

Dunkl operators, special functions and harmonic analysis

Paderborn University

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Abstracts

(as of August 8, 2016)

1 Invited talks

Some facts about Dunkl's kernel of type A

Bechir Amri

University of Tunis El Manar, Tunisia

As originally constructed by Dunkl, the investigation of the intertwining operator between Dunkl operators and ordinary derivatives is still ambiguous. Except for a few cases, finding an explicit form remains an open problem. One of the most important contributions is due to Rösler who proved that the Dunkl intertwining operator attached with a finite reflection group and nonnegative multiplicity function is positive and can be expressed as an integral transform with positive kernel. Dunkl kernels $E_k(\cdot, y)$, first defined by Dunkl by means of the intertwining operator, are the joint eigenfunctions of the Dunkl operators. The basic fact derived from the main result of Rösler is the validity of the representation of E_k by an integral of Laplace type,

$$E_k(x, y) = \int e^{\langle x, y \rangle} d\mu_x(y),$$

where μ_x is a compactly supported probability measure.

This talk focuses on Dunkl kernels associated to root systems of type A. We present in the first part a recursive integral representation for the Dunkl-Bessel function. The main ingredients of the proof are the recursive integral representation for Jack polynomials due to Okounkov and Olshanski and the limit transition between the Opdam hypergeometric function and the Dunkl-Bessel function. The second part of the talk is about the Dunkl kernel for the A_2 -root system, where we give a new formula for E_k as an integral involving the ordinary modified Bessel function. This provides another proof of a Dunkl formula for the intertwining operator.

References

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- [2] B. Amri, *On the integral representations for Dunkl kernels of type A_2* , To appear in Journal of Lie theory.
- [3] C. F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc., 311 (1989), no. 1, 167-183.
- [4] C. F. Dunkl, *Integral kernels with reflection group invariance*, Canadian J. Math. 43 (1991), 1213–1227.
- [5] C. F. Dunkl, *Intertwining operators associated to the group S_3* , Trans. Amer. Math. Soc., 347 (1995), no. 9, 3347–3374.
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Relaxation dynamics in scaled Dunkl processes

Sergio Andraus

The University of Tokyo, Japan

Dunkl processes are the Markov processes obtained from generalizing the heat equation by replacing the spatial partial derivatives in the Laplacian with Dunkl operators. A simple observation indicates that under an appropriate scaling, the probability distribution of a Dunkl process converges to a static distribution at time tending to infinity. This implies that we can regard the Dunkl process as a

relaxation process, and we study the mechanisms which drive the relaxation. We find that the relaxation occurs in two stages, driven by the drift and jump terms in the Kolmogorov backward equation. Our results stem from the derivation of approximating forms for the Dunkl kernel.

Random walks on affine buildings

Jean-Philippe Anker

Université d'Orléans, France

We are interested in the global behavior of isotropic random walks on affine buildings. In joint work with Bruno Schapira (Marseille) and Bartosz Trojan (Wrocław), we have obtained sharp global estimates (same upper and lower bound) for the transition density of nearest neighbor isotropic random walks on thick affine buildings of type \tilde{A}_2 . Our talk will consist in a survey about random walks on affine buildings and in an outline of our approach, which is based on the inversion formula for the spherical Fourier transform.

Double affine Hecke algebras, character varieties of knots and the Jones polynomial

Yuri Berest

Cornell University, USA

Let G be a complex reductive algebraic group. In this talk, we will discuss a general conjecture that there is a natural action of the double affine Hecke algebra of type G on a (quantized) character variety of the complement of a knot in S^3 . We will explain a motivation and give an explicit construction in the case of $SL_2(\mathbb{C})$, using the Dunkl-Cherednik and Demazure-Lusztig operators of type $C^\vee C_1$. We will show that the classical limit ($q = \pm 1$) of our conjecture follows from (and essentially reduces to) a known conjecture in knot theory due to G. Brumfiel and H. Hilden (1990). The main implication is the existence of 3-variable polynomial knot invariants that specialize to the famous Jones polynomial and its colored versions introduced by Witten, Reshetikhin and Turaev.

The talk is based on joint work with P. Samuelson.

References

- [1] Y. Berest and P. Samuelson, *Double affine Hecke algebras and generalized Jones polynomials*, *Compositio Math.* (2016), doi:10.1112/S0010437X16007314 (to appear); arXiv:1402.6032.
- [2] Y. Berest and P. Samuelson, *Affine cubic surfaces and character varieties of knots*, in preparation.

Hypergeometry, the Torus, and Representations of the Symmetric Groups

Charles F. Dunkl

University of Virginia, USA

For each irreducible module of the symmetric group \mathcal{S}_N , there is a basis of parametrized nonsymmetric Jack polynomials in N variables taking values in the module. These polynomials are simultaneous eigenfunctions of a commutative set of operators, self-adjoint with respect to certain Hermitian forms. There exists a matrix-valued measure on the N -torus for which the polynomials are mutually orthogonal, provided that the parameter κ lies in an interval depending on the representation. The absolutely continuous part of the measure satisfies a first-order system of differential equations, defined on $(\mathbb{C} \setminus \{0\})^N$. The system is Frobenius integrable and its solutions are analytic with singularities on the hyperplanes $\bigcup_{i < j} \{x \in \mathbb{C}^N : x_i = x_j\}$. The particular solution of the system defined by the measure has unitary monodromy and a smoothness property at the hyperplanes. For the two-dimensional representations of \mathcal{S}_3 and \mathcal{S}_4 the solutions are expressible in ${}_2F_1$ -hypergeometric functions, for $-\frac{1}{3} < \kappa < \frac{1}{3}$.

