi:\lambda) \arrow[r, "\mathcal{A}(S_2:S_1:\xi:\lambda)"] \arrow[d, shift right, "\omega^{S_1}_{\xi,\lambda}"'] & \mathcal{D}'(S_2:\xi:\lambda) \arrow[d, shift right, "\omega^{S_2}_{\xi,\lambda}"'] \\
C^\infty(S_1:\xi:\lambda)' \arrow[r, "A(S_1:S_2:\xi:\lambda)^*"] \arrow[u, shift right, "\theta^{S_1}_{\xi,\lambda}"'] & C^\infty(S_2:\xi:\lambda)' \arrow[u, shift right, "\theta^{S_2}_{\xi,\lambda}"']
\end{tikzcd}$$

We recall from (B.3) that every distribution $\mu\in\mathcal{D}'(S:\xi:\lambda)$ extends to a continuous linear functional on $\mathcal{V}_{S,\xi,\lambda}$. Therefore, if $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ satisfies (B.4), then in view of Corollary B.5 the assignment

$$\phi\mapsto\mu\Big(\int_{N_{S_2}\cap \overline{N}_{S_1}}\phi(n\,\cdot\,)\,dn\Big)$$

defines for every $\mu\in\mathcal{D}'(S_1:\xi:\lambda)$ a distribution in $\mathcal{D}'(G,V_\xi)$.

Proposition B.6. *Let* $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ *satisfy (B.4). For every* $\mu\in\mathcal{D}'(S_1:\xi:\lambda)$ *the distribution* $\mathcal{A}(S_2:S_1:\xi:\lambda)\mu\in\mathcal{D}'(S_2:\xi:\lambda)$ *is given by*

$$\big[\mathcal{A}(S_2:S_1:\xi:\lambda)\mu\big](\phi)=\mu\Big(\int_{N_{S_2}\cap \overline{N}_{S_1}}\phi(n\,\cdot\,)\,dn\Big)\qquad\big(\phi\in\mathcal{D}(G,V_\xi)\big).$$

For the proof of the proposition we need the following lemma.

Lemma B.7. *Let $\phi \in C^\infty(S : \xi^\vee : -\lambda)$ and consider ϕ as an element of $\mathcal{D}'(S : \xi : \lambda)$. Then*

$$\big(\omega^S_{\xi,\lambda}\phi\big)(f) = \int_K \Big(\phi(k), f(k)\Big)\, dk \qquad \big(f \in C^\infty(S : \xi : \lambda)\big).$$

Proof. Let $f \in C^\infty(S : \xi : \lambda)$. Then

$$\begin{aligned}\big(\omega^S_{\xi,\lambda}\phi\big)(f) &= \int_G \Big(\phi(x), \psi_0(x) f(x)\Big)\, dx \\ &= \int_{M_S}\int_{A_S}\int_{N_S} a^{2\rho_S}\psi_0(man)\, dn\, da\, dm \int_K \Big(\phi(k), f(k)\Big)\, dk.\end{aligned}$$

The claim in the lemma now follows from the observation that

$$\begin{aligned}\int_{M_S}\int_{A_S}\int_{N_S} a^{2\rho_S}\psi_0(man)\, dn\, da\, dm &= \int_{M_S}\int_{A_S}\int_{N_S}\int_K a^{2\rho_S}\psi_0(mank)\, dn\, da\, dm\, dk \\ &= \int_G a_S^{2\rho_S}(x)\psi_0(x)\, dx = 1.\end{aligned}$$

□

Proof of Proposition B.6. Since (B.7) is continuous and $C^\infty(S_1 : \xi^\vee : -\lambda)$ is a dense subspace of $\mathcal{D}'(S_1 : \xi : \lambda)$, it suffices to prove the identity only for functions $\mu \in C^\infty(S_1 : \xi^\vee : -\lambda)$. Let μ be such a function and let $\phi \in \mathcal{D}(G, V_\xi)$. Then

$$\begin{aligned}&\big[\mathcal{A}(S_2 : S_1 : \xi : \lambda)\mu\big](\phi) \\ &\quad = \omega^{S_1}_{\xi,\lambda}(\mu)\Big(x \mapsto \int_{M_S}\int_{A_S}\int_{N_{S_2}}\int_{N_{S_1}\cap\overline{N}_{S_2}} a^{-\lambda+\rho_{S_1}}\xi(m^{-1})\phi(man\overline{n}x)\, d\overline{n}\, dn\, da\, dm\Big).\end{aligned}$$

It follows from Lemma B.7 that the right-hand side is equal to

$$\begin{aligned}&\int_K \Big(\mu(k), \int_{M_S}\int_{A_S}\int_{N_{S_2}}\int_{N_{S_1}\cap\overline{N}_{S_2}} a^{-\lambda+\rho_{S_1}}\xi(m^{-1})\phi(man\overline{n}k)\, d\overline{n}\, dn\, da\, dm\Big)\, dk \\ &\quad = \int_K\int_{M_S}\int_{A_S}\int_{N_{S_2}}\int_{N_{S_1}\cap\overline{N}_{S_2}} a^{-\lambda+\rho_{S_1}}\Big(\mu(k), \xi(m^{-1})\phi(man\overline{n}k)\Big)\, d\overline{n}\, dn\, da\, dm\, dk.\end{aligned}$$

Since the multiplication maps

$$\big(N_{S_1}\cap N_{S_2}\big)\times\big(\overline{N}_{S_1}\cap N_{S_2}\big) \to N_{S_2}, \quad \big(N_{S_1}\cap N_{S_2}\big)\times\big(N_{S_1}\cap\overline{N}_{S_2}\big) \to N_{S_1}$$

are diffeomorphisms with Jacobian equal to 1, we can rewrite this repeated integral as

$$\begin{aligned}&\int_K\int_{M_S}\int_{A_S}\int_{\overline{N}_{S_1}\cap N_{S_2}}\int_{N_{S_1}} a^{-\lambda+\rho_{S_1}}\Big(\mu(k), \xi(m^{-1})\phi(ma\overline{n}nk)\Big)\, dn\, d\overline{n}\, da\, dm\, dk \\ &\quad = \int_K\int_{M_S}\int_{A_S}\int_{N_{S_1}}\Big(\mu(mank), \int_{N_{S_2}\cap\overline{N}_{S_1}} \phi(\overline{n}mank)d\overline{n}\Big)\, dn\; da\, dm\, dk \\ &\quad = \int_G \Big(\mu(x), \int_{N_{S_2}\cap\overline{N}_{S_1}} \phi(\overline{n}x)d\overline{n}\Big)\, dx.\end{aligned}$$

This proves the proposition. □

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