Fourier analysis of Whittaker functions on a real reductive group

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Whittaker functions

Setting

- G real reductive group
- K maximal compact, $G = KAN_0$ Iwasawa decomposition
- $\chi : N_0 \to \mathrm{U}(1)$ unitary character, regular (!)

$$\text{i.e.: } \forall \alpha \in \Sigma(\mathfrak{n}_0, \mathfrak{a}) \text{ simple: } d\chi(e)|_{\mathfrak{g}_\alpha} \neq 0.$$

Whittaker functions

$$\mathcal{M}(G/N_0, \chi) := \{ f : G \xrightarrow{\text{meas}} \mathbb{C} \mid f(xn) = \chi(n)^{-1} f(x) \quad (x \in G, n \in N_0)\}$$

$$L^2(G/N_0, \chi) := \{ f \in \mathcal{M}(G/N_0, \chi) \mid |f| \in L^2(G/N_0)\}$$

- Left reg^r rep^n: $L = \mathrm{Ind}_{N_0}^{G}(\chi)$ is unitary

Whittaker Plancherel formula

Abstractly

- $\mathrm{Ind}_{N_0}^{G}(\chi) = \int_{\widehat{G}}^{\oplus} m_\pi \pi \, d\mu(\pi).$

Concrete realization

- Harish-Chandra, Announcement 1982.

  Details in Collected Papers Vol 5 (posthumous), 141- 307, eds. R. Gangolli, V.S. Varadarajan, Springer 2018.
  Final step not clear.

- N.R. Wallach, Independent treatment.

  Real reductive groups II, Acad. Press 1992.
  Erroneous estimate. Repair addressed in arXiv:1705.06787.

- Today: final step in HC through new inversion theorem.

Discrete part of decomposition

Discrete part

$\pi \in \widehat{G}$ (unitary dual) is said to appear discretely in $L^2(G/N_0, \chi)$ if it can be realized as a closed subrepresentation.

Theorem (HC, W)

If $\pi \in \widehat{G}$ appears discretely in $L^2(G/N_0, \chi)$, then it appears discretely in $L^2(G)$, i.e., it belongs to the discrete series of G.

Corollary

If $\pi \in \widehat{G}$ appears discretely in $L^2(G/N_0, \chi)$, then its infinitesimal character is real and regular, while $\mathrm{rk}(\mathfrak{k}) = \mathrm{rk}(\mathfrak{g})$.

This result is crucial for the separation of tempered spectra in the Whittaker Plancherel decomposition.

Schwartz functions

Define $\rho \in \mathfrak{a}^*$ by $\rho(X) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|_{N_0})$. Let $\mathfrak{Z} := \mathrm{center}\, U(\mathfrak{g})$

Definition (Schwartz space)

$\mathcal{C}(G/N_0, \chi)$: the space of $f \in C^\infty(G/N_0, \chi)$ s.t. $\forall u \in U(\mathfrak{g}), N \in \mathbb{N}$,

$$|L_u f(kan)| \leq C_{u,N}\,(1 + |\log(a)|)^{-N} a^{-\rho} \qquad (kan \in KAN_0).$$

For (τ, V_τ) a finite dimensional unitary representation of K,

$$\mathcal{C}(\tau, G/N_0, \chi) := (\mathcal{C}(G/N_0, \chi) \otimes V_\tau)^K$$

$$\mathcal{A}_2(\tau, G/N_0, \chi) := \{f \in \mathcal{C}(\tau, G/N_0, \chi) \mid \dim \mathfrak{Z} f < \infty\}.$$

Theorem (HC, W)

$$\mathcal{A}_2(\tau, G/N_0, \chi) = L^2_d(\tau, G/N_0, \chi).$$

The space is finite dimensional.

Parabolic subgroups

- $\Sigma = \mathrm{Roots}(\mathfrak{g}, \mathfrak{a}), \quad \Sigma^+ := \{\alpha \in \Sigma \mid \mathfrak{g}_\alpha \subset \mathfrak{n}_0\}, \;\Delta \subset \Sigma^+$ simple roots,
- $W(\mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$.
- $P_0 := Z_K(A) A N_0$, minimal psg.
- $\mathcal{P}(A)$: (finite) set of psg's $P \supset A$.
- $\mathcal{P}_{st} := \{P \in \mathcal{P}(A) \mid P \supset P_0\}$ (standard psg's).
- For P a psg: Langlands deco: $P = M_P A_P N_P$.

