Resonances of the Laplacian on noncompact Riemannian symmetric spaces of low rank

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Statement of the problem

$X = G/K$ is a Riemannian symmetric space of the noncompact type, where:
G = connected noncompact real semisimple Lie group with finite center
K = maximal compact subgroup of G

Examples:

- $H^n(\mathbb{R}) = \mathrm{SO}_0(1,n)/\mathrm{SO}(n)$ real hyperbolic space
- $\mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$, $q \geq p \geq 1$, Grassmannian of p subspaces of $\mathbb{C}^{p+q}$ (complex hyperbolic space if $p = 1$)

Δ= (positive) Laplacian on X, with continuous spectrum $\sigma(\Delta) = [\rho_X^2, +\infty[$ with $\rho_X^2 > 0$.

The resolvent of Δ

$$R_\Delta(u) = (\Delta - u)^{-1}$$

is a bdd operator on $L^2(X)$ depending holomorphically on $u \in \mathbb{C} \setminus \sigma(\Delta)$, i.e.

$$\mathbb{C} \setminus \sigma(\Delta) \ni u \longrightarrow R_\Delta(u) = (\Delta - u)^{-1} \in \mathcal{B}(L^2(X)).$$

is a holomorphic operator-valued function.

As operator on $L^2(X)$, the resolvent R_Δ has no extension across $\sigma(\Delta)$.

Letting R_Δ act on a smaller dense subspace of $L^2(X)$, e.g. $C_c^\infty(X)$, a meromorphic continuation of R_Δ across $\sigma(\Delta)$ is possible.

Theorem (Strohmaier, Mazzeo-Vasy, 2005)

Let X be an arbitrary Riemannian symmetric space of the noncompact type. There are $\Omega \subsetneq \mathbb{C}$ open with $\sigma(\Delta) \subset \Omega$ and M Riemann surface above Ω such that

$$R_\Delta : \Omega \setminus \sigma(\Delta) \ni u \longrightarrow R_\Delta(u) \in \mathrm{Hom}(C_c^\infty(X), C_c^\infty(X)')$$

admits holomorphic extension to M.

$\rightsquigarrow$ Ω is not large enough to find resonances.

Special cases showing that there might be resonances are classical:

Theorem (Guillopé-Zworski, 1995)

For $X = H^n(\mathbb{R})$ and $\Omega = \mathbb{C}$, the resolvent R_Δ has:

- ◇ *holomorphic extension, if n is odd*
- ◇ *meromorphic extension (with infinitely many poles) if n even.*

Problem 1: For general $X = G/K$, does R_Δ admit a meromorphic extension to a Riemann surface above $\Omega = \mathbb{C}$?
If so: what are the poles? What are the residues?

The poles of the meromorphically extended R_Δ are called the (quantum) resonances of the Laplacian.

(Quantum) resonances

In physics:

- Quantum mechanical systems which are bound can only assume certain discrete values of energy (=energy levels) which are constant in time.
- Quantum mechanical systems which are unbound might have states with energy that a certain starting time can assume certain discrete values, but are not constant in time, usually decreasing exponentially (=metastable states).
- Energy at a metastable state is described by a complex number ζ (a resonance):

  $\mathrm{Re}\,\zeta$ = energy at the starting time

  $\mathrm{Im}\,\zeta$ = rate of exponential time decreasing of the energy.
- The resonances are the poles of the meromorphic extension of the resolvent

$$\mathbb{C} \setminus \sigma(H) \ni u \longrightarrow R_H(u) = (H - u)^{-1}$$

  of the Hamiltonian H, with continuous spectrum $\sigma(H)$, describing the unbound system.

In mathematics:

- **Classical situation:** Resonances for Schrödinger operators $H = \Delta_{\mathbb{R}^n} + V$ where:
  - ◇ $\Delta_{\mathbb{R}^n} = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the positive Euclidean Laplacian
  - ◇ V is a potential

  (V chosen so that H is s.a. and $\sigma(H) \subset [0, +\infty[$ is continuous; e.g. $V = 0$).

- **Geometric scattering:** Resonances for the Laplacian Δ of complete non-compact Riemannian manifolds (with bounded geometry).

