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Naturally reductive homogeneous spaces – classification and special geometries

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Homogeneous manifolds

→Link algebraic theory of Lie groups and geometric notions such as isometry and curvature

- É. Cartan 1926

→classification of (Riemannian) symmetric spaces

However classification without further assumptions seems to be impossible

Substancial work in special cases:

→very small dimensions, positive curvature, isotropy irreducible...

Our class: naturally reductive homogeneous spaces

joint with I. Agricola, T. Friedrich and R. Storm

Homogeneous space

- (M, g) plus $G \subseteq \mathrm{Iso}(M)$ s.t. G acts on M transitively and effectively.

  H stabilizer of a point $p \in M$, $M = G/H$.

→Assume M with good properties: connected, simply connected, complete.

Thm. [Ambrose-Singer]

(M, g) is homogeneous iff $\exists T \in \Lambda^2 M \otimes TM$ s. t.

→$\nabla T = 0 = \nabla R$ (AS)

∇ metric connection with torsion T

$[\nabla = \nabla^g + \frac{1}{2} T]$

Constructive proof!

Torsion tensors

- $\mathcal{T} = \Lambda^2 M \otimes TM$

$\rightarrow$Under the action of $\mathrm{O}(n)$ [Cartan]

$$\mathcal{T} = \Lambda^3 M \oplus TM \oplus \mathcal{C}$$

$\rightarrow$Eight classes of homogeneous spaces

- $T \in TM$ [Triceri-Vanhecke]

$\rightarrow$$(M^n, g)$ is the hyperbolic space of dim n

Naturally reductive: $T \in \Lambda^3 M$ [assume $T \neq 0$]

Naturally reductive spaces

Dfn. G/H is *naturally reductive* if $\mathfrak{h}$ admits a *reductive* complement $\mathfrak{m}$ in $\mathfrak{g}$:

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle \; = \; 0$$

→The PFB $G \to G/K$ induces a metric connection ∇ with torsion

$$g(T(X,Y),Z) := T(X,Y,Z) = -\langle [X,Y]_{\mathfrak{m}}, Z\rangle,$$

the so-called *canonical connection* of the nat. red. homog. space.

Conversely, **Nomizu construction**

given (M,g,T) as in (AS) we can recover $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$:

→$\mathfrak{h}$ is the holonomy algebra, $\mathfrak{m}$ is identified with T_pM, for some p

→$[A+X, B+Y] = ([A,B]_{\mathfrak{h}} - R(X,Y)) + (AY - BX - T(X,Y))$

Examples

- any Lie group with a biinv. metric (b.t.w. flat) [Cartan-Schouten]

- construction/classification of left-invariant nat. red. metrics on compact Lie groups [D'Atri-Ziller]

- all isotropy irreducible homogeneous manifolds

- Spheres can carry several nat.red.structures, for example

$$S^{2n+1} = \mathrm{SO}(2n+2)/\mathrm{SO}(2n+1) = \mathrm{SU}(n+1)/\mathrm{SU}(n),$$

$$S^6 = G_2/\mathrm{SU}(3), \; S^7 = \mathrm{Spin}(7)/G_2, \; S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7).$$

But if (M, g) not loc. isometric to sphere or Lie group then admits at most ONE nat. red. structure. [Olmos-Reggiani]

Known classifications

- In dimension 3 [Triceri-Vanhecke]

  →space forms: $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ (in infinite number)

  →one of the following (with suitable left inv. metric): $SU(2)$, $\widetilde{SL}(2,\mathbb{R})$, H^3

- In dimension 4 [Kowalski-Vanhecke]

  →M is loc. $\mathbb{R} \times N^3$ with N^3 nat. red.

- In dimension 5 [Kowalski-Vanhecke]

  →$\mathrm{SU}(3)/\mathrm{SU}(2)$ or $\mathrm{SU}(2,1)/\mathrm{SU}(2)$

  →H^5

  →$(K_1 \times K_2)/SO(2)$, where K_1 and K_2 are $\mathrm{SU}(2)$, $\widetilde{SL}(2,\mathbb{R})$ or H^3

Our approach

- Look at the parallel torsion as the fundamental object.

- For 'non-degenerate' torsion, the connection in (AS) is the characteristic connection for some known geometry (almost contact, almost Hermitian...).

An important tool

- $\sigma_T := \frac{1}{2}\sum_{i=1}^{n}(e_i \lrcorner T) \wedge (e_i \lrcorner T) \; = \overset{X,Y,Z}{\mathfrak{S}} \; g(T(X,Y), T(Z,V))$ $(= 0$ if $n \leq 4)$

  * $\overset{X,Y,Z}{\mathfrak{S}} \; \mathcal{R}(X,Y,Z,V) = \sigma_T(X,Y,Z,V)$
  * $dT = 2\sigma_T$

Thm. [Agricola-Friedrich-F.]

M irreducible, $n \geq 5$, $\nabla T = 0$, $\sigma_T = 0$, then M is a simple compact Lie group (with biinv. metric) or its dual noncompact symmetric space.