Associated parabolics

For $P, Q \in \mathcal{P}(A)$ define: $P \sim Q$ iff $\mathfrak{a}_P$ and $\mathfrak{a}_Q$ are $W(\mathfrak{a})$-conjugate. If so,

$$W(\mathfrak{a}_Q \mid \mathfrak{a}_P) := \{T \in \mathrm{Hom}(\mathfrak{a}_P, \mathfrak{a}_Q) \mid \exists w \in W(\mathfrak{a}) : T = w|_{\mathfrak{a}_P}\}$$

Parabolic induction and Whittaker integrals

For $P = M_P A_P N_P \in \mathcal{P}_{st}$, put $\mathcal{A}_{2,P} := \mathcal{A}_2(\tau|_{K_P}, M_P/M_P \cap N_0, \chi|_{M_P \cap N_0})$.

For $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$, $\mathrm{Re}\lambda >_P 0$,

$$\mathrm{Ind}_P^G(\cdot \otimes -\lambda) : \mathcal{A}_{2,P} \ni \psi \mapsto \mathrm{Wh}(P, \lambda, \cdot, \psi) \in \mathcal{A}_{\mathrm{temp}}(\tau, G/N_0, \chi)$$

Remark

The above Whittaker integral is essentially a finite sum of generalized matrix coefficients (defined by Jacquet integrals) of $\mathrm{Ind}_P^G(\sigma \otimes -\lambda \otimes 1)$, with $\sigma \in \widehat{G}_{\mathrm{ds}}$ appearing in $\mathcal{A}_{2,P}$. (Analogue of Eisenstein integral.)

Viewpoint

The Whittaker integral $\mathrm{Wh}(P, \lambda)$ is viewed as a (K-fixed) element of

$$\mathcal{A}_{\mathrm{temp}}(G/N_0, \chi) \otimes \mathrm{Hom}(\mathcal{A}_{2,P}, V_\tau)$$

depending holomorphically on $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$ in the region $\mathrm{Re}\lambda >_P 0$.

Classical Whittaker functions

Example

- $G = \mathrm{SL}(2, \mathbb{R})$, $\tau \in \mathrm{SO}(2)^\wedge$
- $\mathrm{Wh}(P, \lambda, \psi)$ is essentially a classical Whittaker function in the variable $a^{-\alpha} \in (0, \infty)$.
- satisfies 2^{nd} order ODE on $(0, \infty)$ with regular singularity at 0
- this ODE has irregular singularity at ∞;

For $a^{-\alpha} \to \infty$

- generic solution W of ODE:

$$\forall k \geq 0 : \quad |W(a)| \geq a^{-k\alpha} \qquad \text{(very fast growth)}.$$

- representation theory selects the special solution

$$\forall k \geq 0 : \quad \mathrm{Wh}(P, \lambda, \psi)(a) = \mathcal{O}(a^{k\alpha}) \;\; \text{(very fast decay)}.$$

Holomorphic extension

Theorem (W)

$\mathrm{Wh}(P,\lambda)$, *initially defined for* $\mathrm{Re}\lambda >_P 0$, *extends to entire holom*^{c} *function of* $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$ *with values in* $C^\infty(G/N_0,\chi) \otimes \mathrm{Hom}(\mathcal{A}_{2,P}, V_\tau)$.

Remark: HC: there exists a merom^c extension, regular on $i\mathfrak{a}^*_P$.

Theorem ($\sim$): Uniformly tempered estimates

Let $\varepsilon > 0$ *be suff*^{t}*ly small. If* $u \in U(\mathfrak{g})$ *then* $\exists C, N, r > 0$ *s.t.*

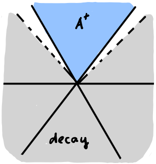
$$|Wh(P,\lambda,u;ka)| \leq C(1+|\lambda|)^N(1+|\log a|)^N e^{r|\mathrm{Re}\lambda||\log a|}a^{-\rho},$$

for all $k \in K$, $a \in A$, $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$ *with* $|\mathrm{Re}\lambda| < \varepsilon$.

