Loos Symmetric Cones

Jimmie Lawson
Louisiana State University

July, 2018

Dedication

I would like to dedicate this talk to Joachim Hilgert, whose 60th birthday we celebrate at this conference and with whom I researched and wrote a big blue book (along with Karl Hofmann) concerning *Lie Groups, Convex Cones, and Semigroups*. Since those earlier days my research concerning cones has veered in different directions, and I would like to report on one of those today.

Loos Symmetric Spaces

The 1969 approach of Ottmar Loos to symmetric spaces axiomatizes a binary operation $(a, b) \mapsto a \bullet b$ for which $S_a : M \to M$ defined by $S_a b = a \bullet b$ may be viewed as a *symmetry* or *point reflection* of M through a.

Let M be a Banach manifold, a smooth manifold modeled on some Banach space E (where smooth, as usual, means C^∞).

Definition

We say $(M, \bullet)$ is a *Loos symmetric space* if M is a Banach manifold, and $(x, y) \mapsto x \bullet y : M \times M \to M$ is a smooth map with the following properties for all $a, b, c \in M$:

(S1) $a \bullet a = a \quad (S_a a = a)$;

(S2) $a \bullet (a \bullet b) = b \quad (S_a S_a = \mathrm{id}_M)$;

(S3) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c) \quad (S_a S_b = S_{S_a b} S_a)$;

(S4) Every $a \in M$ has a neighborhood U such that $a \bullet x = x$ implies $a = x$ for $x \in U$.

Loos Symmetric Spaces

The 1969 approach of Ottmar Loos to symmetric spaces axiomatizes a binary operation $(a, b) \mapsto a \bullet b$ for which $S_a : M \to M$ defined by $S_a b = a \bullet b$ may be viewed as a *symmetry* or *point reflection* of M through a.

Let M be a Banach manifold, a smooth manifold modeled on some Banach space E (where smooth, as usual, means C^∞).

Definition

We say $(M, \bullet)$ is a *Loos symmetric space* if M is a Banach manifold, and $(x, y) \mapsto x \bullet y : M \times M \to M$ is a smooth map with the following properties for all $a, b, c \in M$:

(S1) $a \bullet a = a \quad (S_a a = a)$;

(S2) $a \bullet (a \bullet b) = b \quad (S_a S_a = \mathrm{id}_M)$;

(S3) $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c) \quad (S_a S_b = S_{S_a b} S_a)$;

(S4) Every $a \in M$ has a neighborhood U such that $a \bullet x = x$ implies $a = x$ for $x \in U$.

Symmetric Cones

Symmetric cones are typically defined as open convex self-dual cones in Euclidean space which have a transitive group of symmetries.

Our aim in this talk is to use a modified Loos approach to extend the study of symmetric cones to open cones in Banach spaces. The primary motivating examples are the cones of positive elements in C^*-algebras and the cone of invertible squares of a Jordan-Banach algebra (JB-algebra).

Three types of geometry come into play in our study: differential geometry, reflection or symmetric geometry, and the metric geometry of cones.

Symmetric Cones

Symmetric cones are typically defined as open convex self-dual cones in Euclidean space which have a transitive group of symmetries.

Our aim in this talk is to use a modified Loos approach to extend the study of symmetric cones to open cones in Banach spaces. The primary motivating examples are the cones of positive elements in C^*-algebras and the cone of invertible squares of a Jordan-Banach algebra (JB-algebra).

Three types of geometry come into play in our study: differential geometry, reflection or symmetric geometry, and the metric geometry of cones.

Sprays

A spray is a useful variant for the notion of a connection on a manifold in the Banach manifold setting.

