Horn's problem, and Fourier analysis

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Symmetries in Geometry, Analysis, and Spectral Analysis,
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Horn's problem, and Horn's conjecture

A and B are $n \times n$ Hermitian matrices, and $C = A + B$.
Assume that the eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$ of A,
and the eigenvalues $\beta_1 \geq \cdots \geq \beta_n$ of B are known.
Horn's problem : What can be said about the eigenvalues $\gamma_1 \geq \cdots \geq \gamma_n$ of $C = A + B$?

Weyl's inequalities (1912)

$$\begin{array}{lcll} \gamma_{i+j-1} & \leq & \alpha_i + \beta_j & \text{for } i + j \leq n + 1, \\ \gamma_{i+j-n} & \geq & \alpha_i + \beta_j & \text{for } i + j \geq n + 1. \end{array}$$

Horn's conjecture (1962) The set of possible eigenvalues $\gamma_1, \ldots, \gamma_n$ for $C = A + B$ is determined by a family of inequalities of the form

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j,$$

for certain admissible triples (I, J, K) of subsets of $\{1, \ldots, n\}$.
Klyachko has proven Horn's conjecture,
and described these admissible triples **(1998)**.

$n = 3$, $\alpha = (3.5, 1.4, -4.9)$, $\beta = (2, -0.86, -1.14)$.

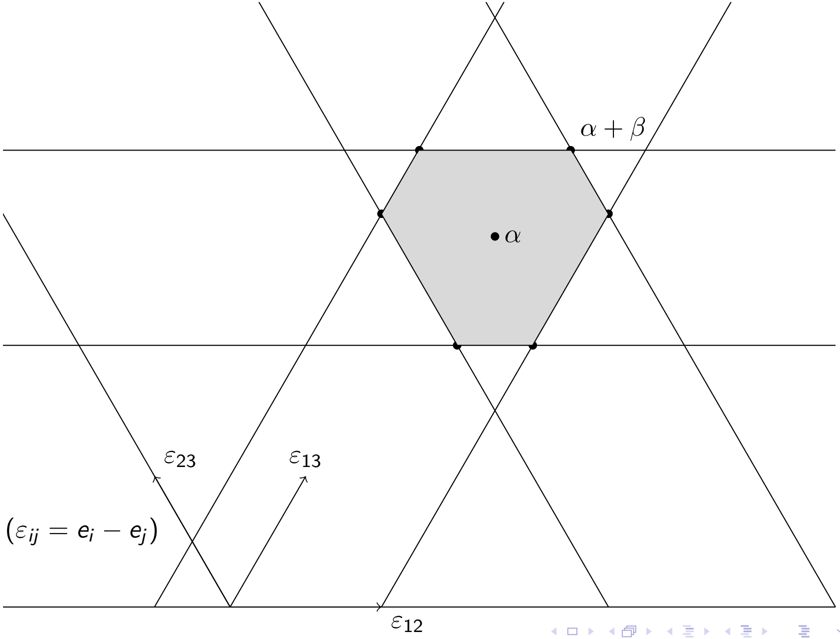
Weyl'inequalities gives

$$\begin{aligned} a_1 &\leq \gamma_1 \leq b_1 \\ a_2 &\leq \gamma_2 \leq b_2 \\ a_3 &\leq \gamma_3 \leq b_3 \end{aligned}$$

In the plane

$$x_1 + x_2 + x_3 = 0,$$

these inequalities determine a hexagon.


$\alpha + \beta$
α
ε_{23}
ε_{13}
$(\varepsilon_{ij} = e_i - e_j)$
ε_{12}

One observes that the vertices of this hexagon are the points $\alpha + \sigma(\beta)$ $(\sigma \in \mathfrak{S}_3)$. This is a special case of the following

Theorem (Lidskii-Wielandt) The set $\mathcal{H}(\alpha, \beta)$ of possible $\gamma = (\gamma_1, \ldots, \gamma_n)$ satisfies

$$\mathcal{H}(\alpha, \beta) \subset \alpha + C(\beta),$$

where $C(\beta)$ is the convex hull of the points $\sigma(\beta)$ $(\sigma \in \mathfrak{S}_n)$.

We consider Horn's problem from a probabilistic viewpoint.
The set of Hermitian matrices X with spectrum $\{\alpha_1, \ldots, \alpha_n\}$ is an orbit $\mathcal{O}_\alpha$ for the natural action of the unitary group $U(n)$:
$X \mapsto UXU^*$ $(U \in U(n))$.
Assume that
the random Hermitian matrix X is uniformly distributed on the orbit $\mathcal{O}_\alpha$,
and the random Hermitian matrix Y uniformly distributed on $\mathcal{O}_\beta$.