- Bernstein-Sato type functional equation for Jacquet integrals.
- Uniformly moderate estimates.
- Wallach's method of improving estimates along max psg's, with parameters.

C-function, Normalized Whittaker function

- $W(P, \lambda)$ is $\mathfrak{Z}$-finite,
- top order asymptotic behavior of exp^l type along $\mathrm{cl}(A^+)$,
- rapid decay outside $\mathrm{cl}(A^+)$.



Lemma

Let $P \in \mathcal{P}_{st}$. For $\psi \in \mathcal{A}_{2,P}$, $\mathrm{Re}\lambda \in \mathfrak{a}_P^{*+}$, $m \in M_P$, $a \to \infty$ in A_P^+,

$$Wh(P, \lambda)(ma)\psi \sim a^{\lambda - \rho_P}[C_P(\lambda)\psi](m),$$

with $C_P(\lambda) \in \mathrm{End}(\mathcal{A}_{2,P})$, merom^c in $\lambda \in \mathfrak{a}_{P\mathbb{C}}^*$ (reg^r for $\mathrm{Re}\lambda \in \mathfrak{a}_P^{*+}$).

Definition (HC)

$$\mathrm{Wh}^\circ(P, \lambda) := \mathrm{Wh}(P, \lambda) \circ C_P(\lambda)^{-1}.$$

Functional equations, Maass-Selberg relations

Lemma (Functional equations: HC)

Let $P, Q \in \mathcal{P}_{st}$, $P \sim Q$. Then for all $s \in W(\mathfrak{a}_Q|\mathfrak{a}_P)$,

$$\mathrm{Wh}^\circ(Q, s\lambda) \circ C^\circ_{Q|P}(s, \lambda) = \mathrm{Wh}^\circ(P, \lambda), \quad (\lambda \in \mathfrak{a}^*_{P\mathbb{C}}),$$

with $C^\circ_{Q|P}(s, \lambda) \in \mathrm{Hom}(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$ a uniquely determined merom^c function of $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$.

Thm (Maass-Selberg relations, HC)

For all $s \in W(\mathfrak{a}_Q|\mathfrak{a}_P)$, $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$,

$$C^\circ_{Q|P}(s, -\bar{\lambda})^* \circ C^\circ_{Q|P}(s, \lambda) = I_{\mathcal{A}_{2,P}}$$

In particular, for $\lambda \in i\mathfrak{a}^*_P$, the map $C^\circ_{Q|P}(s, \lambda)$ is unitary.

Theorem (HC) $\lambda \mapsto \mathrm{Wh}^\circ(P, \lambda)$ is regular on $i\mathfrak{a}^*_P$.

Fourier transform

Dualized Whittaker function ($\sim$)

$$\mathrm{Wh}^*(P, \lambda, x) := \mathrm{Wh}^\circ(P, -\bar{\lambda}, x)^* \in \mathrm{Hom}(V_\tau, \mathcal{A}_{2,P}).$$

Fourier transform

For $f \in \mathcal{C}(\tau, G/N_0, \chi)$, $P \in \mathcal{P}_{st}$, $\lambda \in i\mathfrak{a}_P^*$,

$$\mathcal{F}_P f(\lambda) := \int_{G/N_0} \mathrm{Wh}^*(P, \lambda, x) f(x)\, dx \in \mathcal{A}_{2,P}.$$

Theorem ($\sim$)

$$\mathcal{F}_P : \mathcal{C}(\tau, G/N_0, \chi) \to \mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P},$$

continuous linearly.

Remark: HC proves this for $\mathcal{F}_P$ restricted to $C_c^\infty(\tau, G/N_0, \chi)$.

Proof

this follows from the uniformly tempered estimates.

Wave packets

Definition

For $P \in \mathcal{P}_{st}$, $\psi \in \mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$, $x \in G$,

$$\mathcal{W}_P\psi(x) := \int_{i\mathfrak{a}_P^*} \mathrm{Wh}^\circ(P, \lambda, x)\psi(\lambda)\, d\lambda.$$

Theorem ($\sim$)

$$\mathcal{W}_P : \mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P} \to \mathcal{C}(\tau, G/N_0, \chi)$$

is continuous linear.