On PBW subalgebras of rational Cherednik algebras

Misha Feigin

University of Glasgow, Great Britain

I am going to consider two subalgebras of the rational Cherednik algebra $H_c(W)$ associated with a Coxeter group W . The first one is the degree 0 part of $H_c(W)$ while the second one is generated by Dunkl angular momenta and the group algebra $\mathbb{C}W$. These subalgebras are given by quadratic relations and satisfy Poincaré-Birkhoff-Witt property. They are deformations of semidirect product of quotients of universal enveloping algebras of $gl(n)$ and $so(n)$ with $\mathbb{C}W$, and they are related to quantisation of functions on the variety of matrices of rank at most 1 and on the cone over Grassmanian of two-planes respectively. The centres of these subalgebras give quantum Hamiltonians related to Calogero-Moser integrable systems. The talk is based on joint work with T. Hakobyan [1].

References

- [1] M. Feigin, T. Hakobyan, *On Dunkl angular momenta algebra*, Journal of High Energy Physics, 11 (2015) 107.

Dunkl theory in the study of random matrices

Peter Forrester

University of Melbourne, Australia

In mathematical physics Dunkl theory is perhaps best known for its application to quantum many body problems of the Calogero-Sutherland type. It is also the case that Dunkl theory has relevance to random matrix theory. The aim of this talk is to show how this comes about.

Some Properties of Dunkl-subharmonic functions

Léonard Gallardo

Université de Tours, France

I will introduce the notion of Dunkl-subharmonic functions on \mathbb{R}^d via a volume mean value property. I will then give some of its properties, in particular the strong maximum property and give finally some typical examples like the Dunkl-Newton potential of a nonpositive Radon measure on \mathbb{R}^d .

This is part of a joint work with Chaabane Rejeb.

References

- [1] L. Gallardo and C. Rejeb. *A new mean value property for harmonic functions relative to the Dunkl-Laplacian operator and applications*. Trans. Amer. Math. Soc., Vol. 368, Number 5, May 2016, p.3727-3753 (electronically published on May 22, 2015, DOI:<http://dx.doi.org/10.1090/tran/6671>).

The \mathbb{Z}_2^n Dirac-Dunkl operator and a higher rank Bannai-Ito algebra

Vincent Genest

MIT, Cambridge, USA

In this talk, I will discuss the n -dimensional Dirac-Dunkl operator associated with the reflection group \mathbb{Z}_2^n . I will exhibit the symmetries of this operator, and describe the invariance algebra they generate. The symmetry algebra will be identified as a rank- n generalization of the Bannai-Ito algebra. Moreover, I will explain how a basis for the kernel of this operator can be constructed using a generalization of the Cauchy-Kovalevskaja extension in Clifford analysis, and how these basis functions form a basis for irreducible representations of Bannai-Ito algebra. Finally, I will conjecture on the role played by the multivariate Bannai-Ito polynomials in this framework.

Galois theory aspects of Calogero-Moser space as limits of Dunkl operators

Iain Gordon

University of Edinburgh, Great Britain

I will discuss a recent conjecture of Bonnafé-Rouquier which proposes a Kazhdan-Lusztig cell theory for all finite complex reflection groups using the Galois theory of Calogero-Moser phase space. This conjecture is a puzzle even in the case of the symmetric group. I will explain an approach to the conjecture (still to be confirmed!) by using recent work of Mukhin-Tarasov-Varchenko on the Bethe Ansatz for Gaudin Hamiltonians.

This is joint work with Adrien Brochier and features thesis work of Noah White.

Unitary representations of Cherednik algebras

Stephen Griffeth

Universidad de Talca, Chile

I will survey recent results on unitary representations of Cherednik algebras, and especially their relationship with finite integrable systems, Dunkl operators, and the algebraic combinatorics of linear subspace arrangements.

Resonances and resonant states for (locally) symmetric spaces

Joachim Hilgert

Paderborn University, Germany

Given a differential operator, one considers meromorphic continuations of the resolvent operator. Resonances can then be defined as poles of such a continuation, whereas (generalized) resonant states are defined to be members of the (generalized) eigenspaces of the residue at a resonance. In the context of (locally) symmetric spaces resonances have been studied for Laplace-Beltrami operators (quantum resonances) as well as for the generators of geodesic flows (Ruelle resonances). Most published results deal with spaces of negative sectional curvature, i.e. symmetric spaces of rank one. There are some isolated results on higher rank which show that new phenomena occur. For example one may have infinite rank residue operators.

In this talk we present a brief survey of recent advances achieved by various authors as well as some of our own results.