  Motivations: scattering, dynamical systems, spectral analysis...

  Very active field of research.

Why studying resonances on symmetric spaces?

- ◇ well understood geometry
- ◇ well developed Fourier analysis: HF (=Helgason-Fourier) transform
- ◇ radial part of Δ on a Cartan subspace is a Schrödinger operator
- ◇ tools from representation theory

Some usual renormalizations

$X = G/K$ Riemannian symmetric space of the noncompact type.

- Translate the spectrum $[\rho_X^2, +\infty)$ to $[0, +\infty)$
  i.e. consider $\Delta - \rho_X^2$ instead of Δ
- Change variables $u = z^2$ $\rightsquigarrow$ choice of square root: $\sqrt{-1} = i$
  $u \in \mathbb{C} \setminus [0, +\infty[$ corresponds to $z \in \mathbb{C}^+ = \{w \in \mathbb{C} : \mathrm{Im}\, w > 0\}$.
- Define

$$R(z) = R_{\Delta - \rho_X^2}(z^2) = (\Delta - \rho_X^2 - z^2)^{-1}$$

So $R : \mathbb{C}^+ \to \mathcal{B}(L^2(X))$ is a holomorphic operator-valued function.

Goal:

Meromorphic continuation across $\mathbb{R}$ of $R : \mathbb{C}^+ \to \mathrm{Hom}\,(C_c^\infty(X), C_c^\infty(X)')$

$\rightsquigarrow$

$C^\infty(X)$ instead of $C_c^\infty(X)'$
for $X = G/K$ symmetric
(Paley-Wiener theorem)

Residue operators

Suppose we have a meromorphic continuation of $R : \mathbb{C}^+ \to \mathrm{Hom}(C_c^\infty(X), C^\infty(X))$ across $\mathbb{R}$, i.e.

- a Riemann surface $\begin{array}{c} M \\ \downarrow \pi \\ \Omega \end{array}$ with $\Omega \subset \mathbb{C}$ open, $\Omega \cap \mathbb{R} \neq \emptyset$

- $\widetilde{R} : M \to \mathrm{Hom}(C_c^\infty(X), C^\infty(X))$ meromorphic and extending a lift of R to M:

$$\begin{array}{ccc} M & \xrightarrow{\widetilde{R}} & \mathrm{Hom}(C_c^\infty(X), C^\infty(X)) \\ \uparrow & \nearrow_{R} & \\ \Omega \cap \mathbb{C}^+ & & \end{array}$$

$\rightsquigarrow$ $\forall f, g \in C_c^\infty(X)$: $\langle \widetilde{R}(\cdot) f, g \rangle_{L^2(X)}$ lifts and extends to M the function $\langle R(\cdot) f, g \rangle_{L^2(X)}$

- z_0 is a resonance (=pole of $\widetilde{R}$).

The residue operator at z_0 is the linear operator

$$\mathrm{Res}_{z_0} \widetilde{R} : C_c^\infty(X) \to C^\infty(X)$$

"defined" for $f \in C_c^\infty(X)$ by

$$\mathrm{Res}_{z_0} \widetilde{R}(f) : X \ni y \longrightarrow \mathrm{Res}_{z=z_0} \big[\widetilde{R}(z)(f)\big](y) \in \mathbb{C}$$

["defined": residues are computed wrt charts in M, so up to nonzero constant multiples]

Well-defined: the subspace $\mathrm{Res}_{z_0} := \widetilde{R}(C_c^\infty(X))$ of $C^\infty(X)$.

The rank of the residue operator at z_0 is $\dim\big(\mathrm{Res}_{z_0}\big)$.

Problem 2: Find image and rank of the residue operator at z_0.

Additional properties appear as X is endowed with a G-invariant Riemannian metric.

The Laplacian Δ of X is G-invariant

$\rightsquigarrow$ $R(z)$ and its mero extension $\widetilde{R}(z)$ are G-invariant

$\rightsquigarrow$ the residue operator at a resonance z_0 is a G-invariant operator $C_c^\infty(X) \to C^\infty(X)$

$\rightsquigarrow$ its image $\mathrm{Res}_{z_0} \subset C^\infty(X)$ is a G-module
(a K-spherical representation of G in our case)

Problem 3: Which (spherical) representations of G do we obtain?
Rank of residue operator $\equiv$ dimension of the corresponding representation
Irreducible? Unitary?