What is the joint distribution of the eigenvalues of the sum $Z = X + Y$?

This distribution is a probability measure on $\mathbb{R}^n$ that we will determine explicitly.

Orbits for the action of $U(n)$ on $\mathcal{H}_n(\mathbb{C})$

Spectral theorem : The eigenvalues of a matrix $A \in \mathcal{H}_n(\mathbb{C})$ are real and the eigenspaces are orthogonal.
The unitary group $U(n)$ acts on $\mathcal{H}_n(\mathbb{C})$ by the transformations

$$X \mapsto UXU^*$$

For $A = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$, consider the orbit

$$\mathcal{O}_\alpha = \{UAU^* \mid U \in U(n)\}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n.$$

By the spectral theorem

$$\mathcal{O}_\alpha = \left\{X \in \mathcal{H}_n(\mathbb{C}) \mid \mathrm{spectrum}(X) = \{\alpha_1, \dots, \alpha_n\}\right\}$$

Orbital measures

The orbit $\mathcal{O}_\alpha$ carries a natural probability measure:
the *orbital measure* μ_α,
image of the normalized Haar measure ω of the compact group $U(n)$ by the map $U \mapsto UAU^*$. For a function f on $\mathcal{O}_\alpha$,

$$\int_{\mathcal{O}_\alpha} f(X)\mu_\alpha(dX) = \int_{U(n)} f(UAU^*)\omega(dU).$$

A $U(n)$-invariant measure μ on $\mathcal{H}_n(\mathbb{C})$ can be seen as an integral of orbital measures:
it can be written

$$\int_{\mathbb{H}_n(\mathbb{C})} f(X)\mu(dX) = \int_{\mathbb{R}^n} \Big(\int_{U(n)} f(U\mathrm{diag}(t_1, \ldots, dt_n)U^*)\omega(dU) \Big) \nu(dt),$$

where ν is a $\mathfrak{S}_n$-invariant measure on $\mathbb{R}^n$, called the *radial part* of μ.

If μ is a $U(n)$-invariant probability measure,
and X a random Hermitian matrix with law μ,
then the joint distribution of the eigenvalues of X
is the radial part ν of μ.

Assume that the random Hermitian matrix X is uniformly distributed
on the orbit $\mathcal{O}_\alpha$, i.e. with law μ_α,
and Y uniformly distributed on $\mathcal{O}_\beta$, i.e. with law μ_β,
then the law of the sum $Z = X + Y$ is the convolution product $\mu_\alpha * \mu_\beta$,
and the joint distribution of the eigenvalues of Z
is the radial part $\nu_{\alpha,\beta}$ of the measure $\mu = \mu_\alpha * \mu_\beta$.

Hence the problem is to determine this radial part $\nu_{\alpha,\beta}$.

Fourier-Laplace transform

For a bounded measure μ on $\mathcal{H}_n(\mathbb{C})$,

$$\mathcal{F}\mu(Z) = \int_{\mathcal{H}_n(\mathbb{C})} e^{\mathrm{tr}\,(ZX)} \mu(dX).$$

If μ is $U(n)$-invariant, then $\mathcal{F}\mu$ is $U(n)$-invariant as well, and hence determined by its restriction to the subspace of diagonal matrices.

For $Z = \mathrm{diag}(z_1, \ldots, z_n)$, $T = \mathrm{diag}(t_1, \ldots, t_n)$, define

$$\mathcal{E}(z, t) := \int_{U(n)} e^{\mathrm{tr}\,(ZUTU^*)} \omega(dU).$$

Then $\mathcal{F}\mu_\alpha(Z) = \mathcal{E}(z, \alpha)$.

If μ is $U(n)$-invariant,

$$\mathcal{F}\mu(Z) = \int_{\mathbb{R}^n} \mathcal{E}(z,t)\nu(dt),$$

where ν is the radial part of μ.