Character Spaces of Commutative Hypergroups

Rupert Lasser

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Let K be a commutative hypergroup. A special feature is the fact that there are three character spaces of K , which are in general different: the Gelfand structure space of the L^1 -algebra, the symmetric Gelfand structure space of the L^1 -algebra and the structure space of the hypergroup C^* -algebra. The nature of these character spaces is properly understood only in some special cases. We shall present conditions, which characterize membership in one of the character spaces and recent results on regularity properties depending on the character space considered. Further new results concerning the Pontryagin duality will be discussed.

Differential-difference operators as glues between representation theories

Hiroshi Oda

Takushoku University, Tokyo, Japan

Let $G = KAN$ be a real reductive Lie group and \mathbf{H} a graded Hecke algebra defined by the restricted root datum of G . The representation theory of G and that of \mathbf{H} resemble very much. For example, the \mathbf{H} -counterpart of the Helgason-Fourier transform for $C^\infty(G/K)$ is the Opdam-Cherednik transform for $C^\infty(A)$ [Op].

Some years ago I proved the Chevalley restriction theorem for G/K holds for not only the K -invariants but also the elements belonging to some little K -types [O1]. Along with this, the classical

radial part formula for $C^\infty(G/K)$ is reformulated by Dunkl-Cherednik operators so as to apply to these K -types. Recall these operators endow $C^\infty(A)$ with an \mathbf{H} -module structure. In general, If a pair of G - and \mathbf{H} -modules is given, then we can discuss whether the two modules satisfy the generalized Chevalley restriction theorem and the generalized radial part formula in an abstract level. If they do, we say they make a *radial pair* [O2]. The radial pairs constitutes a category which precisely explains many resemblances between two representation theories.

If time permits, we discuss some functors between the category of G - and \mathbf{H} -modules.

References

- [O1] H. Oda, *Generalization of Harish-Chandra's basic theorem for Riemannian symmetric spaces of non-compact type*, Adv. Math. **281** (2007), 549–596.
- [O2] H. Oda, *Differential-difference operators and radial part formulas for non-invariant elements*, preprint, arXiv:1402.3231.
- [Op] E. M. Opdam, *Harmonic analysis for certain representations of graded Hecke algebras*, Acta Math. **175**(1995), no. 1, 75–121.

Two-parameter deformations of the Hermite semigroup and Dunkl operators

Bent Ørsted

Aarhus University, Denmark

This lecture is based on the joint work [1] and [2] where the aim is to study some further aspects of the Dunkl transform. It turns out that some additional deformations are possible in addition to the multiplicity parameter introduced by C. Dunkl; in particular we obtain some new and natural generalizations of the Fourier transform - in some sense interpolating between two minimal representations of two different simple Lie groups.

References

- [1] Ben Saïd, Salem; Kobayashi, Toshiyuki; Ørsted, Bent: Laguerre semigroup and Dunkl operators. Compos. Math. **148** (2012), no. 4, 1265–1336.
- [2] Ben Saïd, Salem; Kobayashi, Toshiyuki; Ørsted, Bent: Generalized Fourier transforms $\mathcal{F}_{k,a}$. C. R. Math. Acad. Sci. Paris **347** (2009), no. 19-20, 1119–1124.

A relativistic conical function and its offspring

Simon Ruijsenaars

University of Leeds, United Kingdom

We first review one-dimensional potential scattering, tying in the repulsive and attractive potentials

$$g(g-1)/\sinh^2(x), \quad -g(g-1)/\cosh^2(x), \quad g > 1,$$

with the conical/Mehler function specialization of the hypergeometric function. After linking this to the hyperbolic N -particle Calogero-Moser systems of A_{N-1} type, we present our relativistic version of the latter. Specializing again to the reduced 2-particle setting, we sketch results concerning a relativistic generalization of the conical function. In particular, we discuss how it supplies unitary eigenfunction transforms for the repulsive and attractive cases, and how its asymptotics results in

scattering amplitudes satisfying the Yang-Baxter equations. Other topics touched on are product formulas, $SL(2, \mathbb{Z})$ and DAHA.

Some of the results involve joint work with Martin Hallnäs and Steven Haworth.

Macdonald's conjectures for hypergeometric functions

Siddhartha Sahi

Rutgers University, New Brunswick, USA

Hypergeometric functions for symmetric matrices were introduced in the 1950's by Herz [3], who gave an inductive definition using the Laplace transform. Subsequently Constantine [1] obtained an explicit series expansion in terms of the zonal polynomials. These have found considerable applications in multivariate statistics, especially in the theory of non-central distributions. In the 1980's I.G. Macdonald proposed a one-parameter generalization of this theory, replacing zonal polynomials by Jack polynomials. However many of the results in this more general theory are still conjectural; indeed Macdonald's manuscript, though widely circulated and recently posted on the arxiv [4], remains unpublished.