Definition of a Spray

Let M be a Banach manifold and $\pi : TM \to M$ its tangent bundle. A *second-order vector field on M* is a vector field $F : TM \to TTM$ satisfying $T(\pi) \circ F = \mathrm{id}_{TM}$. Let $s \in \mathbb{R}$ and $s_{TM} : TM \to TM$ denote the scalar multiplication by s in each tangent space. A second order vector field F on TM is called a *spray* if

$$F(sv) = T(s_{TM})(sF(v)) \quad \text{for all } s \in \mathbb{R},\ v \in TM.$$

The Exponential Function and Parallel Transport

A spray F gives rise to integral curves in TM, *geodesics* in M (π-projections of the integral curves), and an exponential function. The domain $\mathcal{D}_{\exp} \subseteq TM$ of the *exponential function* is the set of all points $v \in T_x M$, $x \in M$, for which the maximal integral curve $\gamma_v : J \to TM$ of F with $\gamma_v(0) = v$ satisfies $1 \in J$; in this case $\exp_x(v) := \pi(\gamma_v(1))$.

Let $\alpha : [s, t] \to M$ be a piecewise smooth curve. We write

$$P_s^t(\alpha) : T_{\alpha(s)} M \to T_{\alpha(t)} M$$

for the corresponding linear map given by *parallel transport* along α.

A spray F gives rise to integral curves in TM, *geodesics* in M (π-projections of the integral curves), and an exponential function. The domain $\mathcal{D}_{\exp} \subseteq TM$ of the *exponential function* is the set of all points $v \in T_x M$, $x \in M$, for which the maximal integral curve $\gamma_v : J \to TM$ of F with $\gamma_v(0) = v$ satisfies $1 \in J$; in this case $\exp_x(v) := \pi(\gamma_v(1))$.

Let $\alpha : [s, t] \to M$ be a piecewise smooth curve. We write

$$P_s^t(\alpha) : T_{\alpha(s)}M \to T_{\alpha(t)}M$$

for the corresponding linear map given by *parallel transport* along α.

Neeb's Theorem (K.-H. Neeb, 2002)

Let $(M, \bullet)$ be a Loos symmetric space.

(i) Identifying $T(M \times M)$ with $T(M) \times T(M)$, then

$$v \bullet w := T(\mu)(v, w) \text{ where } \mu(x, y) := x \bullet y$$

defines a Loos symmetric space on TM.

(ii) The function $F : TM \to TTM,\ F(v) := -T(S_{v/2} \circ Z)(v)$ defines a spray on M, where $Z : M \to TM$ is the zero section and $S_{v/2}$ is the point symmetry for $v/2$ from part (i).

(iii) $\mathrm{Aut}(M, \bullet) = \mathrm{Aut}(M, F)$, and F is uniquely defined as the only spray invariant under all symmetries S_x, $x \in M$.

(iv) (M, F) is geodesically complete (all geodesics extend to $\mathbb{R}$).

(v) Let $\alpha : \mathbb{R} \to M$ be a geodesic and call the maps $\tau_{\alpha,s} := S_{\alpha(s/2)} \circ S_{\alpha(0)}$, $s \in \mathbb{R}$, translations along α. Then these are automorphisms of $(M, \bullet)$ with

$$\tau_{\alpha,s}(\alpha(t)) = \alpha(t + s) \quad \text{and} \quad d\tau_{\alpha,s}(\alpha(t)) = P_t^{t+s}(\alpha)$$

for all $s, t \in \mathbb{R}$.

Neeb's Theorem (K.-H. Neeb, 2002)

Let $(M, \bullet)$ be a Loos symmetric space.

(i) Identifying $T(M \times M)$ with $T(M) \times T(M)$, then

$$v \bullet w := T(\mu)(v, w) \text{ where } \mu(x, y) := x \bullet y$$

defines a Loos symmetric space on TM.

(ii) The function $F : TM \to TTM,\ F(v) := -T(S_{v/2} \circ Z)(v)$ defines a spray on M, where $Z : M \to TM$ is the zero section and $S_{v/2}$ is the point symmetry for $v/2$ from part (i).

(iii) $\mathrm{Aut}(M, \bullet) = \mathrm{Aut}(M, F)$, and F is uniquely defined as the only spray invariant under all symmetries S_x, $x \in M$.