Taking $\mu = \mu_\alpha * \mu_\beta$,

$$\mathcal{E}(z,\alpha)\mathcal{E}(z,\beta) = \int_{\mathbb{R}^n} \mathcal{E}(z,t)\nu_{\alpha,\beta}(dt).$$

This is the product formula of the spherical functions for the Gelfand pair (G, K).

$$G = U(n) \ltimes \mathcal{H}_n(\mathbb{C}), \quad K = U(n).$$

The group G acts on $\mathcal{H}_n(\mathbb{C})$ by the transformations

$$g \cdot X = UXU^* + A \quad \big(g = (U, A)\big).$$

The spherical functions are given by

$$\varphi_z(g) = \mathcal{E}(z, t),$$

where $t_1, \ldots, t_n$ are the eigenvalues of the matrix $g \cdot 0$. They satisfy the functional equation:

$$\int_K \varphi_z(g_1 U g_2)\omega(dU) = \varphi_z(g_1)\varphi_z(g_2) \quad (g_1, g_1 \in G).$$

With the identification

$$\varphi_z(g_1) = \mathcal{E}(z, \alpha), \quad \varphi_z(g_2) = \mathcal{E}(z, \beta),$$

the functional equation becomes

$$\mathcal{E}(z, \alpha)\mathcal{E}(z, \beta) = \int_{\mathbb{R}^n} \mathcal{E}(z, t)\nu_{\alpha,\beta}(dt).$$

Harish-Chandra-Itzykson-Zuber formula

A is an Hermitian matrix with eigenvalues $\alpha_1, \ldots, \alpha_n$, and B with eigenvalues $\beta_1, \ldots, \beta_n$.

$$\int_{U(n)} e^{\mathrm{tr}\,(AUBU^*)} \omega(dU) = \delta_n! \frac{1}{V_n(\alpha) V_n(\beta)} \det\left(e^{\alpha_i \beta_j}\right)_{1 \leq i,j \leq n}$$

V_n is the Vandermonde polynomial: for $x = (x_1, \ldots, x_n)$,

$$V_n(x) = \prod_{i<j} (x_i - x_j)$$

and

$$\delta_n = (n-1, n-2, \ldots, 2, 1, 0), \quad \delta_n! = (n-1)!(n-2)! \ldots 2!$$

Heckman's measure

Consider the projection $q : \mathcal{H}_n(\mathbb{C}) \to D_n$ onto the subspace D_n of real diagonal matrices.

Horn's theorem The projection $q(\mathcal{O}_\alpha)$ of the orbit $\mathcal{O}_\alpha$ is the convex hull of the points $\sigma(\alpha)$

$$q(\mathcal{O}_\alpha) = C(\alpha) := \mathrm{Conv}(\{\sigma(\alpha) \mid \sigma \in \mathfrak{S}_n\})$$

$(\sigma(\alpha) = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}))$

Heckman's measure is the projection $M_\alpha = q(\mu_\alpha)$
of the orbital measure μ_α.
It is a probability measure on $\mathbb{R}^n$, symmetric, i.e. $\mathfrak{S}_n$-invariant,
with compact support: $\mathrm{support}(M_\alpha) = C(\alpha)$.

Fourier-Laplace transform of a bounded measure M on $\mathbb{R}^n$:

$$\widehat{M}(z) = \int_{\mathbb{R}^n} e^{(z|x)} M(dx)$$

The Fourier-Laplace transform of Heckman's measure M_α is the restriction to D_n of the Fourier-Laplace transform of the orbital measure μ_α:
for $Z = \mathrm{diag}(z_1, \ldots, z_n)$,

$$\widehat{M_\alpha}(z) = \mathcal{F}\mu_\alpha(Z)$$

Therefore $\widehat{M_\alpha}(z) = \mathcal{E}(z, \alpha)$,
and by the Harish-Chandra-Itzykson-Zuber formula,

$$\widehat{M_\alpha}(z) = \delta_n! \frac{1}{V_n(z) V_n(\alpha)} \det\big(e^{z_i \alpha_j}\big)_{1 \leq i,j \leq n}$$

Define the skew-symmetric measure

$$\eta_\alpha = \frac{\delta_n!}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\delta_{\sigma(\alpha)}$$

($\varepsilon(\sigma)$ is the signature of the permutation σ).
Fourier-Laplace of η_α:

$$\widehat{\eta_\alpha}(z) = \frac{\delta_n!}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) e^{(z|\sigma(\alpha))} = \frac{\delta_n!}{V_n(\alpha)} \det\big(e^{z_i \alpha_j}\big)_{1 \le i,j \le n}$$