Remark: HC proves this for $\mathcal{W}_P$ restricted to $C_c^\infty(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$.

Proof requires

- the uniformly tempered estimates
- theory of constant term with parameter
- families of type $\mathrm{II}_{\mathrm{hol}}(\Lambda)$ (as in previous joint work with Carmona and Delorme for reductive symmetric space G/H).

Plancherel formula

If $P, Q \in \mathcal{P}_{st}$, $P \sim Q$ then from the MS rel^s: $\|\mathcal{F}_P f(\lambda)\| = \|\mathcal{F}_Q f(\lambda)\|$.

Plancherel identity (HC)

With suitable normalization of the Lebesgue measures on $i\mathfrak{a}_P^*$,

$$\|f\|^2_{L^2(\tau, G/N_0, \chi)} = \sum_{P \in \mathcal{P}_{st}/\sim} \|\mathcal{F}_P f\|^2_{L^2(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}},$$

for f in the linear span $\mathcal{W} \subset \mathcal{C}(\tau, G/N_0, \chi)$ of the wavepackets $\mathcal{W}_Q(\psi)$, for $Q \in \mathcal{P}_{st}$, and $\psi \in C_c^\infty(i\mathfrak{a}_Q^*) \otimes \mathcal{A}_{2,Q}$.

Problem of the final step: Is $\mathcal{W}$ dense in $L^2(\tau, G/N_0, \chi)$?

Theorem ($\sim$)

Yes! More precisely, for $f \in \mathcal{C}(\tau, G/N_0, \chi)$ we have

$$f = \sum_{P \in \mathcal{P}_{st}/\sim} \mathcal{W}_P \circ \mathcal{F}_P(f).$$

Series expansion

Strategy for the final step: use Paley-Wiener shift argument and residue calculus as known from the theory of symmetric spaces (previous joint work with Schlichtkrull).

Let $P = P_0$ be minimal. Then $\mathrm{Wh}(P, \lambda) \in C^\infty(\tau, G, \chi) \otimes \mathcal{A}^*_{2,P}$ is holomorphic in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. The function is $\mathfrak{Z}$-finite, hence satisfies a cofinite system of differential equations, which has regular singularities at infinity in the direction of A^+.

Expansion at infinity

$$\mathrm{Wh}(P, \lambda) = \sum_{s \in W(\mathfrak{a})} \mathrm{Wh}_+(P, s\lambda) C_{P|P}(s, \lambda)$$

where $\mathrm{Wh}_+(P, \lambda) \in C^\infty(\tau, G, \chi) \otimes \mathcal{A}^*_{2,P}$ is merom^c in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, and

$$\mathrm{Wh}_+(P, \lambda)(a) = a^{\lambda - \rho} \sum_{\mu \in \mathbb{N}\Delta} a^{-\mu} \Gamma_\mu(\lambda), \qquad (a \in A),$$

with $\Gamma_\mu(\lambda) \in \mathrm{Hom}(\mathcal{A}_{2,P}, V_\tau)$ meromorphic, $\Gamma_0(\lambda)(\psi) = \psi(e)$.

Fourier inversion

Key theorem ($\sim$)

$$f(x) = \mathcal{T}_\eta(f)(x) := |W(\mathfrak{a})| \int_{i\mathfrak{a}^*+\eta} \mathrm{Wh}_+(P, \lambda, x)\, \mathcal{F}_P f(\lambda)\, d\lambda,$$

$\forall f \in C_c^\infty(\tau, G/N_0, \chi)$, $\forall x \in G$, *provided* $\eta \in \mathfrak{a}^*$, $\eta >>_{\bar{P}} 0$.

NB: For generic $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the function $\mathrm{Wh}_+(P, \lambda)$ is globally def^d on X, but may exhibit super exp^l growth in directions diff^t from $\mathrm{cl}(A^+)$.

Ideas of proof

- $\mathcal{T}_\eta : C_c^\infty(\tau, G/N_0, \chi) \to C^\infty(\tau, G/N_0, \chi)$.
- $\exists D \in \mathfrak{Z} : D\mathcal{T}_\eta = D\mathcal{T}_0 = D \circ \mathcal{W}_P \circ \mathcal{F}_P$.
- By PW shift $\eta \to \infty$ in $\bar{P}$-dominant direction: $\mathrm{supp} D\mathcal{T}_\eta f \subset K \mathrm{supp} f$.

