Intrinsic Diophantine Approximation on Reductive Homogeneous Spaces

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Based on joint work with

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Homogeneous spaces : general set-up

- Let G be a non-compact locally compact second countable group,

- Γ a discrete lattice subgroup of G, i.e. the space $\Gamma \setminus G$ admits a G-invariant probability measure,

- and let H be a closed non-compact subgroup of G. (Below H will be unimodular, so G/H has a G-invariant measure).

- Assume that almost every Γ-orbit in G/H is dense. (Below Γ will be ergodic on G/H).

- Let $\|g\|$ denote a suitable gauge on G,

- and let ρ denote a suitable metric on $X = G/H$,

Approximation on homogeneous spaces

• We would like to analyze the efficiency of approximation by Γ-orbits on the homogeneous space $X = G/H$, as follows.

• Let $x \in X$ and suppose that for all for all $x_0 \in X$, for all $\epsilon < \epsilon_0$ we can solve the inequality : $\rho(\gamma x, x_0) < \epsilon$,

• with $\gamma \in \Gamma$ satisfying $\|\gamma\| < B\left(\frac{1}{\epsilon}\right)^{\zeta}$ (and ϵ_0, ζ, B depending on the pair (x, x_0)).

• Thus $\zeta(x, x_0)$ gives a rate of approximation of a general point $x_0 \in X = G/H$ by the Γ-orbit of x.

• Define the approximation exponent $\kappa(x, x_0)$ as the infimum of $\zeta(x, x_0)$ such that the foregoing inequalities have solutions satisfying the bound stated above, for all sufficiently small ϵ.

Basic problems : quantifying the denseness exponent

• Problem I : finiteness. Determine when does there exist a finite constant $\zeta(\Gamma, G/H)$ which bounds the rate of approximation by almost every lattice orbit. Determine when does there exist a finite uniform bound for every lattice orbit, without exception.

• Problem II : Explicit bounds. Establish an upper bound and a lower bound for the rate of approximation, and explicate their dependence on G, H, and Γ explicitly.

• Problem III : Optimality. Formulate a simple, easily verifiable and widely applicable criterion for when the upper and lower bounds coincide, giving rise to the optimal rate of approximation by lattice orbits on the homogeneous space.

Example : Integral Diophantine approximation on affine homogeneous rational algebraic varieties

- Let $G = G(\mathbb{R})$ be an algebraic subgroup of $SL_n(\mathbb{R})$, defined over $\mathbb{Q}$,

- Let Γ be a lattice subgroup of $G(\mathbb{R})$, for example the lattice of integral points $G(\mathbb{Z})$ in $G(\mathbb{R})$.

- Consider an affine rational subvariety $X \subset \mathbb{R}^n$ which is invariant and homogeneous under the G-action, so that $X = G/H$,

- Choose norms on $\mathbb{R}^n$ and $M_n(\mathbb{R})$, and restrict them to X and to G.

- Consider the Diophantine inequality $\|\gamma x - x_0\| < \epsilon$, with $\gamma \in \Gamma = G(\mathbb{Z})$ satisfying the norm bound $\|\gamma\| \leq B\epsilon^{-\zeta}$, and the Diophantine approximation exponent $\kappa(x, x_0) = \inf \zeta(x, x_0)$.

- $\kappa(x, x_0)$ is a $\Gamma \times \Gamma$-invariant function, hence almost surely a constant κ when the action is ergodic. Thus κ depends on G, Γ and X only.

Some examples of rational homogeneous varieties

- The unit sphere: $x^2 + y^2 + z^2 = 1$
- The one-sheeted hyperboloid : $x^2 + y^2 - z^2 = 1$
- The two-sheeted hyperboloid : $x^2 + y^2 - z^2 = -1$
- Variety of unimodular matrices : $\det \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix} \right) = xw - yz = 1$
- General rational quadratic varieties.
- Variety of orthogonal matrices : $U^{tr} U = I$
- Affine spaces.
- reductive group varieties, principal homogeneous spaces, symmetric varieties, spherical varieties.......and much more.