Overview of results

General X of real rank one:

- R. Miatello and C. Will (2000):

meromorphic continuation of the resolvent (in the context of Damek-Ricci spaces).

- J. Hilgert and A.P. (2009):

meromorphic continuation of the resolvent (using HF transform).

◇ no resonances if $X = H^n(\mathbb{R})$ with n odd.

◇ (infinitely many) resonances for $X \neq H^n(\mathbb{R})$ with n odd.

◇ **Finite rank** residue operators, image: irreducible finite dim K-spherical reps of G.

General X of real rank ≥ 2: (R. Mazzeo and A. Vasy (2005), A. Strohmaier (2005))

◇ analytic continuation of the resolvent of Δ from $\mathbb{C}^+$ across $\mathbb{R}$

$$\begin{cases} \text{to an open domain in } \mathbb{C}, & \text{if the real rank of } X \text{ is odd} \\ \text{to a logarithmic cover of an open domain in } \mathbb{C}, & \text{if the real rank of } X \text{ is even} \end{cases}$$

The open domain is **not large enough** to find resonances.

◇ **If any**, resonances are along the negative imaginary axis.

◇ **No resonances** in the even multiplicity case (= Lie algebra of G has one conjugacy class of Cartan subalgebras)

Specific $X = G/K$ of real rank 2: (J. Hilgert, A.P., T. Przebinda)

Complete answers to the three problems:

◇ for almost all rank 2 irreducible X

◇ for direct products $X = X_1 \times X_2$, with X_1, X_2 of rank one.

The resolvent of Δ on $X = G/K$

Explicit formula for the resolvent $R(z)$ of Δ on $C_c^\infty(X)$ via HF transform:

For $z \in \mathbb{C}^+$

$$R(z) = (\Delta - \rho_X^2 - z^2)^{-1} : C_c^\infty(X) \ni f \to R(z)f \in C^\infty(X)$$

is given by

$$[R(z)f](y) \asymp \int_{\mathfrak{a}^*} \frac{1}{\langle\lambda,\lambda\rangle - z^2} (f \times \varphi_{i\lambda})(y) \frac{d\lambda}{c(i\lambda)c(-i\lambda)} \qquad (y \in X),$$

where

$\mathfrak{a}^*$ = dual of a Cartan subspace $\mathfrak{a}$ $\rightsquigarrow$ real rank of $X := \dim \mathfrak{a}^*$

$\langle\cdot,\cdot\rangle$ = inner product on $\mathfrak{a}^*$ induced by the Killing form of the Lie algebra of G

$\rightsquigarrow$ extend $\langle\cdot,\cdot\rangle$ to the complexification $\mathfrak{a}^*_{\mathbb{C}}$ of $\mathfrak{a}^*$ by $\mathbb{C}$-bilinearity

φ_λ = spherical function on X of spectral parameter $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$

$\rightsquigarrow$ spherical functions = (normalized) K-invariant joint eigenfunctions of the commutative algebra of G-invariant diff ops on X

$f \times \varphi_{i\lambda}$ = convolution on X of f and $\varphi_{i\lambda}$

$\rightsquigarrow$ by the Paley-Wiener thm for the HF-transform: entire and rapidly decreasing in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$

$c(\lambda)$ = Harish-Chandra's c-function

$\frac{1}{c(i\lambda)c(-i\lambda)}$ = Plancherel density for the HF-fransform

The Plancherel density $[c(i\lambda)c(-i\lambda)]^{-1}$

$\mathfrak{a}$ (=Cartan subspace) $\curvearrowright \mathfrak{g}$ (=Lie algebra of G) by adjoint action $\mathrm{ad}\, H$ with $H \in \mathfrak{a}$
Σ = roots of $(\mathfrak{g}, \mathfrak{a})$
Σ^+ = choice of positive positive roots in Σ

$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \mathrm{ad}\, H(X) = \alpha(H)X$ for all $H \in \mathfrak{a}\}$ = root space of $\alpha \in \Sigma$
$m_\alpha = \dim_{\mathbb{R}} \mathfrak{g}_\alpha$ = multiplicity of the root α
$\rho = 1/2 \sum_{\alpha\in\Sigma^+} m_\alpha \alpha \in \mathfrak{a}^*$

