Bernstein components for p-adic groups

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G: reductive group over a non-archimedean local field F
$\mathrm{Rep}(G)$: category of smooth complex G-representations

Bernstein decomposition

Direct product of categories $\mathrm{Rep}(G) = \prod_{\mathfrak{s}} \mathrm{Rep}(G)^{\mathfrak{s}}$
where $\mathfrak{s}$ is determined by a supercuspidal representation σ of a Levi subgroup M of G

We suppose that M and σ are given

Questions

- What does $\mathrm{Rep}(G)^{\mathfrak{s}}$ look like? Is it the module category of an explicit algebra?
- Can one classify $\mathrm{Irr}(G)^{\mathfrak{s}} = \mathrm{Irr}(G) \cap \mathrm{Rep}(G)^{\mathfrak{s}}$?
- Can one describe tempered/unitary/square-integrable representations in $\mathrm{Rep}(G)^{\mathfrak{s}}$?

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I. Bernstein components and a rough version of the new results

Bernstein components

$P = MU$: parabolic subgroup of G with Levi factor M
$I_P^G : \mathrm{Rep}(M) \to \mathrm{Rep}(P) \to \mathrm{Rep}(G)$: normalized parabolic induction

Definition

$\pi \in \mathrm{Irr}(G)$

- π is supercuspidal if it does not occur in $I_P^G(\sigma)$ for any proper parabolic subgroup P of G and any $\sigma \in \mathrm{Irr}(M)$
- Supercuspidal support $\mathrm{Sc}(\pi)$: a pair (M, σ) with $\sigma \in \mathrm{Irr}(M)$, such that π is a constituent of $I_P^G(\sigma)$ and M is minimal for this property

$X_{\mathrm{nr}}(M)$: group of unramified characters $M \to \mathbb{C}^\times$
$\mathcal{O} \subset \mathrm{Irr}(M)$: a $X_{\mathrm{nr}}(M)$-orbit of supercuspidal irreps
$\mathfrak{s} = [M, \mathcal{O}]$: G-association class of $(M, \mathcal{O})$

Definition

$\mathrm{Irr}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Irr}(G) : \mathrm{Sc}(\pi) \in [M, \mathcal{O}]\}$
$\mathrm{Rep}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Rep}(G) : \text{all irreducible subquotients of } \pi \text{ lie in } \mathrm{Irr}(G)^{\mathfrak{s}}\}$

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Iwahori-spherical component

I: an Iwahori subgroup of G

$$\mathrm{Rep}(G)^I = \{(\pi, V) \in \mathrm{Rep}(G) : V \text{ is generated by } V^I\}$$

The foremost example of a Bernstein component,
for $\mathfrak{s} = (M, X_{\mathrm{nr}}(M))$ where M is a minimal Levi subgroup of G

Theorem (Borel, Iwahori–Matsumoto, Morris)

$\mathcal{H}(G, I) := C_c(I \setminus G / I)$ with the convolution product

- $\mathrm{Rep}(G)^I$ is equivalent with $\mathrm{Mod}(\mathcal{H}(G, I))$
- $\mathcal{H}(G, I)$ is isomorphic with an affine Hecke algebra

When G is F-split, $M = T$ and these affine Hecke algebras are understood very well from Kazhdan–Lusztig

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Centre of a Bernstein component

$N_G(M)$ acts on $\mathrm{Rep}(M)$ by $(g \cdot \sigma)(m) = \sigma(g^{-1}mg)$

$$W(M, \mathcal{O}) = \{g \in N_G(M) : g \text{ stabilizes } \mathcal{O}\}/M$$

$\mathbb{C}[\mathcal{O}]$: ring of regular functions on the complex torus $\mathcal{O}$

Theorem (Bernstein, 1984)

The centre of $\mathrm{Rep}(G)^{\mathfrak{s}}$ is $\mathbb{C}[\mathcal{O}]^{W(M,\mathcal{O})}$

$\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O})] := \mathbb{C}[\mathcal{O}] \otimes_{\mathbb{C}} \mathbb{C}[W(M, \mathcal{O})]$ with multiplication from $W(M, \mathcal{O})$-action on $\mathcal{O}$:

$$(f \otimes w)(f' \otimes w') = f\, w(f') \otimes ww'$$

Main result (first rough version)