Lang's problems : intrinsic rational Diophantine approximation

- In his 1965 "Report on Diophantine approximation" Lang raised the problem of establishing the Diophantine approximation properties of typical points on rational homogeneous algebraic varieties embedded in $\mathbb{R}^d$.

- Lang raised specifically the problems of establishing an analog of Dirichlet's principle (namely an exponent of Diophantine approximation), an analog of Khinchin's theorem, and also an analog of W. Schmidt's (then recent) solution-counting theorem.

- The approximation process is allowed to use only rational points on the variety itself, so it amounts to intrinsic rational Diophantine approximation.

Intrinsic approximation on algebraic varieties

- Lang's problems have the following concrete formulation.

- Consider an affine homogeneous algebraic variety X defined over $\mathbb{Q}$, namely an affine variety acted upon transitively by a linear algebraic group G defined over $\mathbb{Q}$.

- Denote by $\mathrm{X}(\mathbb{R}) \subset \mathbb{R}^N$ the set of real solutions, and by $\mathrm{X}(\mathbb{Q}) \subset \mathbb{Q}^N$ the set of rational solutions. Analyze quantitatively the system of intrinsic Diophantine inequalities :

  $$\|x - r\| \le \epsilon \quad \text{and} \quad D(r) \le \epsilon^{-\kappa+\eta} \quad \text{with} \quad 0 < \epsilon < \epsilon_x \ , \quad \eta > 0$$

  with $x \in \mathrm{X}(\mathbb{R})$ and $r \in \mathrm{X}(\mathbb{Q})$, and $D(r)$ the reduced denominator,

- namely, if $r = (\frac{p_1}{q_1}, \dots, \frac{p_N}{q_N})$ with p_i/q_i reduced fractions, then $D(r) = \mathrm{lcm}(q_1, \dots, q_N)$.

Intrinsic DA with integrality constraints

- Every affine rational variety has a dense set of rational points, but may of course have only a discrete set of integral points (e.g. the unit spheres, Riemannian symmetric spaces).
- We therefore consider a natural generalization of Lang's problem, which includes both integral and rational DA, and which comes up in several important applications.
- Namely, consider intrinsic DA on a rational homogeneous variety by a constrained set of rational points, constituting a proper subset of the set of all rational points on the variety, still dense.
- The simplest example is to approximate using the set $\mathrm{X}(\mathbb{Z}[1/p])$ consisting of rational solutions whose denominators are restricted to be powers of p, where p is prime. Or more generally using $\mathrm{X}(\mathbb{Z}[1/p_1, \dots, 1/p_s \cdots])$, with $p_1, \dots, p_s \cdots$ distinct primes.

Intrinsic DA as lattice-orbit approximation problems

- The Γ-orbit approximation problem on the homogeneous $X = G/H$ formulated initially encompasses all of the examples considered above, as follows.

- Integral Diophantine approximation : $H(\mathbb{R}) \subset G(\mathbb{R})$ real algebraic group def$/\mathbb{Q}$, unimodular and non-compact, $\Gamma = G(\mathbb{Z})$ the lattice of integral points, $X = G/H$ a non-compact affine rational variety.

- p-adic-integral Diophantine approximation : $G = G(\mathbb{R}) \times G(\mathbb{Q}_p)$, $H = G(\mathbb{Q}_p)$ (isotropic over the p-adic field $\mathbb{Q}_p$), $\Gamma = G(Z[\frac{1}{p}])$ an irreducible lattice, $X = G(\mathbb{R}) \times G(\mathbb{Q}_p)/G(\mathbb{Q}_p) = G(\mathbb{R})$ a (possibly compact) rational variety (a group variety in this case).

- S-integral Diophantine approximation : More generally, S a finite set of primes,G an S-algebraic $\mathbb{Q}$-group, H an (isotropic) S-algebraic subgroup $\Gamma = G(\mathbb{Z}[S^{-1}])$ an irreducible lattice.

- Rational Diophantine approximation : $G(\mathbb{A})$ and $H(\mathbb{A})$ algebraic groups of rational adéles (G simply connected), $\Gamma = G(\mathbb{Q})$ the lattice subgroup of rational points.