By the Harish-Chandra-Itzykson-Zuber formula

$$\widehat{\eta_\alpha}(z) = V_n(z)\widehat{M_\alpha}(z).$$

Proposition

$$V_n\Big(-\frac{\partial}{\partial x}\Big) M_\alpha = \eta_\alpha$$

Elementary solution of $V_n\left(\frac{\partial}{\partial x}\right)$

Proposition Define the distribution E_n on $\mathbb{R}^n$

$$\langle E_n, \varphi \rangle = \int_{\mathbb{R}_+^{\frac{n(n-1)}{2}}} \varphi\Big(\sum_{i<j} t_{ij}\varepsilon_{ij}\Big) dt_{ij}$$

$(\varepsilon_{ij} = e_i - e_j)$ Then

$$V_n\left(\frac{\partial}{\partial x}\right) E_n = \delta_0.$$

Proof: An elementary solution of the first order differential operator $\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}$ is the Heaviside distribution

$$\langle Y_{ij}, \varphi \rangle = \int_0^\infty \varphi(t\varepsilon_{ij}) dt$$

Hence

$$E_n = \prod_{i<j}^{*} Y_{ij}$$

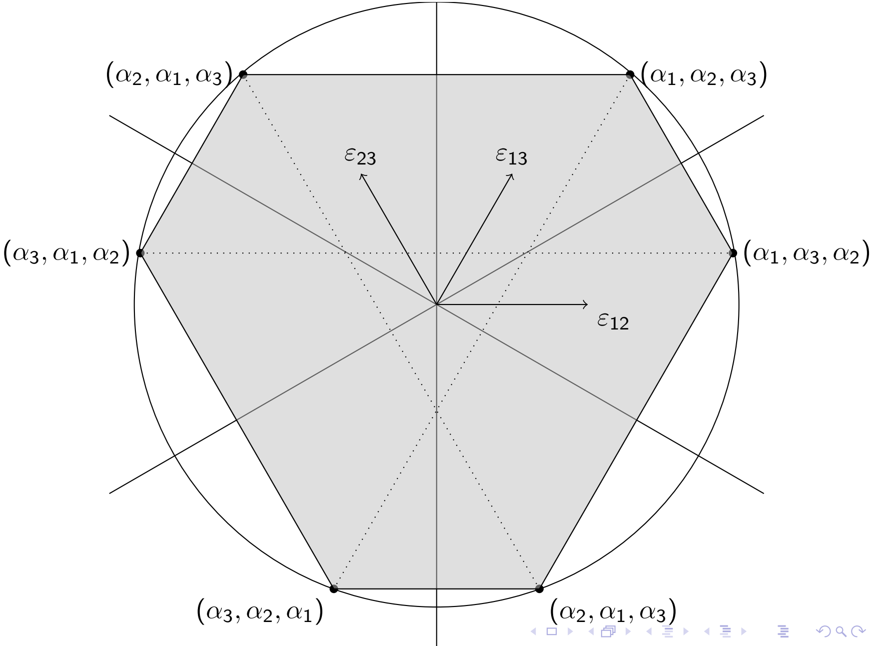
is an elementary solution of $V_n\left(\frac{\partial}{\partial x}\right)$.

Theorem

$$M_\alpha = \check{E}_n * \eta_\alpha$$

$(\check{\varphi}(x) = \varphi(-x)$, $\langle \check{E}_n, \varphi \rangle = \langle E_n, \check{\varphi} \rangle)$
Heckman's measure M_α is supported by the hyperplane
$x_1 + \cdots + x_n = \alpha_1 + \cdots + \alpha_n$.

Next figure is for $n = 3$,
drawn in the plane $x_1 + x_2 + x_3 = \alpha_1 + \alpha_2 + \alpha_3$.



$(\alpha_2, \alpha_1, \alpha_3)$
$(\alpha_1, \alpha_2, \alpha_3)$
ε_{23}
ε_{13}
$(\alpha_3, \alpha_1, \alpha_2)$
$(\alpha_1, \alpha_3, \alpha_2)$
ε_{12}
$(\alpha_3, \alpha_2, \alpha_1)$
$(\alpha_2, \alpha_1, \alpha_3)$

The radial part $\nu_{\alpha,\beta}$

Recall
X is a random Hermitian matrix on $\mathcal{O}_\alpha$ with law μ_α,
and Y on $\mathcal{O}_\beta$ with law μ_β.
The joint distribution of the eigenvalues of $Z = X + Y$ is the radial part $\nu_{\alpha,\beta}$ of $\mu_\alpha * \mu_\beta$.

