Special automorphic forms on exceptional groups

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Overview

- **This talk is about**: Analogue of Siegel modular forms to certain exceptional algebraic groups

Siegel modular forms

- Special automorphic forms for the group Sp_{2n}
- They have a "robust" Fourier expansion
- Are connected to arithmetic

Modular forms on exceptional groups

- Theory initiated by Gross-Wallach, studied by Gan-Gross-Savin, Loke, Weissman, and the speaker
- Special automorphic forms for the groups $G_2, D_4, F_4, E_{6,4}, E_{7,4}, E_{8,4}$
- **Theorem 1**: They have a "robust" Fourier expansion
- **Theorem 2**: Examples of said modular forms that are "arithmetic" in the sense that they have $\overline{\mathbf{Q}}$-valued Fourier expansions

Siegel modular forms

The symplectic group

- $\mathrm{Sp}_{2n} = \{g \in \mathrm{GL}(2n) : {}^t g \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix} g = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}\}$
- $Sp_{2n} \supseteq U(n) \simeq \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a + ib \in U(n) \right\}$

The symmetric space

- $S_n := n \times n$ symmetric matrices
- $\mathcal{H}_n = \{Z = X + iY : X, Y \in S_n(\mathbf{R}), Y > 0\}$ the Siegel upper half-space
- $\mathcal{H}_n \simeq \mathrm{Sp}_{2n}(\mathbf{R})/U(n)$ the symmetric space

$\mathrm{Sp}_{2n}(\mathbf{R})$ acts on $Sp_{2n}(\mathbf{R})/U(n) = \mathcal{H}_n$ via

$$g \cdot Z = (aZ + b)(cZ + d)^{-1}$$

if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $n \times n$ block form.

Siegel modular forms: continued

Siegel modular form of weight $\ell > 0$:

Definition and basic properties

- $f : \mathcal{H}_n \to \mathbf{C}$ **holomorphic** such that
- $f((aZ+b)(cZ+d)^{-1}) = \det(cZ+d)^{\ell} f(Z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ some congruence subgroup of $\mathrm{Sp}_{2n}(\mathbf{Z})$
- **Fourier expansion**:

$$f(Z) = \sum_{T \in S_n(\mathbf{Q}), T \geq 0} a_f(T) e^{2\pi i \,\mathrm{tr}(TZ)}$$

with $a_f(T) \in \mathbf{C}$ and $T \geq 0$ means "T is positive semi-definite".

- If $n = 1$, these are classical modular forms for SL_2

If f a Siegel modular form, can consider $f \in H^0(\Gamma \backslash \mathcal{H}_n, \mathcal{L}^{\ell})$

- a global section of a holomorphic line bundle $\mathcal{L}^{\ell}$ on $\Gamma \backslash \mathcal{H}_n$

Siegel modular forms automorphically

$$\varphi : \mathrm{Sp}_{2n}(\mathbf{Q})\backslash \mathrm{Sp}_{2n}(\mathbf{A}) \to \mathbf{C} \text{ with}$$

The definition

1. $\varphi(gk) = z(k)^{-\ell}\varphi(g)$ for all $k \in U(n)$, $z : U(n) \overset{\det}{\to} U(1) \subseteq \mathbf{C}^\times$
2. $\mathcal{D}_{CR,\ell}\varphi \equiv 0$: φ annihilated by linear differential operator $\mathcal{D}_{CR,\ell}$ so that f_φ on $\mathcal{H}_n$ satisfies the Cauchy-Riemann equations

The Fourier expansion

$$\begin{aligned}\varphi_f\left(\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} Y^{1/2} & \\ & Y^{-1/2}\end{pmatrix}\right) &= \varphi_f(n(X)m) \\ &= \sum_{T \in S_n(\mathbf{Q}), T \geq 0} a_\varphi(T) e^{2\pi i\,\mathrm{tr}(TX)} e^{-2\pi\,\mathrm{tr}(TY)}\end{aligned}$$

where $iY = m \cdot i$ in $\mathcal{H}_n$ and $a_\varphi(T) \in \mathbf{C}$.