$\mathrm{Rep}(G)^{\mathfrak{s}}$ looks like $\mathrm{Rep}(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O})])$

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Approach with progenerators

Π: progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$
so $\Pi \in \mathrm{Rep}(G)^{\mathfrak{s}}$ is finitely generated, projective and $\mathrm{Hom}_G(\Pi, \rho) \neq 0$ for every $\rho \in \mathrm{Irr}(G)^{\mathfrak{s}}$

Lemma (from category theory)

$$\begin{array}{ccc} \mathrm{Rep}(G)^{\mathfrak{s}} & \longrightarrow & \mathrm{End}_G(\Pi) - \mathrm{Mod} \\ \rho & \mapsto & \mathrm{Hom}_G(\Pi, \rho) \\ V \otimes_{\mathrm{End}_G(\Pi)} \Pi & \leftarrow\!\shortmid & V \end{array}$$

is an equivalence of categories

Setup of talk

Investigate the structure and the representation theory of $\mathrm{End}_G(\Pi)$, for a suitable progenerator Π of $\mathrm{Rep}(G)^{\mathfrak{s}}$
Draw consequences for $\mathrm{Rep}(G)^{\mathfrak{s}}$

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Draw consequences for $\mathrm{Rep}(G)^{\mathfrak{s}}$

Comparison with types

$J \subset G$ compact open subgroup, $\lambda \in \mathrm{Irr}(G)$
Suppose: (J, λ) is a $\mathfrak{s}$-type, so

$$\mathrm{Rep}(G)^{\mathfrak{s}} = \{\pi \in \mathrm{Rep}(G) : \pi \text{ is generated by its } \lambda\text{-isotypical component}\}$$

Bushnell–Kutzko: $\mathrm{Rep}(G)^{\mathfrak{s}}$ is equivalent with $\mathcal{H}(G, J, \lambda)$-Mod

Consequences

- $\mathcal{H}(G, J, \lambda)$ and $\mathrm{End}_G(\Pi)$ are Morita equivalent
- In many cases $\mathrm{End}_G(\Pi)$ is Morita equivalent with an affine Hecke algebra

Problems:

- It is not known whether every Bernstein component admits a type
- Even if you have (J, λ), it can be difficult to analyse $\mathcal{H}(G, J, \lambda)$

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II. The structure of supercuspidal Bernstein components

based on work of Roche

Underlying tori

$\sigma \in \mathrm{Irr}(G)$ supercuspidal
$\mathcal{O} = \{\sigma \otimes \chi : \chi \in X_{\mathrm{nr}}(G)\}$
Covering $X_{\mathrm{nr}}(G) \to \mathcal{O} : \chi \mapsto \sigma \otimes \chi$

Example: $GL_2(F)$

χ_-: quadratic unramified character of $GL_2(F)$
It is possible that $\sigma \otimes \chi_- \cong \sigma$,
see the book of Bushnell–Henniart
Then $\mathbb{C}^\times \cong X_{\mathrm{nr}}(G) \to \mathcal{O}$ is a degree two covering

$X_{\mathrm{nr}}(G, \sigma) := \{\chi \in X_{\mathrm{nr}}(G) : \sigma \otimes \chi \cong \sigma\}$, a finite group
$X_{\mathrm{nr}}(G)/X_{\mathrm{nr}}(G, \sigma) \to \mathcal{O}$ is bijective, this makes $\mathcal{O}$ a complex algebraic torus (as variety)

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A progenerator

G^1: subgroup of G generated by all compact subgroups
$\mathrm{ind}_{G^1}^G(\mathrm{triv}, \mathbb{C}) = \mathbb{C}[G/G^1] \cong \mathbb{C}[X_{\mathrm{nr}}(G)]$

Lemma (Bernstein)

For $(\sigma, E) \in \mathrm{Irr}(G)$ supercuspidal
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Some endomorphisms of $E \otimes_{\mathbb{C}} \mathbb{C}[X_{\mathrm{nr}}(G)]$