One key conjecture concerns the asymptotics of the Laplace kernel, which in the Jack setting is given only by a certain power series expansion. The difficulty is somewhat akin to having to prove rapid decay of the function $\exp(-x)$ based on its series expansion alone! We will describe a proof of Macdonald's conjecture using ideas from the theory of Jack polynomials. Yet another conjecture concerns the Laplace transform of a Jack polynomial. We give a proof of this as well, using ideas from the theory of Macdonald polynomials. Finally, as a further application of the theory, we derive a generalization of Ramanujan's "master theorem" [2] to the setting of Jack polynomials.

This is joint work with Gestur Olafsson.

References

- [1] A. Constantine: Some non-central distribution problems in multivariate analysis. *Ann. Math. Statist.* **34** (1963), 1270–1285,
- [2] H. Ding, K. Gross and D. Richards: Ramanujan's master theorem for symmetric cones. *Pacific J. Math.* **175** (1996), 447–490.
- [3] C. Herz: Bessel functions of matrix argument. *Ann. of Math.* **61** (1955), 474–523
- [4] I. G. Macdonald: Hypergeometric Functions I. arXiv: 1309.4568

Finite fields and (q,t) -combinatorics

Dennis Stanton

University of Minnesota, USA

Some of Charles Dunkl's early work on q -Hahn and q -Krawtchouk polynomials used classical groups over finite fields. In this talk I will discuss enumeration problems for finite fields which are analogous to those on the symmetric group. By considering invariant theory for these classical groups, there are natural t -analogues of these q -analogues, thus (q, t) -analogues. Some recent conjectures in this area will be presented.

Rank dependence of qKZ equations and dense loop models

Jasper Stokman

University of Amsterdam, the Netherlands

The qKZ equations form a consistent system of difference equations in n variables. They depend on a choice of representation of the affine Hecke algebra H_n of type A_{n-1} . The qKZ equations play an important role in quantum harmonic analysis and in integrable models from statistical physics. In the first part of the talk I will discuss how solutions of the qKZ equations for different values of n are related. In the second part I will give an application of these results to the dense loop model. This is a percolation model whose configurations are tilings of a half-infinite cylinder by two types of plaquettes. A configuration in this model gives rise to a collection of nonintersecting paths and loops on the half-infinite cylinder, with the paths connecting to the boundary of the half-infinite cylinder. The model is governed by a Markov process with a unique stationary state, which describes the connectivity probabilities at the boundary. The qKZ equations come into play because the stationary state turns out to be a distinguished solution of the qKZ equations for an appropriate choice of representation of the affine Hecke algebra. As a consequence we obtain explicit formulas relating the stationary states of the dense loop model for different sizes of the circumference of the cylinder.

This is joint work with Kayed Al-Qasemi and Bernard Nienhuis.

Mixed norm estimates for the Cesàro means associated with Dunkl-Hermite expansions

Sundaram Thangavelu

Indian Institute of Science, Bangalore, India

In this talk I plan to discuss my joint work with Pradeep Boggarapu and Luz Roncal on mixed norm estimates for the Cesàro means associated with Dunkl-Hermite expansions on \mathbb{R}^d . These expansions arise when one considers the Dunkl-Hermite operator (or Dunkl harmonic oscillator) $H_\kappa := -\Delta_\kappa + |x|^2$, where Δ_κ stands for the Dunkl-Laplacian. It is shown that the desired mixed norm estimates are equivalent to vector-valued inequalities for a sequence of Cesàro means for Laguerre expansions with shifted parameter. In order to obtain such vector-valued inequalities, we develop an argument to extend these Laguerre operators for complex values of the parameters involved and apply a version of three lines lemma.

A Schur-Weyl-like construction of $L(k^N)$ for the DAHA

Monica Vazirani

UC Davis, USA

Building on the work of Calaque-Enriquez-Etingof, Lyubashenko-Majid, and Arakawa-Suzuki, Jordan constructed a functor from quantum D -modules on special linear groups to representations of the double affine Hecke algebra (DAHA) in type A . When we input quantum functions on $SL(N)$ the output is $L(k^N)$, the irreducible DAHA representation indexed by an $N \times k$ rectangle. For the specified parameters, $L(k^N)$ is Y -semisimple, i.e. one can diagonalize the Dunkl operators. We give an explicit combinatorial description of this module via its Y -weight basis.

This is joint work with David Jordan.

The q -oscillator algebra and the Askey-Wilson polynomials

Luc Vinet

Université de Montréal, Canada

The most general representations of the q -oscillator algebra $AB - qBA = 1$ with nondegenerate 3-diagonal operators A and B are constructed. They are shown to lead to the generic Askey-Wilson polynomials as solutions of the generalized eigenvalue problem $(A - \mu B - \nu)\psi = 0$. The special case $\mu = 0$ corresponds to the big q -Jacobi polynomials. This means that the Askey-Wilson polynomials can be obtained by applying the tridiagonalization procedure to the big q -Jacobi polynomials.

Based on joint work with S. Tsujimoto (Kyoto) and A. Zhedanov (Donetsk).

Analysis on the unit sphere with Dunkl weight functions

Yuan Xu

University of Oregon, USA

Much of the classical harmonic analysis on the unit sphere relies on spherical harmonics and related differential operators. A large portion of it can be extended to the weighted setting for a family of weighted functions that are invariant under a reflection group, first studied by Dunkl, for which the ordinary spherical harmonics are replaced by Dunkl's h -spherical harmonics. In this talk, we explain the main development and recent advances of this theory.