Automorphically

- $\pi = \otimes_v \pi_v$ with π_∞ a holomorphic discrete series representation

Classical modular forms

Suppose G a reductive $\mathbf{Q}$-group;

- $K \subseteq G(\mathbf{R})$ a maximal compact subgroup.
- If $G(\mathbf{R})/K$ is a Hermitian tube domain
  - e.g. $G = \mathrm{GSp}_{2n}, \mathrm{GU}(n,n), \mathrm{SO}(2,n), GE_{7,3}$
- Then there is a notion of "modular forms" on G

Modular forms

Automorphic forms for $G(\mathbf{A})$ that give rise to sections of holomorphic line bundles on $G(\mathbf{R})/K$

- These automorphic forms have a robust notion of Fourier expansion
- Are closely connected to arithmetic: E.g., there is a basis of the space of modular forms, all of whose Fourier coefficients are in $\overline{\mathbf{Q}}$

Usefulness of Holomorphic modular forms

1. **Classical generating functions**: Theta functions whose Fourier coefficients are representation numbers of quadratic forms
2. **Construction of cusp forms**: The Ikeda lift and its generalizations
3. **L-values**: Particularly useful for applications to Deligne's conjecture on special values of L-functions
4. ***p*-adic *L*-functions**: p-adic interpolation of L-values and applications
5. **Geometric generating functions (Kudla program)**: Generating functions whose coefficients are certain cohomology classes

Contents

Most G do not have $G(\mathbf{R})/K$ Hermitian

- **Classical groups**: Fixing the Dynkin type, can change the real form so that G/K Hermitian: E.g., $\mathrm{GU}(n,n)$ (type A), $\mathrm{SO}(2,n)$ (type B and D), Sp_{2n} (type C)
- **Exceptional groups**: For G of Dynkin type G_2, F_4, E_8, no real form has G/K Hermitian

Question

Given an exceptional Dynkin type, can one single out a class of special automorphic forms, similar to the holomorphic modular forms on tube domains?

Answer

Gross-Wallach: Look at G with

1. $G(\mathbf{R})$ possessing discrete series (rank K equals rank G)
2. $\pi = \pi_f \otimes \pi_\infty$ with π_∞ a discrete series
3. Moreover, take π_∞ with smallest possible GK-dimension among the discrete series and simplest minimal K-type

The groups

The reductive groups

- Exceptional: G split of type G_2 or F_4, or $E_{6,4}$, $E_{7,4}$, $E_{8,4}$ with real rank four
- Classical: G isogenous to $\mathrm{SO}(4, n)$ with $n \geq 3$

The maximal compact subgroups:

- Suppose G is adjoint, of the above type;
- $K \subseteq G(\mathbf{R})$ the maximal compact subgroup
- $K = (\mathrm{SU}(2) \times L)/\mu_2$ for some L
- e.g. $G = G_2$, $K = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mu_2$; the first $\mathrm{SU}(2)$ is the "long-root" $\mathrm{SU}(2)$
- So, always a normal $\mathrm{SU}(2)$
- **Compare** If H reductive group over $\mathbf{R}$ with Hermitian symmetric space, than K_H (maximal compact of $H(\mathbf{R})$) has a normal $\mathrm{U}(1)$

The modular forms

Gross-Wallach: The groups G have **quaternionic discrete series**

- There is a discrete series π_ℓ of the groups G above, with minimal $K = (\mathrm{SU}(2) \times L)/\mu_2$-type $Sym^{2\ell}(V_2) \boxtimes 1 =: V_\ell$

Modular forms of weight ℓ

Definition 1 (Gan-Gross-Savin)

Suppose $\ell \geq 1$ an integer and $\varphi : \pi_\ell \to \mathcal{A}(G)$ a $G(\mathbf{R})$-equivariant morphism. Then φ is a modular form of weight ℓ.

Equivalent definition

Definition 2

Suppose $\ell \geq 1$ is an integer and $F : G(\mathbf{Q})\backslash G(\mathbf{A}) \to V_\ell^\vee$ an automorphic form satisfying $F(gk) = k^{-1} \cdot F(g)$ for all $g \in G(\mathbf{A})$ and $k \in K \subseteq G(\mathbf{R})$. Then F is a modular form of weight ℓ if $D_\ell F = 0$ for a certain linear differential operator D_ℓ.