Notation: For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $\alpha \in \Sigma$ set $\lambda_\alpha = \frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$

Harish-Chandra c-function:

$\Sigma^+_* = \{\beta \in \Sigma^+ : 2\beta \notin \Sigma\}$ (the unmultipliable positive roots)

$$c_\beta(\lambda) = \frac{2^{-2\lambda_\beta}\Gamma(2\lambda_\beta)}{\Gamma\left(\lambda_\beta + \frac{m_{\beta/2}}{4} + \frac{1}{2}\right)\Gamma\left(\lambda_\beta + \frac{m_{\beta/2}}{4} + \frac{m_\beta}{2}\right)} \quad \text{for } \beta \in \Sigma^+_*$$

$$c(\lambda) = c_0 \prod_{\beta\in\Sigma^+_*} c_\beta(\lambda)$$

where c_0 is a normalizing constant so that $c(\rho) = 1$.

Many rules: e.g. if both β and $\beta/2$ are roots, then $m_{\beta/2}$ is even and m_β is odd.
Many simplifications using classical formulas for Γ*:* e.g. $\Gamma(ix)\Gamma(-ix) = \frac{i\pi}{x \sinh(\pi x)}$.

Example: If G/K of even multiplicities, then $[c(i\lambda)c(-i\lambda)]^{-1}$ is a polynomial

$\widetilde{\rho}_\beta = \frac{1}{2}\left(\frac{m_{\beta/2}}{2} + m_\beta\right)$

Lemma

Set:

$\Pi(\lambda) = \prod_{\beta\in\Sigma_*^+} \lambda_\beta$,

$P(\lambda) = \prod_{\beta\in\Sigma_*^+} \left(\prod_{k=0}^{(m_{\beta/2})/2-1} \left[i\lambda_\beta - \left(\frac{m_{\beta/2}}{4} - \frac{1}{2}\right) + k\right] \prod_{k=0}^{2\widetilde{\rho}_\beta - 2} [i\lambda_\beta - (\widetilde{\rho}_\beta - 1) + k] \right)$,

$Q(\lambda) = \prod_{\substack{\beta\in\Sigma_*^+ \\ m_\beta \text{ odd}}} \coth(\pi(\lambda_\beta - \widetilde{\rho}_\beta))$.

(empty products are equal to 1)

Then:

$$[c(\lambda)c(-\lambda)]^{-1} \asymp \Pi(\lambda)P(\lambda)Q(\lambda).$$

Hence: $[c(i\lambda)c(-i\lambda)]^{-1}$ *has at most first order singularities along the hyperplanes*

$$\mathcal{H}_{\beta,k,\pm} = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \lambda_\beta = \pm i(\widetilde{\rho}_\beta + k)\}$$

where $\beta \in \Sigma_*^+$ *has multiplicity* m_β ***odd*** *and* $k \in \mathbb{Z}_{\geq 0}$.

$\Sigma_{*,\mathrm{odd}}^+ = \{\alpha \in \Sigma_*^+ : m_\alpha \text{ is odd}\}$

Extension of the resolvent of Δ on $X = G/K$

Suppose: real rank of X=dim $\mathfrak{a}^* =: n \geq 2$.

Let $f \in C_c^\infty(X)$ and $y \in X$ be fixed.

Recall

$$[R(z)f](y) \asymp \int_{\mathfrak{a}^*} \underbrace{\frac{1}{\langle\lambda,\lambda\rangle - z^2}}_{\text{singularities along } \mathbb{C}\text{-spheres radius } \pm z} (f \times \varphi_{i\lambda})(y) \underbrace{\frac{d\lambda}{c(i\lambda)c(-i\lambda)}}_{\text{singularities along hyperplanes } \mathcal{H}_{\beta,k,\pm}}$$