Zeros of Ramanujan Type Entire Functions

Ruiming Zhang

Northwest A&F University, Yangling, China

In this talk I shall present some results on the location of zeros for a class of order zero entire functions appeared in an joint work with professor Mourad Ismail [1]. This function class includes some well-known special functions such as Ramanujan function $A_q(z)$, the second q -Bessel function $J_\nu^{(2)}(z; q)$, the third q -Bessel function $J_\nu^{(3)}(z; q)$, and the confluent basic hypergeometric series. I shall show that, under not very restrictive conditions, these entire functions have only positive (negative) zeros.

This talk is based on my recent work [2].

References

- [1] M. E. H. Ismail and R. Zhang, q -Bessel Functions and Rogers-Ramanujan Type Identities, Proceedings of the American Mathematical Society, in production.
- [2] R. Zhang, Zeros of Ramanujan Type Entire Functions, Proceedings of American Mathematical Society, accepted.

2 Contributed Talks

Generalized Dunkl Primitives, real Paley–Wiener theorems and Roe’s theorem for the Dunkl transform on the real line

Nils Byrial Andersen

Aabenraa Statsskole, Denmark

We construct generalized Dunkl Primitives (anti-Dunkl operators) for tempered distributions. We then consider the behaviour of L^p -norms of iterated Dunkl primitives (and derivatives), with a view towards real Paley–Wiener theorems. As a result we also obtain the optimal version of Roe’s theorem for the Dunkl transform on the real line.

The talk is based on joint work with Marcel de Jeu, Leiden University.

A Paley-Wiener theorem about the spectral parameter and applications

Salem Ben Saïd

Université de Lorraine, Nancy, France

The inversion formula for the Dunkl transform in polar coordinates is

$$f(x) = \int_0^\infty f_k^s(x) ds, \quad \text{for } f \in \mathcal{D}(\mathbb{R}^d),$$

where f_k^s are “projections” of f into the eigenspaces of the Dunkl Laplacian Δ_k corresponding to the eigenvalue $-s^2$. Here the parameter k comes from Dunkl’s theory of differential-difference operators. A natural question is how the properties of the function f relate to the properties of f_k^s ? In this talk we will answer this question whenever $\text{supp}(f) \subset \overline{B(O, R)}$. As a first application of this Paley-Wiener type theorem, we will provide a spectral version of de Jeu’s Paley-Wiener theorem for the Dunkl transform. The second application concerns a support theorem for general types of Dunkl spherical means.

Characterization of $(\eta, \gamma, k, 2)$ -Dini-Lipschitz functions in terms of their Helgason Fourier transform

Radouan Daher

University Hassan II of Casablanca

In this talk, using a generalized translation operator, we obtain an analog of Younis Theorem 5.2 in [M. S. Younis, Fourier transforms of Dini-Lipschitz functions, Int. J. Math. Math. Sci. 9 (2),(1986), 301–312.] for the Helgason Fourier transform of a set of functions satisfying the $(\eta, \gamma, k, 2)$ -Dini-Lipschitz condition in the space L^2 for functions on noncompact rank one Riemannian symmetric spaces.

The talk is based on joint work with Salah El Ouadih.

Inequalities for extreme zeros of classical orthogonal polynomials

Kathy Driver

University of Cape Town, South Africa

We prove that interlacing properties of zeros can be applied to obtain bounds for the largest and smallest zeros of classical orthogonal polynomials.

This is joint work with Kerstin Jordaan.

Potential theory associated with the Dunkl Laplacian

Kods Hassine

Monastir University

We give a potential theoretical approach to study the Dunkl Laplacian Δ_k . By introducing the Green kernel relative to Δ_k , we prove that the Dunkl Laplacian generates a balayage space and we investigate the associated family of harmonic measures. Therefore, by mean of harmonic kernels, we give a characterization of all Δ_k -harmonic functions on large class of open subsets U of \mathbb{R}^d . We also establish existence and uniqueness result of a solution of the corresponding Dirichlet problem.

Combinatorial and Analytic Properties of the n -Dimensional Hermite Polynomials

Mourad Ismail

University of Central Florida, USA and King Saud University, Saudi Arabia

We investigate some combinatorial and analytic properties of the n -dimensional Hermite polynomials introduced by Hermite in the late 19-th century. We derive recurrence relations for these polynomials using their combinatorial interpretation. We also establish a new linear generating function and a Kibble–Slepian formula for the n -dimensional Hermite polynomials which generalize the Kibble–Slepian formula for the univariate Hermite polynomials and the Poisson kernel (Mehler formula) for the n -dimensional Hermite polynomials, respectively.

The talk is based on joint work with Plamen Simeonov.