(iv) (M, F) is geodesically complete (all geodesics extend to $\mathbb{R}$).

(v) Let $\alpha : \mathbb{R} \to M$ be a geodesic and call the maps $\tau_{\alpha,s} := S_{\alpha(s/2)} \circ S_{\alpha(0)}$, $s \in \mathbb{R}$, translations along α. Then these are automorphisms of $(M, \bullet)$ with

$$\tau_{\alpha,s}(\alpha(t)) = \alpha(t + s) \quad \text{and} \quad d\tau_{\alpha,s}(\alpha(t)) = P_t^{t+s}(\alpha)$$

for all $s, t \in \mathbb{R}$.

We consider the category with objects Loos symmetric spaces and morphisms smooth maps that are homomorphisms with respect to the operation $\bullet$. Note that $\mathbb{R}$ equipped with the operation $s \bullet t = 2s - t$ is an object in this category.

Proposition

Let $(M, \bullet)$ be a Loos symmetric space. Let $\alpha : \mathbb{R} \to M$ be a map. The following are equivalent.

1. α is a maximal geodesic.
2. There exists $x \in M$ and $v \in T_x M$ such that $\alpha(t) = \exp_x(tv)$ for all $t \in \mathbb{R}$.
3. α is a morphism in the category of Loos symmetric spaces.
4. α is a *continuous* homomorphism from $(\mathbb{R}, \bullet) \to (M, \bullet)$.

Geodesics

We consider the category with objects Loos symmetric spaces and morphisms smooth maps that are homomorphisms with respect to the operation $\bullet$. Note that $\mathbb{R}$ equipped with the operation $s \bullet t = 2s - t$ is an object in this category.

Proposition

Let $(M, \bullet)$ be a Loos symmetric space. Let $\alpha : \mathbb{R} \to M$ be a map. The following are equivalent.

1. α is a maximal geodesic.
2. There exists $x \in M$ and $v \in T_x M$ such that $\alpha(t) = \exp_x(tv)$ for all $t \in \mathbb{R}$.
3. α is a morphism in the category of Loos symmetric spaces.
4. α is a *continuous* homomorphism from $(\mathbb{R}, \bullet) \to (M, \bullet)$.

Midpoints of Symmetry

In the Loos axioms the first two axioms $S_a a = a$ and $S_a S_a = \mathrm{id}_M$ have obvious intuitive geometric content. The third axiom (S3) may be rewritten as $S_a(b \bullet c) = (S_a b) \bullet (S_a c)$, which shows that S_a is a morphism in the Loos symmetric space category.

For our purposes we need a stronger version of (S4), namely:

(S4*) the equation $x \bullet a = b$ has a unique solution x.

Applying (S2) we see that $x \bullet b = a$ and thus that S_x is the unique point reflection carrying a to b and vice-versa. It is thus appropriate to call the solution x a *midpoint of symmetry* for a and b. We denote this unique solution of $x \bullet a = b$ by $a\#b$ and note that $a\#b = b\#a$.

Midpoints of Symmetry

In the Loos axioms the first two axioms $S_a a = a$ and $S_a S_a = \mathrm{id}_M$ have obvious intuitive geometric content. The third axiom (S3) may be rewritten as $S_a(b \bullet c) = (S_a b) \bullet (S_a c)$, which shows that S_a is a morphism in the Loos symmetric space category.

For our purposes we need a stronger version of (S4), namely:

(S4*) the equation $x \bullet a = b$ has a unique solution x.

Applying (S2) we see that $x \bullet b = a$ and thus that S_x is the unique point reflection carrying a to b and vice-versa. It is thus appropriate to call the solution x a *midpoint of symmetry* for a and b. We denote this unique solution of $x \bullet a = b$ by $a \# b$ and note that $a \# b = b \# a$.

In addition we often want our symmetric structures to be *pointed*, i.e., to have a designated element that we typically write as ε.

Note. The preceding systems are entirely algebraically defined.