Polar coordinates in $\mathfrak{a}^*$ give

$$R(z) := [R(z)f](y) = \int_0^\infty \frac{1}{r^2 - z^2} F(r) r \, dr$$

where

$$F(r) = F_{f,y}(r) = r^{n-2} \int_{S^{n-1}} (f \times \varphi_{ir\sigma})(y) \, \frac{\omega(\sigma)}{c(ir\sigma)c(-ir\sigma)}$$

and

$\omega(\sigma) =$ pullback to S^{n-1} of the $\mathrm{SO}(n)$-invariant $(n-1)$-form

$$\omega(z) = \sum_{j=1}^n (-1)^{j-1} z_j \, dz_1 \cdots \widehat{dz_j} \cdots dz_n, \qquad z = (z_1, \dots, z_n) \in \mathbb{C}^n \equiv \mathfrak{a}_{\mathbb{C}}^*$$

Set $a = \min\{\widetilde{\rho}_\beta|\beta| : \beta \in \Sigma^+_{*,\mathrm{odd}}\}$ (and $a = +\infty$ if m_β even for all $\beta \in \Sigma^*_+$)

Lemma

- *For every fixed $\sigma \in \mathfrak{a}^*$ with $|\sigma| = 1$, the function $r \mapsto [c(ir\sigma)c(-ir\sigma)]^{-1}$ is holomorphic on $\mathbb{C} \setminus i(]-\infty, -a] \cup [a, +\infty[)$.*
- *The function*

$$\mathbb{C} \setminus i(]-\infty, -a] \cup [a, +\infty[) \ni w \to F(w) \in \mathbb{C}$$

  is holomorphic.
- *Let $U = \mathbb{C}^- \cup \{z \in \mathbb{C} : \mathrm{Re}\, z > 1, 0 \le \mathrm{Im}\, z < 1\}$, where $\mathbb{C}^- = \{z \in \mathbb{C} : \mathrm{Im}\, z < 0\}$.*
  Then $\exists$ holo function $H = H_{f,y} : U \to \mathbb{C}$ such that

$$R(z) = H(z) + i\pi F(z) \qquad \text{for } z \in U \cap \mathbb{C}^+$$

  .

Corollary

- *The mero extension of R across the negative imaginary axis (where the resonances could be) is equivalent to that of F.*
- *If any, the resonances are located on $i]-\infty, -a]$.*

The set $\Sigma^+_{*,\text{odd}}$

Let Σ be an irreducible root system in $\mathfrak{a}^*$ such that $\Sigma^+_{*,\text{odd}} \neq \emptyset$.

- Σ_* is a reduced and irreducible root system. So it has at most two root lengths.
- Roots of same lenght form a unique Weyl group orbit and have therefore same root multiplicity m_β.
- If there is a unique root length, then m_β is constant and $\Sigma^+_{*,\text{odd}} = \Sigma^+_*$. (This happens for $\Sigma = \Sigma_*$ of type A,D or E)
- If there are two root lengths (i.e. for Σ_* of type B,C,F or G), then $\Sigma^+_* = \Phi_1 \sqcup \Phi_2$, where roots in Φ_j have same length, and $\Sigma^+_{*,\text{odd}} \in \{\Sigma^+_*, \Phi_1, \Phi_2\}$.

  $\Sigma^+_* = \Phi_1 \sqcup \Phi_2$ is obtained from the following decompositions:

  $$B_n = (A_1)^n \sqcup D_n \qquad C_n = (A_1)^n \sqcup D_n \qquad F_4^+ = D_4^+ \sqcup D_4^+ \qquad G_2^+ = A_2^+ \sqcup A_2^+$$

Consequences: If $\Sigma^+_{*,\text{odd}} \neq \emptyset$, then:

◇ The hyperplane arrangement $\mathcal{H} = \{\ker \beta : \beta \in \Sigma^+_{*,\text{odd}}\}$ is simplicial (= every connected component of $\mathfrak{a}^* \setminus \cup\mathcal{H}$ is the intersection of $n = \dim \mathfrak{a}^*$ open halfspaces, i.e. is the positive linear span of n lin. indep. vectors).

◇ For some Σ of types B, C or BC, we have $\Sigma^+_{*,\text{odd}} = (A_1)^n$.