- $\mathbb{C}[X_{\mathrm{nr}}(M)] \subset \mathrm{End}_G\big(E \otimes_{\mathbb{C}} \mathbb{C}[X_{\mathrm{nr}}(G)]\big)$, by multiplication operators
- for $\chi \in X_{\mathrm{nr}}(G, \sigma)$: $\sigma \cong \chi \otimes \sigma$
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Structure of endomorphism algebra

For $\chi, \chi' \in X_{\mathrm{nr}}(G, \sigma)$ there exists $\natural(\chi, \chi') \in \mathbb{C}^\times$ such that

$$\phi_\chi \circ \phi_{\chi'} = \natural(\chi, \chi') \phi_{\chi\chi'}$$

This gives a twisted group algebra $\mathbb{C}[X_{\mathrm{nr}}(G, \sigma), \natural]$ inside $\mathrm{End}_G\big(E \otimes_{\mathbb{C}} \mathbb{C}[X_{\mathrm{nr}}(G)]\big)$

Theorem (Roche)

$\mathrm{End}_G\big(E \otimes_{\mathbb{C}} \mathbb{C}[X_{\mathrm{nr}}(G)]\big) \cong \mathbb{C}[X_{\mathrm{nr}}(G)] \rtimes \mathbb{C}[X_{\mathrm{nr}}(G, \sigma), \natural]$

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Properties, from $\mathrm{Rep}(G)^{\mathfrak{s}}$

- $\mathrm{Irr}\big(\mathrm{End}_G\big(E \otimes_{\mathbb{C}} \mathbb{C}[X_{\mathrm{nr}}(G)]\big)\big) \longleftrightarrow X_{\mathrm{nr}}(G)/X_{\mathrm{nr}}(G, \sigma) \longleftrightarrow \mathcal{O}$
- $Z\big(\mathrm{End}_G\big(E \otimes_{\mathbb{C}} \mathbb{C}[X_{\mathrm{nr}}(G)]\big)\big) \cong \mathbb{C}[\mathcal{O}]$

Structure of endomorphism algebra

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Lemma (Roche, Heiermann)

If $\mathrm{Res}^G_{G^1}(\sigma)$ is multiplicity-free, then $\mathrm{End}_G\big(E \otimes_{\mathbb{C}} \mathbb{C}[X_{\mathrm{nr}}(G)]\big)$ is Morita equivalent with the commutative algebra $\mathbb{C}[\mathcal{O}]$

Maybe $\mathrm{Res}^G_{G^1}(\sigma)$ is always multiplicity-free?

Interpretation

$\mathrm{End}_G\big(E \otimes_{\mathbb{C}} \mathbb{C}[X_{\mathrm{nr}}(G)]\big)$ is the endomorphism algebra of a vector bundle over $\mathcal{O} \cong X_{\mathrm{nr}}(G)/X_{\mathrm{nr}}(G,\sigma)$

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III. Structure of non-supercuspidal Bernstein components

Motivated by work of Heiermann for classical p-adic groups

A progenerator

$P = MU$: parabolic subgroup of G, $(\sigma, E) \in \mathrm{Irr}(M)$ supercuspidal
$\mathfrak{s} = [M, \mathcal{O}]$

Theorem (Bernstein)

$\Pi := I_P^G\big(E \otimes_{\mathbb{C}} \mathbb{C}[X_{\mathrm{nr}}(M)]\big)$ is a progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$

This is deep, it relies on second adjointness

Via I_P^G, $\mathbb{C}[X_{\mathrm{nr}}(M)]$ embeds in $\mathrm{End}_G(\Pi)$

Lemma

$\rho \in \mathrm{Irr}(G)^{\mathfrak{s}}$. Suppose that the $\mathrm{End}_G(\Pi)$-module $\mathrm{Hom}_G(\Pi, \rho)$ has a $\mathbb{C}[X_{\mathrm{nr}}(M)]$-weight χ.
Then ρ has supercuspidal support $(M, \sigma \otimes \chi)$.