Scope of the problem

The natural context for intrinsic Diophantine approximation on homogeneous varieties to consider includes:

- arbitrary affine homogenous varieties for linear algebraic groups,
- defined over an arbitrary number field K,
- using K-rational points constrained by arbitrarily prescribed integrality conditions on their denominators,
- achieving simultaneous Diophantine approximation over several completions of the field K,
- with the approximation rates being given by explicit exponents.
- *Obviously, some assumptions are going to be necessary...........*

Lower bound for the integral DA exponent

• A basic ingredient in solving the Diophantine inequalities which approximate a point $x_0 \in X$ is to estimate how many orbit points γx are available in a compact neighborhood Ω of x_0 in the homogeneous affine variety $X = G/H$.

• Define the empirical growth parameter for such points :

$$a = \sup_{\Omega \text{ compact}} \limsup_{T \to \infty} \frac{\log |\{\gamma \in \Gamma\,;\, \|\gamma\| < T,\, \gamma x \in \Omega\}|}{\log T}$$

• Let $H_T = \{h \in H\,;\, \|h\| < T\}$. For $x = [H]$, these orbit points are those in $\Gamma \cap (\Omega \cdot H)$ with norm at most T. Their growth obeys the bounds satisfied by the volume growth of the norm balls H_T in the stability subgroup H, namely

$$T^{a-\eta} \ll m_H(H_T) \ll T^{a+\eta} \quad , \quad a > 0$$

• so the number of points available for the approximation obeys

$$|\{\gamma \in \Gamma\,;\, \|\gamma\| < T,\, \gamma x \in \Omega\}| \ll T^{a+\eta}$$

• Recall G and H are unimodular, and assume that the local behavior of the invariant measure m_X on X is given by

$$m_X(\{\|x - x_0\| < \epsilon\} \sim \epsilon^d \,.$$

• This is certainly the case when $X = G/H$ is a real affine variety, with $d = \dim_{\mathbb{R}}(G/H)$. Then in a compact neighborhood Ω of $x_0 \in X$ there will exist an ϵ-separated net of size roughly ϵ^{-d}.

• There are at most $T^{a+\eta}$ pigeons (points of the form γx in Ω, with $\|\gamma\| < T$), and roughly ϵ^{-d} pigeon holes (disjoint ϵ-balls in Ω.)

• If the Diophantine approximation in Ω is to be successful, then clearly $\epsilon^{-d} \ll T^{a+\eta}$, i.e. $T \gg \epsilon^{-d/a-\eta}$, and we can conclude :

• Theorem 1. (Ghosh-Gorodnik-N.) d/a is a lower bound on the Diophantine exponent κ : it is impossible to approximate points on $X = G/H$ as above by points in generic lattice orbits any faster, namely using lattice elements of smaller size.

Deriving an upper bound for the integral DA exponent

- An upper bound on the Diophantine exponent requires different considerations, namely

- Spectral estimates in the automorphic representation of G on $L^2(\Gamma \setminus G)$, restricted to H, leading to an effective mean ergodic theorem for the action of H on the probability space $\Gamma \setminus G$,

- Dynamical arguments utilizing the speed of equidistribution of H-orbits in $\Gamma \setminus G$, in the form of a "shrinking target argument".

- Effective duality arguments translating a rate of equidistribution for the H-orbits in the action of H on $\Gamma \setminus G$ to a rate of approximation of Γ-orbits the action of Γ on G/H.

The effective mean ergodic theorem

• To formulate our spectral estimate, consider the intersection of norm balls with the stability group H, namely $H_T = \{h \in H \,;\, \|h\| < T\}$.

• Consider the invariant probability measure $m_{\Gamma \setminus G}$ on $Y = \Gamma \setminus G$ and define averaging operators $\pi_Y(\beta_T) : L^2(\Gamma \setminus G) \to L^2(\Gamma \setminus G)$, given by

$$\pi_Y(\beta_T)f(y) = \frac{1}{m_H(H_T)} \int_{h \in H_T} f(yh)dm_H(h)\ ,\ y \in \Gamma \setminus G\,.$$

• Assume that the quantitative mean ergodic theorem for the averaging operators $\pi^0_Y(\beta_T)$ holds, namely :

• there exists $\theta > 0$ such that

$$\|\pi_Y(\beta_T)f - \int_Y fdm\|_{L^2(\Gamma\setminus G)} \le C_\eta m_H(H_T)^{-\theta+\eta}\|f\|_{L^2(\Gamma\setminus G)}$$

for suitable C_η and $t \ge t_0$, and every $\eta > 0$, arbitrarily small.