Suppose $F : G(\mathbf{R}) \to V_\ell^\vee$ satisfies $F(gk) = k^{-1} \cdot F(g)$ for all $g \in G(\mathbf{R})$ and $k \in K$.

- Cartan involution: $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.
- Let $\{X_i\}_i$ be a basis of $\mathfrak{p}$ and $\{X_i^*\}_i$ the dual basis of $\mathfrak{p}^*$.

A formal operator

Define $\widetilde{D}F : G(\mathbf{R}) \to V_\ell^\vee \otimes \mathfrak{p}^*$ via

$$\widetilde{D}F(g) = \sum_i (X_i F)(g) \otimes X_i^*.$$

- $\mathfrak{p} = V_2 \boxtimes W$ as representation of $\mathrm{SU}(2) \times L$
- $V_\ell^\vee \otimes \mathfrak{p} = Sym^{2\ell-1}(V_2) \boxtimes W \oplus Sym^{2\ell+1}(V_2) \boxtimes W$

The operator D_ℓ

Define $pr : V_\ell^\vee \otimes \mathfrak{p}^* \twoheadrightarrow Sym^{2\ell-1}(V_2) \boxtimes W$ and

$$D_\ell := pr \circ \widetilde{D}.$$

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The Heisenberg parabolic

Let G be a group of the quaternionic type, $P = HN \subseteq G$ the Heisenberg parabolic. The unipotent radical N is two-step: $Z \subseteq N$ is one-dimensional and $N/Z = W$ is abelian

Examples

- $G = \mathrm{SO}(4,n)$, $P = (\mathrm{GL}_2 \times \mathrm{SO}(2,n-2))N$ with $N/Z = V_2 \otimes V_n$, V_n quadratic space of signature $(2, n-2)$
- $G = G_2$, $P = \mathrm{GL}_2\, N$ with $N/Z = Sym^3(V_2) \otimes \det(V_2)^{-1}$
- $G = F_4$, $P = \mathrm{GSp}_6\, N$ with N/Z the third fundamental representation of GSp_6
- $G = E_{6,4}$, $P = HN$ with $H \approx \mathrm{GU}(3,3)$ and N/Z the 20-dimensional (twisted) exterior cube representation
- $G = E_{7,4}$, $P = HN$ with H of type D_6 and N/Z the 32-dimensional half-spin representation
- $G = E_{8,4}$, $P = GE_{7,3}N$ with N/Z the 56-dimensional representation of GE_7

The Fourier expansion of modular forms

- Suppose F is a modular form of even weight ℓ on G.
- Consider $F_Z(g) = \int_{[Z]} F(zg)\,dz$, the constant term of F along Z.

Denote by $n : W(\mathbf{R}) \simeq N/Z$, $\langle\, , \,\rangle$ the H-invariant symplectic form on W

Theorem 3

Suppose $\ell \geq 1$ is fixed. For $\omega \in W(\mathbf{Q})$ satisfying "$\omega \geq 0$", there are explicit functions $\mathcal{W}_\omega : H(\mathbf{R}) \to V_\ell^\vee$ with the following property: If F is a modular form on G of weight ℓ, with there are Fourier coefficients $a_F(\omega) \in \mathbf{C}$ so that for $x \in W(\mathbf{R})$ and $h \in H(\mathbf{R})$

$$F_Z(n(x)h) = F_N(h) + \sum_{\omega \geq 0} a_F(\omega) e^{2\pi i \langle \omega, x\rangle} \mathcal{W}_\omega(h)$$

where F_N is the constant term of F along N.

Theorem 3 continued

Moreover,

$$F_N(h) = \Phi_F(h)v_\ell + \beta_F \frac{\zeta(\ell+1)}{\zeta(\ell)} v_0 + \overline{\Phi_F(h)} v_{-\ell}$$

for some holomorphic modular form Φ_F of weight ℓ on $H(\mathbf{R})$ and $\beta_F \in \mathbf{C}$. Here $\{v_\ell, v_{\ell-1}, \ldots, v_0, \ldots, v_{-\ell}\}$ is a certain basis of $V_\ell^\vee$.