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Finite groups associated to $(M, \mathcal{O})$

- $X_{\mathrm{nr}}(M, \sigma)$, acting on $X_{\mathrm{nr}}(M)$
- $W(M, \mathcal{O}) = \{g \in N_G(M) : g \text{ stabilizes } \mathcal{O}\}/M$, acting on $\mathcal{O}$

Every $w \in W(M, \mathcal{O})$ lifts to a $\mathfrak{w} \in \mathrm{Aut}_{\mathrm{alg.var.}}(X_{\mathrm{nr}}(M))$

Lemma

There exists a group $W(M, \sigma, X_{\mathrm{nr}}(M)) \subset \mathrm{Aut}_{\mathrm{alg.var.}}(X_{\mathrm{nr}}(M))$ with

$$1 \to X_{\mathrm{nr}}(M, \sigma) \to W(M, \sigma, X_{\mathrm{nr}}(M)) \to W(M, \mathcal{O}) \to 1$$

Example

$G = GL_6(F), M = GL_2(F)^3, \sigma = \tau^{\boxtimes 3}$, then $X_{\mathrm{nr}}(M) \cong (\mathbb{C}^\times)^3$ and

either $W(M, \sigma, X_{\mathrm{nr}}(M)) = W(M, \mathcal{O}) \cong S_3$

or $\quad W(M, \sigma, X_{\mathrm{nr}}(M)) \cong (\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_3$

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Structure of $\mathrm{End}_G(\Pi)$

$\mathbb{C}(X_{\mathrm{nr}}(M))$: quotient field of $\mathbb{C}[X_{\mathrm{nr}}(M)]$, rational functions on $X_{\mathrm{nr}}(M)$

Main result (precise but weak version)

There exist a 2-cocycle $\natural$ of $W(M, \sigma, X_{\mathrm{nr}}(M))$ and an algebra isomorphism

$$\mathrm{End}_G(\Pi) \underset{\mathbb{C}[X_{\mathrm{nr}}(M)]}{\otimes} \mathbb{C}(X_{\mathrm{nr}}(M)) \cong \mathbb{C}(X_{\mathrm{nr}}(M)) \rtimes \mathbb{C}[W(M, \sigma, X_{\mathrm{nr}}(M)), \natural]$$

This only says something about $\mathrm{Rep}(G)^{\mathfrak{s}} \cong \mathrm{End}_G(\Pi)$-Mod
outside the tricky points of the cuspidal support variety $\mathcal{O}$

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Example: $SL_2(F)$

$M = T, \sigma = \text{triv}, \mathcal{O} = X_{\mathrm{nr}}(T) \cong \mathbb{C}^\times$
$W(M, \sigma, X_{\mathrm{nr}}(M)) = W(G, T) = \{1, s_\alpha\}$

Harish-Chandra's intertwining operator

$$I_{s_\alpha}(\chi) : I_P^G(\chi) \to I_P^G(\chi^{-1}), \quad f \mapsto \left[g \mapsto \int_{U_{-\alpha}} f(u s_\alpha g)\, \mathrm{d}u\right]$$

rational as function of $\chi \in X_{\mathrm{nr}}(T)$

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Singularities of J_{s_α}

at $\chi \in X_{\mathrm{nr}}(T)$ with $\chi(\alpha^\vee(\text{uniformizer of } F)) = q_F^{\pm 1}$
For these χ: $I_P^G(\chi)$ is reducible

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IV. Links with affine Hecke algebras

Sketch of an extended affine Hecke algebra

- Start with $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M,\mathcal{O})]$
- $W(M,\mathcal{O})$ contains a normal reflection subgroup $W(\Sigma_{\mathcal{O}})$
- Twist the multiplication in $\mathbb{C}[W(M,\mathcal{O})]$ by a 2-cocycle $\tilde{\natural}$ of $W(M,\mathcal{O})/W(\Sigma_{\mathcal{O}})$
- For every simple reflection $s_\alpha \in W(\Sigma_{\mathcal{O}})$, replace the relation $(s_\alpha + 1)(s_\alpha - 1) = 0$ in $\mathbb{C}[W(M,\mathcal{O})]$ by
$$(T_{s_\alpha} + 1)(T_{s_\alpha} - q_F^{\lambda(\alpha)}) = 0 \quad \text{for some } \lambda(\alpha) \in \mathbb{R}_{\geq 0}$$
- Adjust the multiplication relations between $\mathbb{C}[\mathcal{O}]$ and the T_{s_α}
- This gives an algebra $\tilde{\mathcal{H}}(\mathcal{O})$ with the same underlying vector space $\mathbb{C}[\mathcal{O}] \otimes \mathbb{C}[W(M,\mathcal{O})]$, $\quad \mathbb{C}[\mathcal{O}]$ is still a subalgebra