Example: G/K or rank 3 and root system Σ of type BC, B or C

$\Sigma^+ = \Sigma_{\mathrm{s}}^+ \sqcup \Sigma_{\mathrm{m}}^+ \sqcup \Sigma_{\mathrm{l}}^+$, where:
$\Sigma_{\mathrm{s}}^+ = \{e_j; 1 \le j \le n\}$, multiplicity m_{s},
$\Sigma_{\mathrm{m}}^+ = \{e_i \pm e_j; 1 \le i \ge j \le n\}$, multiplicity m_{m},
$\Sigma_{\mathrm{l}}^+ = \{2e_j; 1 \le j \le n\}$, multiplicity m_{l}.

$\mathrm{G/K}$	Σ	m_α	$\Sigma_{*,\mathrm{odd}}^+$
$\mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(3)$	A_3	1	Σ^+
$\mathrm{SU}^*(8)/\mathrm{Sp}(8)$	A_3	4	$\emptyset$
$\mathrm{SU}(3,q)/\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(q))$ $(q \ge 3)$	C_3 $(q=3)$ BC_3 $(q>3)$	$(2(q-3),2,1)$	Σ_{l}^+
$\mathrm{SO}_0(3,q)/\mathrm{SO}(3)\times\mathrm{SO}(q)$ $(q>3)$	B_3	$(q-3,1,0)$	Σ_{m}^+ (q odd) $\Sigma_{\mathrm{s}}^+ \sqcup \Sigma_{\mathrm{m}}^+$ (q even)
$\mathrm{SO}^*(12)/\mathrm{U}(6)$	BC_3	$(4,4,1)$	Σ_{l}^+
$\mathrm{Sp}(6,\mathbb{R})/\mathrm{U}(3)$	C_3	$(0,1,1)$	$\Sigma_{\mathrm{m}}^+ \sqcup \Sigma_{\mathrm{l}}^+$
$\mathrm{Sp}(3,q)/\mathrm{Sp}(3)\times\mathrm{Sp}(q)$ $(q \ge 3)$	BC_3	$(4(q-3),4,3)$	Σ_{l}^+
$\mathfrak{e}_{7(-25)}/(\mathfrak{e}_6+\mathbb{R})$	C_3	$(0,8,1)$	Σ_{l}^+

When $\Sigma_{*,\mathrm{odd}}^+ = \Sigma_{\mathrm{l}}^+$, the mero extension of F for G/K can be deduced from that for a direct product of rank-one symmetric spaces.

Direct products of rank-one symmetric spaces

$X = X_1 \times \cdots \times X_n$ where X_j=rank-one Riemannian symmetric noncompact type

(the index j indicates objects associated with X_j)

$$\mathfrak{a}^* = \mathfrak{a}_1^* \oplus \cdots \oplus \mathfrak{a}_n^*, \qquad \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 \oplus \cdots \oplus \langle \cdot, \cdot \rangle_n$$
$$\Sigma = \Sigma_1 \times \cdots \times \Sigma_n \quad \text{with} \quad \Sigma_j \in \{A_1, BC_1\}$$
$$\Delta = \sum_{j=1}^n (\mathrm{id} \otimes \cdots \otimes \Delta_j \otimes \cdots \mathrm{id}), \qquad \sigma(\Delta) = [\rho_X^2, +\infty[, \qquad \rho_X^2 = \rho_{X_1}^2 + \cdots + \rho_{X_n}^2$$
$$c(\lambda) = c_1(\lambda_1) \cdots c_n(\lambda_n), \qquad \lambda = \lambda_1 \cdots + \lambda_n \in \mathfrak{a}_{\mathbb{C}}^* \quad \text{with} \quad \lambda_j \in \mathfrak{a}_{j\mathbb{C}}^*$$

- The Plancherel density of X_j is singular iff $X_j \neq H^n(\mathbb{R})$ with n odd.
- The Plancherel density of X is the product of the Plancherel densities of the X_j's. It has first order singularities along N mutually orthogonal families of hyperplanes parallel to the coordinate axes, where $N = \sharp\{j \in \{1, \ldots, n\} : X_j \neq H^n(\mathbb{R}), n \text{ odd}\}$.