Midpoints of Symmetry

In the Loos axioms the first two axioms $S_a a = a$ and $S_a S_a = \mathrm{id}_M$ have obvious intuitive geometric content. The third axiom (S3) may be rewritten as $S_a(b \bullet c) = (S_a b) \bullet (S_a c)$, which shows that S_a is a morphism in the Loos symmetric space category.

For our purposes we need a stronger version of (S4), namely:

(S4*) the equation $x \bullet a = b$ has a unique solution x.

Applying (S2) we see that $x \bullet b = a$ and thus that S_x is the unique point reflection carrying a to b and vice-versa. It is thus appropriate to call the solution x a *midpoint of symmetry* for a and b. We denote this unique solution of $x \bullet a = b$ by $a \# b$ and note that $a \# b = b \# a$.

In addition we often want our symmetric structures to be *pointed*, i.e., to have a designated element that we typically write as ε.

Note. The preceding systems are entirely algebraically defined.

Dyadic Powers

Let $(A, \bullet, \varepsilon)$ be a pointed set with binary operation $\bullet$ satisfying (S1), (S2), (S3), and (S4*). We define for $x \in A$

$$x^{-1} := S_\varepsilon x, \qquad x^2 := S_x \varepsilon, \qquad x^{1/2} := x \# \varepsilon,$$

and inductively from these definitions all dyadic rational powers may be defined. Furthermore, the dyadic rationals $\mathbb{D}$ equipped with the operation $p \bullet q = 2p - q$ satisfies (S1)-(S4*) and $q \mapsto x^q$ is a $\bullet$-homomorphism from $\mathbb{D}$ to A.

Note. In systems satisfying (S4*) a $\bullet$-homomorphism is a $\#$-homomorphism and vice-versa.

Dyadic Powers

Let $(A, \bullet, \varepsilon)$ be a pointed set with binary operation $\bullet$ satisfying (S1), (S2), (S3), and (S4*). We define for $x \in A$

$$x^{-1} := S_\varepsilon x, \qquad x^2 := S_x \varepsilon, \qquad x^{1/2} := x \# \varepsilon,$$

and inductively from these definitions all dyadic rational powers may be defined. Furthermore, the dyadic rationals $\mathbb{D}$ equipped with the operation $p \bullet q = 2p - q$ satisfies (S1)-(S4*) and $q \mapsto x^q$ is a $\bullet$-homomorphism from $\mathbb{D}$ to A.

Note. In systems satisfying (S4*) a $\bullet$-homomorphism is a $\#$-homomorphism and vice-versa.

Some Examples

The following are some examples of Loos symmetric spaces satisfying (S4*).

(1) Any Banach space E with $a \bullet b = 2a - b$, $a \# b = \frac{1}{2}(a + b)$, and $\varepsilon = 0$, a specific case being $E = \mathbb{R}$.

(2) The manifold of $n \times n$ positive definite matrices $\mathbb{P}_n$ with $A \bullet B = AB^{-1}A$, $A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ (the matrix geometric mean), and $\varepsilon = I$.

Some Examples

The following are some examples of Loos symmetric spaces satisfying (S4*).

(1) Any Banach space E with $a \bullet b = 2a - b$, $a \# b = \frac{1}{2}(a + b)$, and $\varepsilon = 0$, a specific case being $E = \mathbb{R}$.

(2) The manifold of $n \times n$ positive definite matrices $\mathbb{P}_n$ with $A \bullet B = AB^{-1}A$, $A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ (the matrix geometric mean), and $\varepsilon = I$.

(3) More generally, the classical simply connected Riemannian symmetric spaces of nonpositive curvature.

Some Examples

The following are some examples of Loos symmetric spaces satisfying (S4*).

(1) Any Banach space E with $a \bullet b = 2a - b$, $a\#b = \frac{1}{2}(a+b)$, and $\varepsilon = 0$, a specific case being $E = \mathbb{R}$.