Surprising corollary:

Corollary 4

Suppose $\ell \geq 1$ and F a modular form of weight ℓ for G. If F is bounded as a function on $G(\mathbf{A})$ then F is cuspidal.

Fourier coefficients

Suppose F is a modular form on G of even weight ℓ.

Definition 5

The Fourier coefficients of F are the numbers $a_F(\omega)$, β_F, and the Fourier coefficients of Φ_F

Definition 6

Suppose $R \subseteq \mathbf{C}$ is a ring. One says F has Fourier coefficients in R if all the Fourier coefficients are of F are in $R \subseteq \mathbf{C}$.

- **Warning**: Unlike the case of holomorphic modular forms on GL_2, the algebraicity of the Hecke eigenvalues does not imply the algebraicity of the Fourier coefficients.
- There is no *a priori* reason to expect any modular form to have Fourier coefficients in a small ring (e.g., $\mathbf{Z}, \mathbf{Q}, \overline{\mathbf{Q}}$)
- Definitions above crucially use Theorem on Fourier expansion as input

Proof of Theorem 3

Fix $\chi : N(\mathbf{R}) \to \mathbf{C}^\times$ a unitary character.

Proof of Theorem 3

The proof of Theorem 3 proceeds by making a complete and explicit analysis of all moderate growth functions $\mathcal{W}_\chi : G(\mathbf{R}) \to V_\ell^\vee$ satisfying

1. $\mathcal{W}_\chi(gk) = k^{-1} \cdot \mathcal{W}_\chi(g)$ for all $k \in K$ and $g \in G(\mathbf{R})$
2. $\mathcal{W}_\chi(ng) = \chi(n)\mathcal{W}_\chi(g)$ for all $n \in N(\mathbf{R})$ and $g \in G(\mathbf{R})$
3. $D_\ell \mathcal{W}_\chi(g) \equiv 0$.

The analysis implies

Multiplicity one

dim $Hom_{N(\mathbf{R})}(\pi_\ell, \chi) \leq 1$ if χ nontrivial, and is 0 unless $\chi \geq 0$.

For generic χ, this multiplicity one result was previous proved by Wallach (via a different method)

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Modular forms with algebraic Fourier coefficients

Theorem 7

There are examples of modular forms with Fourier coefficients in small rings:

1. *On $E_{8,4}$, the minimal and next-to-minimal modular forms (weight 4 and weight 8) have Fourier coefficients in $\mathbf{Q}$. These modular forms have many Fourier coefficients equal to 0. Uses key input from work of W.T. Gan and G. Savin.*
2. *On $E_{6,4}$, there is a weight 4 modular form with all Fourier coefficients in $\mathbf{Z}$. This example is "distinguished" but not "singular", and is closely connected to "arithmetic invariant theory".*
3. *On $\mathrm{Spin}(8)$ and G_2, there are nonzero cusp forms of arbitrarily large weight with all Fourier coefficients in $\overline{\mathbf{Q}}$. Examples constructed using the theta correspondence $\mathrm{SO}(4,4) \leftrightarrow \mathrm{Sp}_4$.*

- The Theorem says that some modular forms on exceptional groups possess "surprising" arithmeticity.

There is θ-lift:

- $\mathrm{Sp}_4 \leftrightarrow \mathrm{SO}(4,4)$
- Start with holomorphic Siegel modular cusp forms f on $\mathrm{Sp}(4)$ of weight ℓ, get $\theta(f)$ on $\mathrm{SO}(4,4)$
- Rallis: $\theta(f)$ on $\mathrm{SO}(4,4)$ is a cusp form.

Theorem 8

With appropriate Schwartz-Bruhat data for Weil representation, $\theta(f)$ is a nonzero weight ℓ modular form. Moreover, the Fourier coefficients of $\theta(f)$ are neatly described in terms of the Fourier coefficients of the f. In particular, the Fourier coefficients of $\theta(f)$ can be made to be nonzero algebraic integers.