Example: product of two rank-one Riemannian symmetric spaces

J. Hilgert, A.P. and T. Przebinda (2017):

◇ meromorphic continuation of R to suitable Riemann surfaces over $\mathbb{C}$

◇ No resonances if **one** of the two spaces is $H^n(\mathbb{R})$ with n odd,

◇ infinitely many resonances in the other cases

◇ residue operators with **finite rank**

◇ range of the residue operators realized by finite direct sums of tensor products of finite dim irr K-spherical reps of G_1 and G_2

(where $X_1 = G_1/K_1$ and $X_2 = G_2/K_2$ are the symm spaces)

The integral defining F for $X = X_1 \times \cdots \times X_n$

Suppose $X_j \neq H^n(\mathbb{R})$, n odd, exactly for $j = 1, \ldots, N$ with $N \leq n$.

For $j = 1, \ldots, N$ define:

$p_j : \mathbb{C}^n \ni z = (z_1, \ldots, z_n) \to z_j \in \mathbb{C}$,

$L_j = (a_j + b_j\mathbb{Z}_{\geq 0}) \cup (-a_j - b_j\mathbb{Z}_{\geq 0})$ with $a_j > 0$, $b_j > 0$

$L = \bigcup_{j=1}^N p_j^{-1}(L_j) = \bigcup_{j=1}^N \bigcup_{l_j \in L_j} \{z \in \mathbb{C}^n : z_j = l_j\}$

$a = \min\{a_1, \ldots, a_N\}$.

$S^{n-1}(\mathbb{C}) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 = 1\}$ (the complex sphere)

$\omega(z) = \sum_{j=1}^n (-1)^{j-1} z_j \, dz_1 \cdots \widehat{dz_j} \cdots dz_n, \qquad z = (z_1, \ldots, z_n) \in \mathbb{C}^n$

Let $\mathbf{f} : \mathbb{C}^n \to \mathbb{C}$ be meromorphic on $\mathbb{C}^n$ and holomorphic on $\mathbb{C}^n \setminus iL$.

Since $\mathbf{f}(z)\omega(z)$ is a closed form of top complex dimension on $S^{n-1}(\mathbb{C}) \setminus iL$ the function

$$\mathbb{C} \setminus i((-\infty, -a] \cup [a, \infty)) \ni w \to F(w) = \int_{S^{n-1}} \mathbf{f}(wz)\omega(z) \in \mathbb{C}$$

is well defined and holomorphic.

Remark: For the study of the resolvent on X, one chooses

$\mathbf{f}(wz) = w^{n-2}(f \times \varphi_{iwz})(y) \, [c(iwz)c(-iwz)]^{-1}$, having identified $\mathfrak{a}_{\mathbb{C}}^* \ni \lambda \equiv z \in \mathbb{C}^n$.

Fix $v_0 \in]-\infty, -a] \cup [a, \infty[$. Then $S^{n-1}(\mathbb{R}) \cap \frac{1}{v_0} L \neq \emptyset$ is possible and therefore the integral $\int_{S^{n-1}} \mathbf{f}(wz)\omega(z)$, with $w = iv_0$, might diverge.

- Suppose $C_{iv_0} \subseteq S^{n-1}(\mathbb{C}) \setminus \frac{1}{v_0} L$ is a cycle homologous to S^{n-1} in $S^{n-1}(\mathbb{C})$.

  $\rightsquigarrow$ C_{iv_0} is a "deformation" of S^{n-1} within $S^{n-1}(\mathbb{C})$ which is disjoint with $\frac{1}{v_0} L$

Since L is a locally finite family of hyperplanes, $\exists$ an open neighborhood $W \subseteq \mathbb{C}$ of iv_0 such that $C_{iv_0} \subseteq S^{n-1}(\mathbb{C}) \setminus \frac{i}{W} L$. So

$$W \ni w \to \int_{C_{iv_0}} \mathbf{f}(wz)\omega(z) \in \mathbb{C}$$

is well defined and is holomorphic.

- Fix $w_0 \in W \cap \mathbb{C}_{Re>0}$. Suppose we have found finitely many cycles

$$C_k \subseteq S^{n-1}(\mathbb{C}) \setminus \frac{i}{w_0} L \qquad (k = 1, 2, \dots, M)$$

such that $[S^{n-1}] = [C_{iv_0}] + \sum_k [C_k]$ in $H_{n-1}(S^{n-1}(\mathbb{C}) \setminus \frac{i}{w_0} L)$.