(2) The manifold of $n \times n$ positive definite matrices $\mathbb{P}_n$ with $A \bullet B = AB^{-1}A$, $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ (the matrix geometric mean), and $\varepsilon = I$.

(3) More generally, the classical simply connected Riemannian symmetric spaces of nonpositive curvature.

Felix Klein's Erlangen Program (1872) emphasized the role of geometric automorphism groups in the study of geometry. In the case of a Loos symmetric space $(M, \bullet)$ this group is the *displacement group* $G(M)$, the group generated by the *displacements* $S_x S_y$ for $x, y \in M$. By (S2) and (S3) these are all automorphisms of $(M, \bullet)$. In the case M is pointed, the group is also generated by all *basic displacements* $S_x S_\varepsilon$ for $x \in M$. In this setting we denote $S_x S_\varepsilon$ by $Q(x)$. (There are closely related to displacements in Jordan algebras.)

Some Basic Identities:
(1) $Q(Q(x)y) = Q(x)Q(y)Q(x)$;
(2) $Q(x^{-1}) = Q(x)^{-1}$.

The tuple $(M, d, \bullet, \varepsilon)$ is a pointed symmetric metric space of nonpositive Busemann curvature if (M, d) is a metric space, $(M, \bullet)$ satisfies (S1)-(S4*), $(x, y) \to x \bullet y$ is continuous and for all $x, y \in M$
(1) $d(gx, gy) = d(x, y)$ for all displacements $g \in G(M)$;
(2) $d(x^{-1}, y^{-1}) = d(x, y)$;
(3) $d(x^{1/2}, y^{1/2}) \leq \frac{1}{2} d(x, y)$ (the Busemann condition).

Theorem

For $(M, d, \bullet, \varepsilon)$ as given, whenever $x \neq y$, there exists a unique (injective) $\bullet$-homomorphism $\alpha : \mathbb{R} \to M$ such that $\alpha(0) = x$ and $\alpha(1) = y$. Furthermore, α is a metric geodesic in the sense that $d(\alpha(s), \alpha(t)) = |s - t| d(x, y)$ for all $s, t \in \mathbb{R}$.

We call $\alpha(t)$ the *t-weighted mean* of x and y and write it $x \#_t y$. Note $x \bullet y = x \#_{-1} y$ and $x \# y = x \#_{1/2} y$.

Metric spaces of Nonpositive Curvature

The tuple $(M, d, \bullet, \varepsilon)$ is a pointed symmetric metric space of nonpositive Busemann curvature if (M, d) is a metric space, $(M, \bullet)$ satisfies (S1)-(S4*), $(x, y) \to x \bullet y$ is continuous and for all $x, y \in M$
(1) $d(gx, gy) = d(x, y)$ for all displacements $g \in G(M)$;
(2) $d(x^{-1}, y^{-1}) = d(x, y)$;
(3) $d(x^{1/2}, y^{1/2}) \leq \frac{1}{2} d(x, y)$ (the Busemann condition).

Theorem

For $(M, d, \bullet, \varepsilon)$ as given, whenever $x \neq y$, there exists a unique (injective) $\bullet$-homomorphism $\alpha : \mathbb{R} \to M$ such that $\alpha(0) = x$ and $\alpha(1) = y$. Furthermore, α is a metric geodesic in the sense that $d(\alpha(s), \alpha(t)) = |s - t| d(x, y)$ for all $s, t \in \mathbb{R}$.

We call $\alpha(t)$ the *t-weighted mean* of x and y and write it $x \#_t y$. Note $x \bullet y = x \#_{-1} y$ and $x \# y = x \#_{1/2} y$.

Normal Cones

We turn now to our main topic of interest, Loos symmetric structures on cones.

Let E be a Banach space and let Ω be an open cone, a subset that is topologically open and is closed under addition and multiplication by positive scalars. We assume that the closed cone $\overline{\Omega}$ satisfies $\overline{\Omega} \cap \overline{\Omega} = \{0\}$. We define a partial order on E by $x \leq y$ iff $y - x \in \overline{\Omega}$.