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Localization

We analyse the category of those $\mathrm{End}_G(\Pi)$-modules, all whose $\mathbb{C}[X_{\mathrm{nr}}(M)]$-weights lie in a specified subset $U \subset X_{\mathrm{nr}}(M)$
These are related to $\tilde{\mathcal{H}}(\mathcal{O})$-modules with $\mathbb{C}[\mathcal{O}]$-weights in $\{\sigma \otimes \chi : \chi \in U\}$

Polar decomposition

$$X_{\mathrm{nr}}(M) = \mathrm{Hom}(M/M^1, \mathbb{C}^\times) = \mathrm{Hom}(M/M^1, S^1) \times \mathrm{Hom}(M/M^1, \mathbb{R}_{>0})$$
$$= X_{\mathrm{unr}}(M) \times X_{\mathrm{nr}}^+(M)$$

Fix any $u \in \mathrm{Hom}(M/M^1, S^1)$ and define

$$U = W(M, \sigma, X_{\mathrm{nr}}(M))\, u\, X_{\mathrm{nr}}^+(M)$$
$$\tilde{U} = \text{ image of } U \text{ in } \mathcal{O} = W(M, \mathcal{O})\{\sigma \otimes u\chi : \chi \in X_{\mathrm{nr}}^+(M)\}$$

Advantage: by further reduction to $uX_{\mathrm{nr}}^+(M)$ we get rid of $X_{\mathrm{nr}}(M, \sigma)$

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Fix any $u \in \mathrm{Hom}(M/M^1, S^1)$ and define

$$U = W(M, \sigma, X_{\mathrm{nr}}(M))\, u\, X^+_{\mathrm{nr}}(M)$$
$$\tilde{U} = \text{ image of } U \text{ in } \mathcal{O} = W(M, \mathcal{O})\{\sigma \otimes u\chi : \chi \in X^+_{\mathrm{nr}}(M)\}$$

Advantage: by further reduction to $uX^+_{\mathrm{nr}}(M)$ we get rid of $X_{\mathrm{nr}}(M, \sigma)$

Localization

We analyse the category of those $\mathrm{End}_G(\Pi)$-modules, all whose $\mathbb{C}[X_{\mathrm{nr}}(M)]$-weights lie in a specified subset $U \subset X_{\mathrm{nr}}(M)$
These are related to $\tilde{\mathcal{H}}(\mathcal{O})$-modules with $\mathbb{C}[\mathcal{O}]$-weights in $\{\sigma \otimes \chi : \chi \in U\}$

Polar decomposition

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Main result

$\mathcal{O} = \{\sigma \otimes \chi : \chi \in X_{\mathrm{nr}}(M)\}, \mathfrak{s} = [M, \mathcal{O}]$
Π: progenerator of $\mathrm{Rep}(G)^{\mathfrak{s}}$

$\tilde{\mathcal{H}}(\mathcal{O})$ constructed by modification of $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O})]$
(with certain specific parameters $q_F^{\lambda(\alpha)}$)

$u \in \mathrm{Hom}(M/M^1, S^1)$, $U = W(M, \sigma, X_{\mathrm{nr}}(M))\, u\, X_{\mathrm{nr}}^+(M)$
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Theorem

There are equivalences between the following categories

- $\{\pi \in \mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}} : \mathrm{Sc}(\pi) \subset (M, \tilde{U})\}$ (fl : finite length)
- $\{V \in \mathrm{End}_G(\Pi) - \mathrm{Mod}_{\mathrm{fl}}$: all $\mathbb{C}[X_{\mathrm{nr}}(M)]$-weights of V in $U\}$
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Under a mild condition on the 2-cocycle $\tilde{\natural}$ involved in $\tilde{\mathcal{H}}(\mathcal{O})$ (conjecturally always fulfilled):

Corollary

There is an equivalence of categories between

$$\mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}} \quad \text{and} \quad \tilde{\mathcal{H}}(\mathcal{O}) - \mathrm{Mod}_{\mathrm{fl}}$$