Characterizations and harmonic analysis of sieved orthogonal polynomials

Stefan Kahler

Technical University of Munich

Starting with the well-known Askey–Wilson operator \mathcal{D}_q and the averaging operator \mathcal{A}_q , we introduce sieved versions D_k and A_k by letting q approach a k -th root of unity. Motivated by characterization results on continuous q -ultraspherical polynomials given by Ismail–Obermaier (2011) and in our paper [1], we apply these new operators to obtain characterizations of k -sieved symmetric random walk polynomials. In contrast to their ancestors \mathcal{D}_q and \mathcal{A}_q , and in contrast to the special cases $D_1 = \frac{d}{dx}$ and $A_1 = \text{id}$, the kernels of the sieved operators D_k and A_k are no longer finite-dimensional as soon as $k \geq 2$. The infinite-dimensional kernels lead to additional degrees of freedom, and it will be one of the purposes of the talk to see how just these additional degrees of freedom correspond to quite

unexpected phenomena concerning characterizations in terms of appropriate expansions. In doing so, we will get involved with nonlinear systems and shall use certain infinite-dimensional sparse matrix representations to express the connection coefficients to the only invariant under sieving processes (i.e., to the Chebyshev polynomials of the first kind).

In the second part of the talk, which is based on joint work with S. Straub, we will focus on the harmonic analysis of sieved polynomials and on the interplay with certain transfer operators. Such transfer operators can be reconstructed (or defined) via inverse images w.r.t. the Chebyshev polynomials of the first kind, and they are left inverses of the corresponding composition operators. Our above-mentioned result on sparse connection matrices will play a crucial role. We study various actions of such transfer operators, as well as some spectral properties.

References

- [1] Kahler, S.: Characterizations of ultraspherical polynomials and their q -analogues, Proc. Amer. Math. Soc. 144 (2016), no. 1, 87–101.

Matrix-valued orthogonal polynomials associated to quantum algebras

Erik Koelink

Radboud Universiteit Nijmegen

Matrix-valued orthogonal polynomials can be obtained using representation theory and compact symmetric pairs. The structure theory can be used to obtain information on the matrix-valued weight in the orthogonality relations, the recurrence relations for the polynomials, matrix-differential operators, etc. We discuss the quantum version of matrix-valued orthogonal polynomials associated to the quantum symmetric pair associated to $(SU(2) \times SU(2), \text{diag})$ in the context of the work of Letzter, Kolb and others. In particular, we work in the setting of quantised universal enveloping algebras and their representation theory. We will focus on the description of the matrix-valued weight in the orthogonality relations and the matrix-differential operators in this situation. The (conjectural) possibility of a universal element in the quantised universal enveloping algebra for the square root of the weight function will be discussed. The appropriate analogue of the conjecture in the classical (i.e. group) case is valid, and it is discussed in the lecture by Maarten van Pruijssen.

The main results are based on joint work [arXiv:1507.03426](https://arxiv.org/abs/1507.03426), to appear in Ramanujan J., with Noud Aldenhoven and Pablo Román.

Addition type formulas associated with dual product formulas

Tom Koornwinder

University of Amsterdam, the Netherlands

An old problem of Askey suggests to find addition type formulas associated with dual product formulas (i.e., linearization formulas) for special orthogonal polynomials, just as classical addition formulas are associated with product formulas (think about the product formula for Legendre polynomials which writes $P_n(x)P_n(y)$ as an integral involving a single Legendre polynomial P_n). In the lecture a positive answer to this problem will be given in a non-polynomial case, for Jacobi functions [DL] with parameters $(\alpha, -1/2)$. The dual product formula in this case was given by Hallnäs and Ruijsenaars [HR], Theorem 4.3, where the weight function in the integral can be recognized as the weight function in the orthogonality relation for Wilson polynomials. Tentatively, polynomial examples may be found in a similar way.

References

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[HR] M. Hallnäs and S. Ruijsenaars, Product formulas for the relativistic and nonrelativistic conical functions, [arXiv:1508.07191v1](https://arxiv.org/abs/1508.07191v1) (math.CA).

On the Green function and Hardy-Stein identities for the Dunkl Laplacian

Tomasz Luks

Paderborn University, Germany

This talk will be devoted to some new results in the potential theory of the Dunkl Laplacian Δ_k in real Euclidean spaces. In the first part we will introduce and describe basic properties of the Green function of the unit ball for Δ_k . As an application, we will derive the so-called Poisson-Jensen formula for Δ_k -subharmonic functions and Hardy-Stein identities for Hardy spaces of Δ_k . In the second part we will give precise two-sided estimates of the Green function and the Poisson kernel for Δ_k in the case of a one-root system. These estimates contrast sharply with the well known results in the potential theory of the classical Laplacian.

This is a joint work with Piotr Graczyk, University of Angers, and Margit Rösler, Paderborn University.

On The Dunkl Intertwining Operator

Mostafa Maslouhi

University Ibn Tofail, Morocco

We give an integral representation for the operator $V_k \circ e^{\Delta/2}$ for an arbitrary Weyl group and a large class of regular weights k containing those of non negative real parts. Our representing measures are absolute continuous with respect the Lebesgue measure in \mathbb{R}^d .