We further assume that $\overline{\Omega}$ is *normal*, that is, there exists an equivalent norm on E satisfying $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$, and we henceforth assume that this is the one chosen. By a slight abuse of language we also speak of Ω as a normal cone if $\overline{\Omega}$ is.

We turn now to our main topic of interest, Loos symmetric structures on cones.

Let E be a Banach space and let Ω be an open cone, a subset that is topologically open and is closed under addition and multiplication by positive scalars. We assume that the closed cone $\overline{\Omega}$ satisfies $\overline{\Omega} \cap \overline{\Omega} = \{0\}$. We define a partial order on E by $x \leq y$ iff $y - x \in \overline{\Omega}$.

We further assume that $\overline{\Omega}$ is *normal*, that is, there exists an equivalent norm on E satisfying $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$, and we henceforth assume that this is the one chosen. By a slight abuse of language we also speak of Ω as a normal cone if $\overline{\Omega}$ is.

The Thompson Metric

Let Ω be an open normal cone in the Banach space E. For $x, y \in \Omega$ we set

$$\begin{aligned} M(x/y): &= \inf\{t > 0 : x \leq ty\} \\ d_T(x, y): &= \max\{\log(M(x/y), \log(My/x)\}. \end{aligned}$$

The *Thompson* or *part* metric d_T is a complete metric on Ω and the metric topology agrees with the relative topology from E.

A surprising amount can be obtained without smoothness.

Hypotheses

Let Ω be an open normal cone in the Banach space E. Let $(\Omega, \bullet, \varepsilon)$ satisfy (S1)-(S4*) and assume $\bullet$ is a continuous binary operation. We further assume the conditions

- $x^{1/2}(= x\#\varepsilon) \leq \frac{x+\varepsilon}{2}$ for all $x \in \Omega$;
- each basic displacement $Q(x)$ is additive and positively homogeneous on Ω.

Note the the hypothesis $x^{1/2} \leq (1/2)(x+\varepsilon)$ relates the symmetric structure and the partial order (and is equivalent to $(x-\varepsilon)^2 \geq 0$ in C^*-algebras). The preservation of the linear structure of Ω by each $Q(x)$ relates the linear structure of Ω to the displacement group of Ω.

Conclusions (Y. Lim, L.)

Under the hypotheses of the preceding slide, we obtain the following conclusions.

(1) Each displacement $Q \in G(\Omega)$ is an isometry for the Thompson distance and extends to an invertible bounded linear operator on E that is an order isomorphism.

(2) With respect to the Thompson metric $a\#b$ is the metric midpoint of a and b.

Conclusions (Y. Lim, L.)

Under the hypotheses of the preceding slide, we obtain the following conclusions.

(1) Each displacement $Q \in G(\Omega)$ is an isometry for the Thompson distance and extends to an invertible bounded linear operator on E that is an order isomorphism.

(2) With respect to the Thompson metric $a\#b$ is the metric midpoint of a and b.

(3) For each $a \neq b$, the map $t \mapsto a\#_t b$ is a $\bullet$-homomorphism, a maximal metric geodesic, and the unique one carrying 0 to a and 1 to b. Furthermore, the function $(t, x, y) \mapsto x\#_t y : \mathbb{R} \times M \times M \to M$ is continuous.

Conclusions (Y. Lim, L.)

Under the hypotheses of the preceding slide, we obtain the following conclusions.

(1) Each displacement $Q \in G(\Omega)$ is an isometry for the Thompson distance and extends to an invertible bounded linear operator on E that is an order isomorphism.

(2) With respect to the Thompson metric $a\#b$ is the metric midpoint of a and b.

(3) For each $a \neq b$, the map $t \mapsto a\#_t b$ is a $\bullet$-homomorphism, a maximal metric geodesic, and the unique one carrying 0 to a and 1 to b. Furthermore, the function $(t, x, y) \mapsto x\#_t y : \mathbb{R} \times M \times M \to M$ is continuous.