Spectral gaps

• The foregoing estimate holds in great generality. Indeed, the representation of G is $L^2_0(\Gamma \setminus G)$, typically has a spectral gap.

• In that case, a non-amenable closed subgroup $H \subset G$ will typically also have the property that the unitary representation of H in $L^0_2(\Gamma \setminus G)$ has a spectral gap as well.

• If H is (say) any non-compact simple subgroup of G, then the spectral gap implies the estimate stated above on the operator norms on $\pi_Y(\beta_T)$. There are many other cases where such an estimate holds.

• The effective mean ergodic theorem for the averages β_T was established in N-1998 and elaborated further in Gorodnik-N-2006.

• Note also that the MET for $\pi_Y(\beta_T)$ implies of course the ergodicity of H on $\Gamma \setminus G$. By the duality principle for homogeneous spaces, the ergodicity of Γ on G/H follows, and in particular, almost every Γ-orbit in $X = G/H$ is dense.

An upper bound for the Diophantine exponent

• Summarizing our situation : $G \subset SL_n(\mathbb{R})$ is an algebraic group defined over $\mathbb{Q}$, H a closed algebraic subgroup, both unimodular and non-compact, and Γ a discrete lattice in G.

• the rate of growth of lattice points in the norm balls $H_T = \{h \in H\, ;\, \|h\| < T\}$ is the same as the rate of volume growth of H_T (namely $\sim T^a$),

• $d = \dim_{\mathbb{R}}(G/H)$ denote the real dimension the affine subvariety $X = G/H \subset \mathbb{R}^n$,

• and we assume that the averages β_T supported on H_T satisfy the quantitative mean ergodic theorem in $L^2(\Gamma \setminus G)$ with rate θ.

• Theorem 2.(Ghosh-Gorodnik-N) Under the assumptions stated above, the Diophantine exponent satisfies the upper bound $\kappa \leq \frac{d}{2\theta a}$.

• Conclusion : if $2\theta = 1$ then the lower and upper bounds for the Diophantine exponent coincide !

Best possible rate of approximation

- Corollary I. If the rate of convergence in the mean ergodic theorem for the averaging operators β_T acting on $L^2_0(\Gamma \setminus G)$, is as fast as the inverse of the square root of the volume of H_T, then the rate of Diophantine approximation of Γ-orbits on the variety $X = G/H$ is best possible, and given by $\kappa = \frac{d}{a}$, the a-priori pigeon-hole bound !

- Corollary II. If the stability group H is simple (and non-compact), and the restriction of the automorphic representation $\pi^0_{G/\Gamma}$ to H is weakly contained in the regular representation of H, then the Diophantine exponent of the irreducible lattice Γ of G in its action on G/H is best possible, and is given by $\kappa = \frac{d}{a}$.

- The fact that weak-containment in the regular representation imlies the norm estimate given by the parameter $\theta = 1/2$ in this case is a reflection of the Kunze-Stein phenomenon.

- It is a surprising and most useful fact that the foregoing optimal spectral estimate for H holds in considerable generality for a large class of triples (G, H, Γ), with H a non-amenable group.

Subgroup temperedness

• Definition : H is tempered in (Γ, G), if the unitary rep. of H in $L_0^2(\Gamma \setminus G)$ is weakly contained in λ_H.

• Definition : H is tempered in G, if the restriction of any irreducible non-trivial unitary rep. of G to H is weakly contained in λ_H. This implies that H is tempered in (Γ, G) for every lattice Γ.

• There are several general principles which can be used to establish subgroup temperedness for a remarkably wide variety of cases:

• Kazhdan's argument : When $H = SL_2(\mathbb{R}) \ltimes \mathbb{R}^2 \subset G$, any unitary representation of G without $\mathbb{R}^2$-invariant unit vectors is tempered when restricted to $SL_2(\mathbb{R})$.