We show in particular that the operator $V_k \circ e^{\Delta/2}$ extends uniquely as a bounded operator to a large class of functions which are not necessarily differentiable. In the case of non negative weights, this operator is shown to be positivity-preserving.

On harmonic analysis operators in the Laguerre setting

Adam Nowak

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We study several fundamental harmonic analysis operators in the multi-dimensional context of the Dunkl harmonic oscillator and the underlying group of reflections isomorphic to \mathbb{Z}_2^d . Noteworthy, **we admit also negative values of the multiplicity function**. Our investigations include maximal operators, g -functions, Lusin area integrals, Riesz transforms and multipliers of Laplace and Laplace-Stieltjes type. By means of the general Calderón-Zygmund theory we prove that these operators are bounded on weighted L^p spaces, $1 < p < \infty$, and from weighted L^1 to weighted weak L^1 .

The talk is based on a recent joint work with Krzysztof Stempak and Tomasz Z. Szarek.

References

- [1] A. Nowak, K. Stempak, T.Z. Szarek, *On harmonic analysis operators in Laguerre-Dunkl and Laguerre-symmetrized settings*. arXiv:1602.05369

The symmetry algebra of the Dirac-Dunkl operator of type A_{N-1}

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We consider the Dirac-Dunkl operator associated to the root system of type A_{N-1} with corresponding reflection group $G = S_N$, the symmetric group on N elements. This operator is a square root of the Dunkl Laplacian realised in the framework of Clifford analysis. We explicitly determine the symmetries of the Dirac-Dunkl operator and their underlying algebraic structure. Natural candidates for representation spaces are the Dunkl monogenics, which are homogeneous polynomials in the kernel of the Dirac-Dunkl operator. In the three-dimensional case, an explicit basis for the Dunkl monogenics is constructed using a coordinate transformation and a Cauchy-Kovalevskaja extension theorem. By means of this basis, the Dunkl monogenics are shown to support finite-dimensional irreducible representations of the symmetry algebra.

This is joint work with Hendrik De Bie and Joris Van der Jeugt.

Vector valued orthogonal polynomials in several variables

Maarten van Pruijssen

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The relation between Jacobi polynomials and matrix coefficients of compact symmetric pairs is also available for vector valued functions. Indeed, if we impose a multiplicity freeness condition, then the generalized spherical functions of fixed type have the structure of a free module over the polynomial ring of zonal spherical functions. In this way we obtain families of vector valued orthogonal polynomials in several variables, that are simultaneous eigenfunctions of a commutative algebra of differential operators.

We briefly discuss this construction together with the classification of the data for which this construction is possible. We also indicate how the families of polynomials in low dimensional examples of rank one can be deformed and allow for shift operators. Finally, we discuss some details for the symmetric pair $(\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag}(\mathrm{SU}(n+1)))$: the polynomials in this case depend on n variables and take values in a $\binom{n+k}{n} - 1$ -dimensional space, for $k \geq 1$. The weight matrix that determines the pairing for the polynomials can be made explicit by a generalization of a miraculous formula of TOM KOORNWINDER.

This is joint work with Erik Koelink and Pablo Román.

Wiener Tauberian theorem for rank one semisimple Lie groups and for hypergeometric transform

Sanjoy Pusti

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A famous theorem of Norbert Wiener states that for a function $f \in L^1(\mathbb{R})$, span of translates $f(x - a)$ with complex coefficients is dense in $L^1(\mathbb{R})$ if and only if the Fourier transform \hat{f} is nonvanishing on \mathbb{R} . That is the ideal generated by f in $L^1(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ if and only if the Fourier transform \hat{f} is nonvanishing on \mathbb{R} . This theorem is well known as the Wiener Tauberian theorem. This theorem has been extended to abelian groups. The hypothesis (in the abelian case) is on a Haar integrable function which has nonvanishing Fourier transform on all unitary characters. However, back in 1955, Ehrenpreis and Mautner observed that Wiener Tauberian theorem fails even for the commutative Banach algebra of integrable radial functions on $SL(2, \mathbb{R})$.

In this talk we shall discuss about a genuine analogue of the theorem for real rank one, connected noncompact semisimple Lie groups with finite centre and also for the hypergeometric transform.

Riesz type Potential Theory associated to root systems

Chaabane Rejeb

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For a root system R on \mathbb{R}^d and a nonnegative multiplicity function k on R , we consider the Dunkl-heat kernel $p_t(x, y)$ associated to the Dunkl-Laplace operator Δ_k . For $\beta \in]0, d + 2\gamma[$, where $\gamma = \frac{1}{2} \sum_{\alpha \in R} k(\alpha)$, we study the Dunkl-Riesz kernel of index β defined by

$$R_{k,\beta}(x, y) = \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} t^{\beta/2-1} p_t(x, y) dt.$$

and the corresponding Dunkl-Riesz potential $I_{k,\beta}[\mu]$ of a Radon measure μ on R^d . According to the values of β , we study the Δ_k -superharmonicity of these functions and we give some applications like the Δ_k -Riesz measure of $I_{k,\beta}[\mu]$ and the uniqueness principle.