$\implies \mathrm{rad}(D\mathcal{T}_\eta)$ is a differential operator on A

Proof of Fourier inversion

Ideas of proof:

- $\mathrm{rad}(D\mathcal{T}_\eta)$ differential operator commuting with $\mathrm{rad}(\mathfrak{Z})$
- By asymptotic analysis along A_P^+ : $D\mathcal{T}_\eta = D$ on $C_c^\infty(\tau, G/N_0, \chi)$.
- By Holmgren's uniqueness theorem for analytic PDO:

$$DT_\eta f = Df \implies D(\mathcal{T}_\eta f - f) = 0 \implies \mathcal{T}_\eta f - f = 0.$$

Residual kernels

By Fourier inversion, if $f \in C_c^\infty(\tau, G/N_0, \chi)$, $x \in G$,

$$f(x) = |W(\mathfrak{a})| \int_{i\mathfrak{a}^* + \eta} \mathrm{Wh}_+(P, \lambda, x)\, \mathcal{F}_P f(\lambda)\, d\lambda.$$

Shifting η towards zero and organizing residues, one gets

$$f(x) = \sum_{Q \in \mathcal{P}_{st}} [W : N_W(\mathfrak{a}_Q)]\ t(Q)\, T_Q^t f(x),$$

where

$$T_Q^t f(x) = \int_{i\mathfrak{a}_Q^* + \varepsilon_Q} \int_{G/N_0} K_Q^t(\lambda, x, y) f(y)\, dy\, d\lambda_Q.$$

- $\varepsilon_Q \in \mathfrak{a}_Q^{*+}$ sufficiently close to 0.
- $t : \mathcal{P}_{st} \to [0, 1]$ is a weight function describing a certain organization of residue shifts.

Conclusion

Theorem ($\sim$)

$$K_Q^t(\lambda, x, y) = \mathrm{Wh}^\circ(Q,\lambda)(x) \circ \mathrm{Wh}^*(Q,\lambda)(y) = K_{\mathcal{W}_Q \circ \mathcal{F}_Q}.$$

This identification relies on the Maass-Selberg relations. These also imply that the functions $\lambda \mapsto K_Q^t(\lambda, x, y)$ are regular on $i\mathfrak{a}_Q^*$, hence we may let $\varepsilon_Q \to 0$ and then:

Plancherel formula

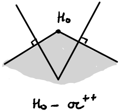
$$f(x) = \sum_{Q \in \mathcal{P}_{st}} [W : N_W(\mathfrak{a}_Q)] t(Q)\, \mathcal{W}_Q \mathcal{F}_Q f(x).$$

- $[W : N_W(\mathfrak{a}_Q)] t(Q)$ gives the weight by which Q contributes to its class in $\mathcal{P}_{st}/\sim$.

Bonus: Paley-Wiener theorem

Definition

A function $f \in \mathcal{C}(\tau, G/N_0, \chi)$ is said to be *cone supported* (notation $\mathcal{C}_{cs}$) if $\exists H_0 \in \mathfrak{a}$ s.t.



$$\mathrm{supp} f \subset K \exp(H_0 - \mathfrak{a}^{++}) N_0.$$

Lemma

If $f \in \mathcal{C}_{cs}(\tau, G/N_0, \chi)$, then $\forall u \in U(\mathfrak{g})$ $\forall m > 0$ $\exists C > 0$:

$$\|L_u f(ka)\| < C\, e^{-m|\log a|} \qquad (\forall k \in K, a \in A).$$

Paley-Wiener theorem

Let $P = P_0$ (minimal). Then ${}^u\mathcal{F}_P$ (unnormalized) is injective on $\mathcal{C}_{cs}(\tau, G/N_0, \chi)$. The image of this space under ${}^u\mathcal{F}_P$ equals the space $\mathrm{PW}(\chi, \tau)$ of holomorphic functions $\varphi : \mathfrak{a}^*_{\mathbb{C}} \to \mathcal{A}_{2,P}$ satisfying

- *certain estimates of Paley–Wiener type;*
- *relations of Arthur–Campoli type.*

Thank you