- Analogue of special θ-lift $\widetilde{SL_2} \leftrightarrow \mathrm{SO}(2,n)$: Doi-Naganuma, Niwa, Shintani, Kudla, Oda, Rallis-Schiffmann

Fourier coefficients of $\theta(f)$

- $W = V_2 \otimes V_4 = e \otimes V_4 \oplus f \otimes V_4$, V_4 quadratic space of signature $(2,2)$, e, f basis of V_2
- If $\omega = e \otimes v_e + f \otimes v_f$, set

$$S(\omega) = \frac{1}{2} \begin{pmatrix} (v_e, v_e) & (v_e, v_f) \\ (v_e, v_f) & (v_f, v_f) \end{pmatrix}.$$

Fourier coefficient formula

If ω is primitive, then $a_{\theta(f)}(\omega) = a_f(S(\omega))$.

- If ω is not primitive, then there is a slightly more complicated formula for $a_{\theta(f)}(\omega)$
- Formula implies that $a_{\theta(f)}(\omega)$ are nonzero algebraic integers if the a_f's are

Corollary 9

Suppose $\ell \geq 16$ is even. Then there are nonzero cuspidal modular forms of weight ℓ on G_2 with all Fourier coefficients in $\overline{\mathbf{Q}}$.

Proof of Corollary.

1. Embed $\iota : G_2 \hookrightarrow SO(4,4)$
2. Set $F = \iota(\theta(f))$
3. One can show that F is still cuspidal modular form of weight ℓ
4. Using crucially the **positive semi-definiteness condition** for the nonvanishing of Fourier coefficients of modular forms, can check that the Fourier coefficients of F are **finite sums** of Fourier coefficients of $\theta(f)$, thus still algebraic integers

□

Remark: Rallis-Schiffmann, Li-Schwermer constructed different cohomological cusp forms on G_2 via $G_2 \subseteq SO(3,4) \leftrightarrow \widetilde{SL_2}$.

Recall:

- H: The Levi subgroup of the Heisenberg parabolic subgroup of G
- W: The abelianized unipotent radical of the Heisenberg parabolic subgroup of G

Rank of Fourier coefficients

- The action of $H(\mathbf{C})$ on $W(\mathbf{C}) = N/Z(\mathbf{C})$ has four nonzero orbits
- If $\omega \neq 0$, $\omega \in W$, one say ω has rank $1, 2, 3$ or 4 depending on the orbit
- The open orbit of H on W consists of those ω of rank four
- The elements of rank one in W form the most degenerate nonzero orbit

Fact If F a modular form on G then F is a cusp form if and only if $F_N = 0$ and $a_F(\omega) = 0$ for all ω of rank $1, 2$ and 3.

Heisenberg Eisenstein series

Suppose $G = E_{8,4}$, P Heisenberg parabolic.

$$\nu : P \to \mathrm{GL}_1$$

generating the character group of P. On $G = E_{8,4}$,

$$|\nu(p)|^{29} = \delta_P(p)$$

for $p \in P$. Suppose

- $\ell \geq 1$ even
- $f(g, \ell; s) \in Ind_{P(\mathbf{A})}^{G(\mathbf{A})}(|\nu|^s)$, certain $Sym^{2\ell}(V_2)$-valued section.
- $E(g, \ell; s) = \sum_{\gamma \in P(\mathbf{Q}) \backslash G(\mathbf{Q})} f(\gamma g, \ell; s)$ absolutely convergent for $Re(s) > 29$.
- If $s = \ell + 1$ in range of absolute convergence, $E(g, s = \ell + 1)$ a **modular form of weight** ℓ for G

Question

Does $E(g, s = \ell + 1)$ have rational Fourier coefficients?

Next to minimal

Motivated by work of Gross-Wallach on continuation of quaternionic discrete series, take $\ell = 8$ and $G = E_{8,4}$.

Proposition

The Eisenstein series $E(g, \ell = 8; s)$ is regular at $s = 9$ (even though outside the range of absolute convergence), and defines square integrable weight 8 modular form at this point.

Set

$$\theta_{ntm}(g) = E(g, \ell = 8; s = 9)$$

Theorem 10 (Savin)

The spherical constituent of the degenerate principal series $Ind_{P(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}(|\nu|^9)$ is "small", i.e., many twisted Jacquet modules are 0. Consequently, the rank three and rank four Fourier coefficients of θ_{ntm} are 0.