Then, by Stokes Theorem, for $w \in \mathbb{C}_{\mathrm{Re}>0}$ near w_0

$$\int_{S^{n-1}} \mathbf{f}(wz)\omega(z) = \int_{C_{iv_0}} \mathbf{f}(wz)\omega(z) + \sum_k \int_{C_k} \mathbf{f}(wz)\omega(z).$$

The first integral on the RHS is holo on W. One hopes to choose the C_k's so that residue computations in z yield a mero function of $w \in W$.

- The homology of $S^{n-1}(\mathbb{C}) \setminus \{\text{hyperplane arrangement}\}$ is not known, unlike the case of $\mathbb{C}^n \setminus \{\text{hyperplane arrangement}\}$ (Goresky-MacPherson).
- Useful description: $S^{n-1}(\mathbb{C})$ can be identified with the tangent bundle

$$TS^{n-1} = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : |u| = 1, u \cdot v = 0\}$$

to S^{n-1} by means of the isomorphism

$$\tau : S^{n-1}(\mathbb{C}) \ni z = x + iy \to \left(\frac{x}{|x|}, y\right) \in TS^{n-1}$$

with inverse

$$\tau^{-1} : TS^{n-1} \ni (u, v) \to \sqrt{1 + |v|^2} u + iv \in S^{n-1}(\mathbb{C}).$$

- The general construction of the cycles is not yet achieved C_{iv_0} and C_k, even in rank 3.
- Easiest possible case of rank 3: $X = X_1 \times X_2 \times X_3$ with $X_1 \neq H^n(\mathbb{R})$, n odd, and $X_2 = X = 3 = H^n(\mathbb{R})$, n odd.

  One family of parallel singular hyperplanes perpendicular to x_1-axis.

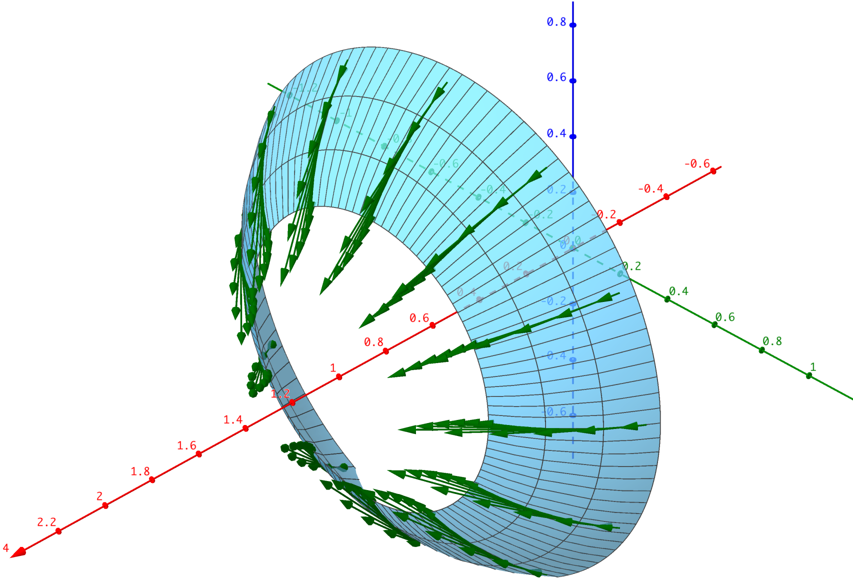
  For $v_0 \in]-\infty, -a] \cup [a, \infty[$: $S^2 \cap \frac{1}{v_0} L \neq \emptyset$ if and only if $\left|\frac{l}{v_0}\right| \leq 1$, and

  $\left|\frac{l}{v_0}\right| < 1 \;\Rightarrow$ intersection is a circle perpendicular to x_1 axis (generic case)

  $\left|\frac{l}{v_0}\right| = 1 \;\Rightarrow$ intersection is a single point $\in \{(\pm 1, 0, 0)\}$.

  Theorem. The resolvent R extends holomorphically to $\mathbb{C}$ (no resonances).

Happy Birthday, Joachim!



Happy Birthday, Joachim!