(4) $d_T(a\#_t b, a\#_t c) \leq t\, d_T(b, c)$ for $a, b, c \in \Omega$. Hence Ω equipped with the Thompson metric has nonpositive curvature in the sense of Busemann.

Conclusions (Y. Lim, L.)

Under the hypotheses of the preceding slide, we obtain the following conclusions.
(1) Each displacement $Q \in G(\Omega)$ is an isometry for the Thompson distance and extends to an invertible bounded linear operator on E that is an order isomorphism.
(2) With respect to the Thompson metric $a\#b$ is the metric midpoint of a and b.
(3) For each $a \neq b$, the map $t \mapsto a\#_t b$ is a $\bullet$-homomorphism, a maximal metric geodesic, and the unique one carrying 0 to a and 1 to b. Furthermore, the function $(t, x, y) \mapsto x\#_t y : \mathbb{R} \times M \times M \to M$ is continuous.
(4) $d_T(a\#_t b, a\#_t c) \leq t\, d_T(b, c)$ for $a, b, c \in \Omega$. Hence Ω equipped with the Thompson metric has nonpositive curvature in the sense of Busemann.

Loos Symmetric Cones Defined (finally)

We enhance our earlier definition by adding smoothness.

Definition

A triple $(\Omega, \bullet, \varepsilon)$ with $\varepsilon \in \Omega$ is a pointed Loos symmetric cone if

1. Ω is an open normal cone in a Banach space E;
2. $(\Omega, \bullet)$ satisfies $a \bullet a = a$, $a \bullet (a \bullet b) = b$; $a \bullet (b \bullet c) = (a \bullet b) \bullet (a \bullet c)$ and $x \bullet a = b$ has unique solution $x = a \# b$;
3. $(a, b) \mapsto a \bullet b$, $(a, b) \mapsto a \# b$ are both smooth;
4. $a^{1/2} (= a \# \varepsilon) \leq \frac{a+\varepsilon}{2}$ for all $a \in \Omega$;
5. each basic displacement $Q(a)$ is additive and positively homogeneous on Ω.

A Finsler Metric

For each $a \in \Omega$ we can define an order unit norm on E by

$$\|x\|_a = \inf\{\lambda > 0 : -\lambda a \leq x \leq \lambda a\},$$

which is equivalent to the given norm on E. Identifying the tangent space $T\Omega$ with $\Omega \times E$, we define a Finsler structure on $T\Omega$ by $\|(a, x)\| = \|x\|_a$. This Finsler structure is invariant under the action of $TQ : T\Omega \to T\Omega$ for each Q in the displacement group.

The Finsler structure gives a method of computing arc length for piecewise differentiable curves in Ω.

Proposition. Let $\alpha(t) = a\#_t b$ for $a \neq b \in \Omega$. For any $t_0 < t_1$ the restriction of α to $[t_0, t_1]$ is a minimal length curve from $\alpha(t_0)$ to $\alpha(t_1)$ of length $d_T(\alpha(t_0), \alpha(t_1))$. It follows that the length metric for the Finsler metric is the Thompson metric.

For each $a \in \Omega$ we can define an order unit norm on E by

$$\|x\|_a = \inf\{\lambda > 0 : -\lambda a \leq x \leq \lambda a\},$$

which is equivalent to the given norm on E. Identifying the tangent space $T\Omega$ with $\Omega \times E$, we define a Finsler structure on $T\Omega$ by $\|(a, x)\| = \|x\|_a$. This Finsler structure is invariant under the action of $TQ : T\Omega \to T\Omega$ for each Q in the displacement group.

The Finsler structure gives a method of computing arc length for piecewise differentiable curves in Ω.
Proposition. Let $\alpha(t) = a\#_t b$ for $a \neq b \in \Omega$. For any $t_0 < t_1$ the restriction of α to $[t_0, t_1]$ is a minimal length curve from $\alpha(t_0)$ to $\alpha(t_1)$ of length $d_T(\alpha(t_0), \alpha(t_1))$. It follows that the length metric for the Finsler metric is the Thompson metric.

