Categorical Lie Groups and Lie's Third Theorem for Locally Exponential Lie Algebras

Lie's Third Theorem states that each finite-dimensional Lie algebra is the Lie algebra of a Lie group, but in infinite dimensions, there are Lie algebras which cannot be the Lie algebra of any Lie group. The first example of such a Banach-Lie algebra was found by van Est and Korthagen in 1964. In our talk we shall explain this phenomenon and how to overcome it by considering categorified Lie groups instead of ordinary Lie groups.

A **categorical group**, also called a **weak 2-group** or a **gr-category**, is a category $\mathcal{G}$, together with two functors

$$\begin{aligned} \otimes &: \mathcal{G} \times \mathcal{G} \to \mathcal{G} && \text{(multiplication)} \\ \text{inv} &: \mathcal{G} \to \mathcal{G} && \text{(inversion)} \end{aligned}$$

and a unit object $\mathbb{1}$ in $\mathcal{G}$, satisfying the corresponding associativity and invertability relations for ordinary groups, but only *up to natural isomorphisms.* For instance, the associativity relation is given by natural isomorphisms

$$\alpha_{g,h,k} : (g \otimes h) \otimes k \to g \otimes (h \otimes k)$$

which we call **associator**. Non-trivial associators allow to treat not strictly associative group multiplications in a coherent way (and in a certain sense this is the sole generalisation that categorical groups allow for in comparison to ordinary groups). Typical examples of non-trivial associators arise from non-trivial group 3-cocycles.

The failure of a Lie algebra to come from a Lie group arises when integrating the Lie algebra to a local Lie group and then solving an associativity constraint. In the approach of van Est and Korthagen, this constraint stems from the existence of Banach-Lie groups with non-vanishing second homotopy group. It thus may be solved by passing to a version of a 2-fold connected cover of a Lie group. This passage leads us naturally (i.e. in terms of 3-cocycles, describing this 2-cover) to the wold of categorical Lie groups. Moreover, the geometrical interpretation of the procedure that we present leads us to bundle gerbes and principal 2-bundles.