Hardy-type inequalities for fractional powers of the Dunkl–Hermite operator

Luz Roncal

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We prove Hardy-type inequalities for fractional powers of the Dunkl–Hermite operator. Consequently, we also obtain Hardy inequalities for the fractional harmonic oscillator as well. In order to get these inequalities, we use h -harmonic expansions to reduce the problem to the Laguerre setting. Then, we push forward an argument developed by R. L. Frank, E. H. Lieb and R. Seiringer, initially developed in the Euclidean setting, to get a Hardy inequality for the fractional-type Laguerre operator. Such technique is based on two facts: first, to get an integral representation for the corresponding fractional operator, and second, to write a proper ground state representation.

Our Hardy inequalities are proven in the cases in which the weight involved is either non-homogeneous or homogeneous. In the first case, the constant arising turns out to be optimal. As a consequence of the Hardy inequalities we also obtain versions of Heisenberg uncertainty inequality for the considered fractional operators.

This is joint work with Ó. Ciaurri (Universidad de La Rioja, Spain) and S. Thangavelu (Indian Institute of Science of Bangalore, India).

An analogue to the Duistermaat-Kolk-Varadarajan estimate for the spherical functions associated with root systems of type A

Patrice Sawyer

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In this talk, we consider the generalized spherical functions ϕ_λ associated to the root systems of type A in order to provide an estimate in the spectral parameter λ . This estimate generalizes, for the root systems of type A, an estimate obtained by Duistermaat, Kolk and Varadarajan.

Matrix-valued commuting differential operators and their joint eigenfunctions

Nobukazu Shimeno

Kwansei Gakuin University, Japan

In this talk, we give an example of vector-valued analogue of the theory of the Heckman-Opdam hypergeometric function associated with a root system. We construct matrix-valued commuting differential operators associated with root system of type A_2 and their joint eigenfunctions. In group case, the differential operators are radial parts of invariant differential operators on a certain homogeneous vector bundle over Riemannian symmetric space

$$SL(3, \mathbb{K})/SO(3, \mathbb{K}) \quad (\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}),$$

and up to a constant multiple there exists a unique Weyl group invariant vector-valued joint eigenfunction that is analytic at the origin, which is the radial part of a matrix coefficient of a principal series representation. Allowing the root multiplicity to be an arbitrary complex number, we give matrix-valued commuting differential operators and connection coefficients (c -functions) for their joint eigenfunctions given by power series.

Harmonic analysis operators in a certain Dunkl setting

Tomasz Szarek

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In the talk we will consider several harmonic analysis operators in the multi-dimensional context of the Dunkl Laplacian (the case when a group of reflections is isomorphic to \mathbb{Z}_2^n). Our investigations include maximal operators, multipliers of Laplace and Laplace-Stieltjes transform type, Riesz transforms, g -functions and Lusin area integrals. We use the general Calderón-Zygmund theory to prove that these objects are bounded in weighted L^p spaces, $1 < p < \infty$, and are of weighted weak type $(1, 1)$. Noteworthy, in our main result we do not assume positivity of the multiplicity function, which is unusual in the Dunkl context.

The talk is based on a joint paper with A.J. Castro [1].

References

- [1] A.J. Castro, T.Z. Szarek, *On fundamental harmonic analysis operators in certain Dunkl and Bessel settings*, J. Math. Anal. Appl., 412 (2014), 943–963.

Orthogonal polynomials on the real line corresponding to a perturbed chain sequence

Anbhu Swaminathan

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Chain sequences related to orthogonal polynomials on the real line are well studied. In this work, we consider the perturbation of the minimal parameter sequence of a given chain sequence and study the properties of the corresponding orthogonal polynomials.

References

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A model for the higher rank Racah algebra

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We propose a generalization of the Racah algebra by considering the tensor product of n copies of $\mathfrak{su}(1,1)$. Its role as symmetry algebra for the \mathbb{Z}_n^2 Dunkl-Laplacian and its connection to the generalized Bannai-Ito algebra are explained. Bases for Dunkl-harmonics are constructed so that each basis consists of joint eigenfunctions of a maximal abelian subalgebra of the generalized Racah algebra. It is explained how to solve the Racah problem for the generalized Racah algebra. The connection coefficients between two bases correspond to the multivariate Racah polynomials as defined by M. V. Tratnik.

This is joint work with Hendrik De Bie, Vincent Genest and Luc Vinet.

On positive integral representations for Bessel functions

Michel Voit

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There exist several multivariate extensions of the classical beta integral representations of the one-dimensional Bessel functions of some index with respect to such functions with lower indices. For instance, for Bessel functions on matrix cones there are such representations by Herz and others, and for certain Bessel functions associated with root systems of type B there are such results by Macdonald. In both cases, the associated representing multivariate beta probability measures converge for sufficiently large indices. Similar to the theory of Gindikin for Riesz measures, one can extend the index set analytically via tempered distributions where these distributions are (probability) measures precisely if the index is sufficiently large or contained in some discrete set. This result leads to examples of pairs of indices of Bessel functions of type B and thus for Dunkl kernels for which no positive integral representation exists. This disproves some older conjectures.

The talk is based on joint work with Margit Rösler.