Dimension 6

→$*\sigma_T$ is a 2-form.

→Can be seen as a skew endomorphism.

Does it induce an almost complex struture?

Not always... Classify it by its rank (=0,2,4,6)

Case A: $\sigma_T = 0$

Thm. [AFF]

A Riem. 6-mnfld with parallel skew torsion T s.t. $\sigma_T = 0$ splits into two 3-dimensional manifolds with parallel skew torsion.

Cor. [AFF]

Any 6-dim. nat. red. homog. space with $\sigma_T = 0$ is loc. isometric to a product of two 3-dimensional nat. red. homog. spaces.

Dimension 6

Case B: $\mathrm{rk}(*\sigma_T) = 2$

Thm. [AFF]

Let (M^6, g, T) be a 6-mnfd with parallel skew torsion s.t. $\mathrm{rk}(*\sigma_T) = 2$. Then $\nabla\mathcal{R} = 0$, i. e. M is nat. red.

Furthermore, M is a product $K_1 \times K_2$ of two Lie groups equiped with a left inv. metric and K_1 and K_2 are $\mathrm{SU}(2)$, $\widetilde{SL}(2, \mathbb{R})$ or H^3

Case C: $\mathrm{rk}(*\sigma_T) = 4$

Cannot occur.

Dimension 6

Case D: $\mathrm{rk}(*\sigma_T) = 6$

$*\sigma_T$ induces an almost complex structure J $\surd$

Characteristic connection. [Friedrich-Ivanov]

(M^{2n}, g, J, Ω) admits ∇ with skew torsion T such that $\nabla J = 0 = \nabla g$ iff N is skew-symmetric.

$\rightarrow T = N + d^J\Omega$

$$N(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

$$d^J\Omega(X,Y,Z) = d\Omega(JX, JY, JZ)$$

Dimension 6

Case D: $\mathrm{rk}(*\sigma_T) = 6$

Thm. [AFF]

Let (M^6, g, T) be a 6-mnfd with parallel skew torsion s.t. $\mathrm{rk}(*\sigma_T) = 6$. Then:

Case D.1 (M^6, g) is nearly Kähler: $(\nabla^g_X J)X = 0$ (∗)

Case D.2 (M^6, g) is nat. red. and:

→a 2-step nilp. group with Lie alg. $\mathbb{R}^3 \times \mathbb{R}^3$ s.t. $[(u_1, v_1), (u_2, v_2)] = (0, u_1 \times u_2)$

→$S^3 \times \mathbb{R}^3$, →$S^3 \ltimes \mathbb{R}^3$ →$S^3 \times S^3$, →$\mathrm{SL}(2, \mathbb{C})$

(∗) If homog. then nat. red. and: S^6, $S^3 \times S^3$, $\mathbb{C}P^3$, $\mathrm{U}(3)/\mathrm{U}(1)^3$ [Butruille 2005]

Other dimensions

Semidirect products

H, N connected Lie groups, N abelian.

$\varphi : H \longrightarrow \mathrm{Aut}(N)$ non-trivial homomorphism.

$H \ltimes_\varphi N$: $\quad H \times N$ with $(h_1, n_1)(h_2, n_2) = (h_1 h_2, n_1 + \varphi(h_1) n_2)$

Induces $\varphi_* : \mathfrak{h} \longrightarrow \mathrm{End}(\mathfrak{n})$ $\quad [(A, u), (B, v)] = ([A, B], \varphi_*(A)(v) - \varphi_*(B)(u))$

Lemma. [Agricola-F.]

$H \ltimes_\varphi N$ admits no bi-invariant metrics.

Corollary.

H compact, N vector space: $H \ltimes_\varphi N$ and $H \times N$ are not isomorphic as Lie groups .

Other dimensions

Tangent Lie groups

- Choose: $H = G$, $N = \mathfrak{g}$
  $\varphi = \mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$ the adjoint representation.

$\rightarrow TG = G \ltimes_{\mathrm{Ad}} \mathfrak{g}$ tangent Lie group

$\rightarrow TG$ is the tangent bundle of G

- Metrics of split signature have been constructed on generic TM^n [Kobayashi-Yano]

- Nad. red. metrics lift to nat. red. metrics (split signature) [Sekizawa]

Tangent Lie groups

Compact Lie group G equipped with a bi-invariant metric.

Thm. [AF]

There is a two-parameter family of almost Hermitian structures $(TG, g_{a.b}, J_{a,b})$:

$\rightarrow$Characteristic connection ∇ is s.t. $\nabla T = 0 = \nabla R$

$\rightarrow \mathfrak{hol}(\nabla) = [\mathfrak{g}, \mathfrak{g}]$

$\rightarrow$Each metric $g_{a,b}$ is isometric to a left-invariant metric on $G \times \mathfrak{g}$.

Further directions

New families were produced by R. Storm

(generalizes a construction of C. Gordon for 2-step nilpotent groups)

R. Storm advanced a classif. of nat. red. spaces in dim. 7 and 8.