More on next-to-minimal modular form

On split E_8

- Analogous "next-to-minimal" automorphic form is spherical
- Studied by Michael B. Green-Stephen D. Miller-Pierre Vanhove
- Also by Dmitry Gourevitch-Henrik P. A. Gustafsson-Axel Kleinschmidt-Daniel Persson-Siddhartha Sahi

Theorem 11

The weight 8 modular form θ_{ntm} has rational Fourier coefficients.

Proof.

1. Savin's result gives vanishing of rank three and four Fourier coefficients
2. Explicit computation (outside range of abs. convergence) gives rationality of rank 1 and rank 2 Fourier coefficients
3. Constant term analyzed using work of H. Kim on weight 8 singular modular form on $GE_{7,3}$

Explicit computation of θ_{ntm}

1. Define special $Sym^{2\ell}(V_2)$-valued Eisenstein series $E_\ell(g)$ on $SO(3, 4k+3)$
2. Prove that the constant term θ_{ntm} from $E_{8,4}$ down to $SO(3,11)$ is $E_8(g)$
3. **Theorem**: the $E_\ell(g)$ have rational Fourier coefficients (in a precise sense)
4. The Fourier coefficients of $E_8(g)$ can be identified with rank 1 and rank 2 Fourier coefficients of θ_{ntm}.

To prove the $E_\ell(g)$ have rational Fourier coefficients:

Jacquet integral

Explicit computation of certain Archimedean Jacquet integral

$$\int_{V_{2,4k+2}(\mathbf{R})} e^{2\pi i (v,x)} f_\ell(wn(x))\, dx.$$

The minimal modular form on $E_{8,4}$

- Defined by Gan as special value

$$\theta_{min}(g) = E(g, \ell = 4; s = 5)$$

(outside the range of absolute convergence). Gan proves that it is square integrable automorphic form

- Analogue on split E_8 studied by Ginzburg-Rallis-Soudry

Theorem 12

θ_{min} is a modular form of weight 4 with Fourier coefficients in $\mathbf{Z}$.

1. Local results (Savin) imply rank 2,3,4 Fourier coefficients are 0
2. Kazhdan-Polischuk: up to constant multiple, the rank 1 FCs are divisor sums $\sigma_4(n)$
3. Theorem: when θ_{min} is normalized to have integer rank one Fourier coefficients, the constant term also has integer coefficients.

A distinguished modular form

Globally, there is an arithmetic invariant on the orbits of $H(\mathbf{Q})$ on $W(\mathbf{Q})$:

$$q : W(\mathbf{Q})^{rk=4} \to \mathbf{Q}^\times/(\mathbf{Q}^\times)^2 = \{ \text{ quadratic etale extensions of } \mathbf{Q}\}.$$

Fact: If F a modular form on G, $\omega \in W(\mathbf{Q})$ and $q(\omega) > 0$ then $a_F(\omega) = 0$. In other words, only ω corresponding to imaginary quadratic fields can have associated nonzero Fourier coefficients

Fix an imaginary quadratic extension $E/\mathbf{Q}$. Associated to E, there is a group G_E over $\mathbf{Q}$ of type $E_{6,4}$.

Theorem 13

There is a weight 4 modular form θ_E on G_E with Fourier coefficients in $\mathbf{Z}$ such that θ_E has nonzero Fourier coefficients of all ranks and

1. *If $\omega \in W(\mathbf{Q})^{rk=4}$ and $q(\omega) \in \mathbf{Q}^\times/(\mathbf{Q}^\times)^2$ does not represent E, then the Fourier coefficient $a_{\theta_E}(\omega) = 0$*

Proof of Theorem 13:

1. Define G_E, which is simply-connected of type $E_{6,4}$
2. Carefully embed G_E in $E_{8,4}$ via $\iota_E : G_E \to E_{8,4}$
3. Define $\theta_E = \iota_E^*(\theta_{min})$, the pull-back of the modular form generating the minimal representation on $E_{8,4}$
4. The Fourier coefficients of θ_E can then be computed from those of θ_{min}
5. θ_{min} only has nonzero Fourier coefficients for the **most degenerate** ω, those ω of rank 1
6. This vanishing of $a_{\theta_{min}}(\omega)$ imposes a strong arithmetic condition on the Fourier coefficients of θ_E.

Thank you

Thank you for your attention!