• Let G be a simple property T group, H a semisimple subgroup. There are universal pointwise upper bounds on the matrix coefficients of G in general unitary representations. These can be restricted to a subgroup H, and when they are in $L^{2+\eta}(H)$ the restricted representation is tempered.

• For example, $L^{2+\epsilon}$-integrability for restrictions holds for the images $\sigma_n(H)$ of all the irreducible representations $\sigma_n : SL_2(\mathbb{R}) \to SL_n(\mathbb{R})$, $n \geq 3$, as observed originally by Margulis (1995).

• This can be greatly generalized. For example, given a simple group H, and an irreducible finite dimensional representation $\sigma : H \to SL_n(\mathbb{R})$, $\sigma(H)$ is a tempered subgroup of $SL_n(\mathbb{R})$, if the rep σ has sufficiently large and generic highest weight.

• Irreducible non-trivial unitary representations of simple groups have matrix coefficients in $L^{2k}(G)$ for some k. Restricting a representation of the direct product G^k to the diagonally embedded copy of G yields matrix coefficients which are in $L^{2+\eta}(G)$. Therefore the diagonally embedded subgroup G is a tempered subgroup of G^k.

• For some lattices in $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ and their low level congruence subgroups, the Selberg eigenvalue conjecture is known to hold. Then $L^2_0(\Gamma \setminus G)$ is a tempered representation of G, namely it is weakly contained in λ_G. This holds for example for $SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z}[i]) \subset SL_2(\mathbb{C})$.

Previous results : isometric actions on compact homogeneous spaces

• For the sphere S^2 viewed as a compact homogeneous space of $SO_3(\mathbb{R})$ or $SU_2(\mathbb{C})$, an upper bound on the exponent of p-adic Diophantine approximation for a suitable lattice in the unit group of a quaternion algebra is a consequence of the celebrated Lubotzky-Phillips-Sarnak construction (1985-6).

• This approach has been subsequently extended to actions of certain lattices on odd-dimensional spheres by Clozel (2005). The work of Oh (2005) on spectral estimates for Hecke operators on homogeneous spaces of compact Lie groups also yields an upper bound on Diophantine exponents.

• For approximation by ALL rational points on the sphere, an upper bound of the exponent was established using elementary methods by Schmutz (2008).

• Kleinbock and Merrill (2013) have established the best possible exponent for rational approximation on the unit spheres in any dimension $n \geq 2$, together with an analog of Khinchine's theorem (and even sharper results). More recently [FKMS] considered general quadratic varieties.

• The proof views the rational points on the sphere as integral points in a suitable light-cone, homogeneous under the isometry group of an indefinite form. Then a method of duality in homogeneous dynamics developed by Dani, Margulis and Klenbock-Margulis is utilized, reducing the problem to analyzing cusp excursions of a suitable diagonalizable group.

• This approach is both elegant and successful, but its scope is unclear. Whether it applies beyond quadratic varieties, or to approximation by points of constrained denominators, or to quantitative results going beyond Khinchin's theorem, such as an analog of Schmidt's theorem, remains to be seen.

From denseness to equidistribution

- Let ν be a Radon measure on $X = G/H$, abs. cont. w.r.t. the G-inv. meas.

- Let R_h (where $h \in \mathbb{N}$ or $h \in \mathbb{R}_+$) be a collection of locally finite subsets of $X = G/H$, namely the intersection of each R_h with any compact subset of X is finite.

- We will assume that $\cup_h R_h$ is a countable dense subset of X.

- The collection of sets R_h is said to be equidistributed in X (with respect to ν, and the rate $v(h)$) if there exists a function $v(h) \to \infty$ such that for every bounded domain $\Omega \subset X$ with boundary of measure zero

$$\frac{|R_h \cap \Omega|}{v(h)} \to \nu(\Omega) \quad \text{as } h \to \infty$$

Locally uniform discrepancy of dense point sets

- In this case, we define the discrepancy of R_h in Ω as

$$\mathcal{D}(R_h, \Omega) := \left| \frac{|R_h \cap \Omega|}{v(h)} - \nu(\Omega) \right|. \quad (1)$$

- The discrepancy measures the deviation of the sets R_h from being fairly deposited in the set Ω.