The Exponential Function Revisited

For a pointed Loos symmetric space $(\Omega, \bullet, \varepsilon)$, we assume that the norm of the Banach space E is the order unit norm $\|\cdot\|_\varepsilon$, we identify $T_\varepsilon\Omega$ with E and write the exponential function $\exp_\varepsilon : T_\varepsilon\Omega \to \Omega$ as $\exp : E \to \Omega$.

Basic Results

(1) $\exp : E \to \Omega$ is a diffeomorphism with inverse "log."
(2) For $a \in \Omega$, $\alpha(t) := a^t = \varepsilon \#_t a = \exp(t \log a)$ is the (spray, Finsler, and metric) geodesic with $\alpha(0) = \varepsilon$ and $\alpha(1) = a$.
(3) $d_T(\varepsilon, b) = \|\log b\|$ and $d_T(a, b) = \|\log(Q(a^{-1/2})b)\|$ for all $a, b \in \Omega$.

Example: The Positive Cone of a C^*-algebra

Let Ω be the cone of positive elements of a C^*-algebra with identity. Then Ω admits the structure of a pointed Loos symmetric cone with

- $E = \Omega - \Omega$, the closed subspace of self-adjoint (hermitian) elements;
- $\varepsilon = 1$;
- $a \bullet b = ab^{-1}a, \quad a\#b = a^{1/2}(a^{-1/2}ba^{-1/2})^{1/2}a^{1/2}$;
- $Q(a)(b) = aba$, $a\#_t b = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$;
- The exponential, log, and power function a^t are all the same in the C^*-algebra and the Loos symmetric cone, as are the norms $\|\cdot\|$.

Inequalities

An interesting problem is what inequalities can be derived in the setting of Loos symmetric cones. They have something of a universal character since they are valid beyond C^*-algebras. The algebraic-geometric nature of Loos symmetric cones also provides new tools, primarily geometric ones, for the study of inequalities.

We list some sample inequalities that can be derived in the context of Loos symmetric cones.

- (The harmonic-geometric-arithmetic mean inequality)
$$2(a^{-1}+b^{-1})^{-1} \leq a\#b \leq \frac{a+b}{2} \text{ for } a, b \in \Omega.$$
- (Loewner-Heinz) If $a \leq b$, then $a^t \leq b^t$ for $0 \leq t \leq 1$..
- $\|\log(Q(a^{-t/2}))b^t)\| \leq t\|\log(Q(a^{-1/2}))b\|$ for $0 < t < 1$.

In the context of C^*-algebras the last inequality is a variant of the Cordes inequality $\|a^t b^t\| \leq \|ab\|^t$ for $0 < t < 1$.

An interesting problem is what inequalities can be derived in the setting of Loos symmetric cones. They have something of a universal character since they are valid beyond C^*-algebras. The algebraic-geometric nature of Loos symmetric cones also provides new tools, primarily geometric ones, for the study of inequalities.

We list some sample inequalities that can be derived in the context of Loos symmetric cones.

- (The harmonic-geometric-arithmetic mean inequality) $$2(a^{-1}+b^{-1})^{-1} \leq a\#b \leq \tfrac{a+b}{2} \text{ for } a, b \in \Omega.$$
- (Loewner-Heinz) If $a \leq b$, then $a^t \leq b^t$ for $0 \leq t \leq 1$..
- $\|\log(Q(a^{-t/2}))b^t)\| \leq t\|\log(Q(a^{-1/2}))b\|$ for $0 < t < 1$.

In the context of C^*-algebras the last inequality is a variant of the Cordes inequality $\|a^t b^t\| \leq \|ab\|^t$ for $0 < t < 1$.

The previous inequalities are presumably only a very small sample of what can be derived and one hopes that a more systematic study would find interesting new derivations of old results and even some newer ones and provide new tools for their study.

THE END