Extras

The above equivalences of categories respect parabolic induction, temperedness and square-integrability of representations

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V. Classification of irreducible representations in $\mathrm{Rep}(G)^{\mathfrak{s}}$

Representations of affine Hecke algebras

- From the equivalence $\mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}} \cong \tilde{\mathcal{H}}(\mathcal{O}) - \mathrm{Mod}_{\mathrm{fl}}$, $\mathrm{Irr}(G)^{\mathfrak{s}}$ can be determined in terms of affine Hecke algebras
- The irreps of an affine Hecke algebra are known in principle, but their classification is involved

Replacing q_F by 1 in affine Hecke algebras

- $q_F = 1$-version of $\tilde{\mathcal{H}}(\mathcal{O})$: $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M,\mathcal{O}), \tilde{\natural}]$
- Its representation theory is easy, with Clifford theory

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Classification of tempered irreps

Assume that $\sigma \otimes u \in \mathrm{Irr}(M)$ is supercuspidal and unitary/tempered

Theorem

There exist (canonical?) bijections between the following sets

- $\{\pi \in \mathrm{Irr}(G)^{\mathfrak{s}} : \pi \text{ tempered}, \mathrm{Sc}(\pi) \in (M, \sigma \otimes u X_{\mathrm{nr}}^{+}(M))\}$
- $\{\tilde{V} \in \mathrm{Irr}(\tilde{\mathcal{H}}(\mathcal{O})) : \tilde{V} \text{ tempered}, \tilde{V} \text{ has a } \mathbb{C}[\mathcal{O}]\text{-weight in } \sigma \otimes u X_{\mathrm{nr}}^{+}(M)\}$
- $\{V \in \mathrm{Irr}(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M, \mathcal{O}), \tilde{\natural}]) : V \text{ tempered, with a } \mathbb{C}[\mathcal{O}]\text{-weight } \sigma \otimes u\}$
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$W(M, \mathcal{O})_{\sigma \otimes u}$ embeds in $W(M, \sigma, X_{\mathrm{nr}}(M))$
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Classification of irreducible representations

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- $\mathrm{Irr}\Big(\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M,\mathcal{O}),\tilde{\natural}]\Big)$ – under a mild condition on $\tilde{\natural}$
- $\big\{(\sigma',\rho) : \sigma' \in \mathcal{O}, \rho \in \mathrm{Irr}(\mathbb{C}[W(M,\mathcal{O})_{\sigma'},\tilde{\natural}])\big\}/W(M,\mathcal{O})$

The last item is also known as a twisted extended quotient

$$(\mathcal{O}//W(M,\mathcal{O}))_\natural$$

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Summary

For an arbitrary Bernstein block $\mathrm{Rep}(G)^{\mathfrak{s}}$ of a reductive p-adic group G:

- $\mathrm{Rep}_{\mathrm{fl}}(G)^{\mathfrak{s}}$ is equivalent with the category of finite length modules of an extended affine Hecke algebra $\tilde{\mathcal{H}}(\mathcal{O})$, whose $q_F = 1$-form is $\mathbb{C}[\mathcal{O}] \rtimes \mathbb{C}[W(M,\mathcal{O}),\tilde{\natural}]$
- Upon tensoring with $\mathbb{C}(X_{\mathrm{nr}}(M))$ over $\mathbb{C}[X_{\mathrm{nr}}(M)]$, or upon taking irreducible representations, $\mathrm{Rep}(G)^{\mathfrak{s}}$ becomes equivalent with $\mathbb{C}[X_{\mathrm{nr}}(M)] \rtimes \mathbb{C}[W(M,\sigma,X_{\mathrm{nr}}(M)),\natural]$

Questions / open problems

- Can one use the above to study unitarity of G-representations?
- Can the parameters $q_F^{\lambda(\alpha)}$ of $\tilde{\mathcal{H}}(\mathcal{O})$ be described in terms of σ or $\mathcal{O}$? Are the $\lambda(\alpha)$ integers?
- Does the 2-cocycle $\natural$ of $W(M,\sigma,X_{\mathrm{nr}}(M))$ descend to $W(M,\mathcal{O}) = W(M,\sigma,X_{\mathrm{nr}}(M))/X_{\mathrm{nr}}(M,\sigma)$?

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