- Taking $\Omega = B(x, \delta) \subset X$, the condition $\lim_{h\to\infty} \mathcal{D}\left(R_h, B(x, \delta)\right) = 0$ means that the pointwise discrepancy of the family of sets R_h at scale δ at the point x vanishes asymptotically.

- A bound of the form $\mathcal{D}(R_h, B(x, \delta)) = o\left(v(h)^{-\tau}\right)$ (with $\tau > 0$) as x varies in a compact subset of X given an effective rate of the vanishing of the locally uniform pointwise discrepancy of the family R_h at the scale δ.

Discrepancy of $\mathbb{Z}[\frac{1}{p}]$-rational points in $G = SL_2(\mathbb{R})$

- Example: consider establishing an effective bound for the pointwise discrepancy of $\mathbb{Z}[\frac{1}{p}]$-points in $G = SL_2(\mathbb{R})$, at arbitrarily small scales.
- Order the elements of $SL_2\left(\mathbb{Z}[\frac{1}{p}]\right) \subset SL_2(\mathbb{R})$ according to the size of the denominator, namely let

$$\mathcal{B}_h = \left\{ r \in SL_2\left(\mathbb{Z}[\frac{1}{p}]\right) \ ; \ D(r) \le p^h \right\} .$$

- For any bounded open domain $\Omega \subset SL_2(\mathbb{R})$, and any fixed h, the intersection of $\mathcal{B}_h$ and Ω is a finite set, and the union of these intersections over $h \in \mathbb{N}_+$ is dense in Ω.
- We let W_h be the closure in $SL_2(\mathbb{Q}_p)$ of $\mathcal{B}_h$, and choose $v(h)$ to be the Haar measure of the compact open subset W_h of $SL_2(\mathbb{Q}_p)$.

Effective estimate of multi-scale uniform pointwise discrepancy

- The equidistribution result underlying the definition of discrepancy, namely

$$\frac{|\mathcal{B}_h \cap \Omega|}{v(h)} \longrightarrow m_G(\Omega) \ , \ h \to \infty$$

  is, in this case, a consequence of the well-known equidistribution result for Hecke points together with a duality argument.

- We would like an effective pointwise estimate jointly in both h and the scale δ, which amounts to a solution-counting result for the Diophantine problem (an analog of W. Schmidt's theorem).

- Theorem 3. (Ghosh-Gorodnik-N, 2016). For arbitrarily small scales $\delta > h^{-\alpha}$ the following effective uniform discrepancy bound holds

$$|\mathcal{B}_h \cap B(g,\delta)| = v(h) m_G(B(g,\delta)) + O\left(v(h)^{-\tau}\right) \ ,$$

  with $\tau > 0$ explicit, and independent of $g \in G$ and $\delta > 0$.

The duality principle

• Our discussion is an instance of the general method of duality in homogenous dynamics, which aims to establish properties of the Γ-orbits in G/H by using properties of the H-orbits in $\Gamma \setminus G$.

• When aiming to establish a rate of approximation for Γ-orbits in G/H when ordered by a norm, the dual property which is most pertinent is the existence of a rate in the mean ergodic theorem for ball averages on H acting on $\Gamma \setminus G$.

• A general quantitative duality principle has been developed in joint work with Alex Gorodnik. It yields conclusions which are considerably more precise than just the existence of a rate of approximation by Γ-orbits.

• For example, it is possible to prove quantitative mean and pointwise ergodic theorems for the discrete averages supported on orbit points when ordered by a norm, although the optimality of the rate is compromised.

The method of duality

• These results sharpen earlier results by Gorodnik and Barak Weiss (2007) on equidistribution of orbit points ordered by a norm.

• The quantitative method of duality applies in considerable generality, for all locally compact groups, closed subgroups H, and discrete lattices Γ,

• subject only to natural and necessary assumptions about

1. the gauge $\|g\|$ on G and the distance ρ on G/H,
2. the local behavior of the invariant measure $m_{G/H}$ on the homogeneous space $G/H = X$.
3. the volume growth of the sets H_T and the lattice points in their vicinity,
4. the spectral theory of H in $L^2_o(\Gamma \setminus G)$.