Theorem

$$\begin{aligned} \nu_{\alpha,\beta} &= \frac{1}{n!}\frac{1}{\delta_n!} V_n(x)\eta_\alpha * M_\beta \\ &= \frac{1}{n!}\frac{1}{\delta_n!}\frac{V_n(x)}{V_n(\alpha)} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\delta_{\sigma(\alpha)} * M_\beta. \end{aligned}$$

The sum has positive and negative terms.
However $\nu_{\alpha;\beta}$ is a probability measure on $\mathbb{R}^n$.
The measure $\nu_{\alpha,\beta}$ is symmetric (invariant by $\mathfrak{S}_n$).

This theorem can be seen as a special case of a result by Graczyk and Sawyer (2002).
A similar result, but slightly different, is given by Rösler (2003).

The set of possible systems of eigenvalues for the sum $Z = X + Y$ is

$$S(\alpha, \beta) = \mathrm{support}(\nu_{\alpha,\beta})$$

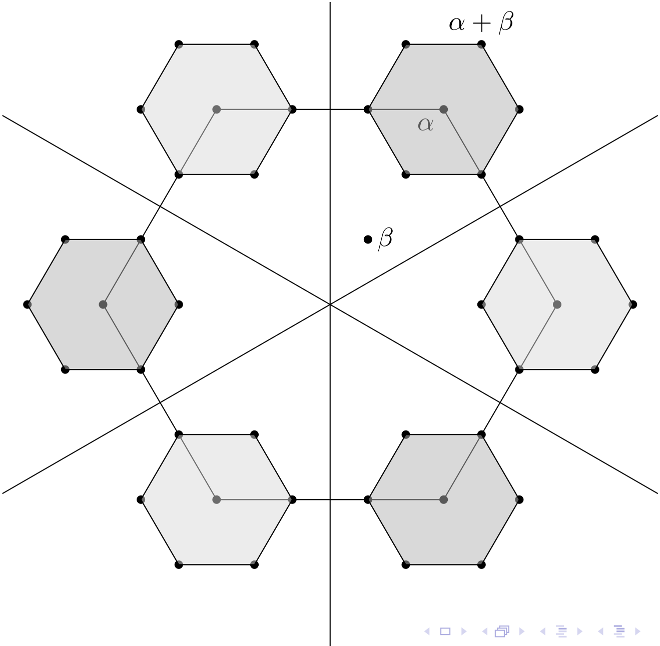
The proof amounts to check that the measure

$$\nu = \frac{1}{n!} \frac{1}{\delta_n!} V_n(x) \eta_\alpha * M_\beta$$

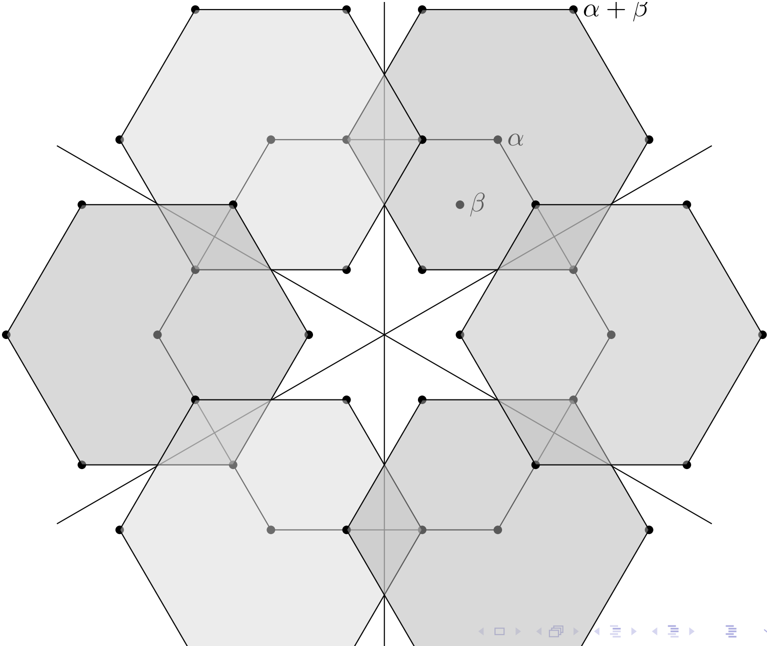
satisfies the relation

$$\int_{\mathbb{R}^n} \mathcal{E}(z, t) \nu(dt) = \mathcal{E}(z, \alpha) \mathcal{E}(z, \beta)$$

Next figure is for $n = 3$, $\alpha = (3, 0, -3)$, $\beta = (1, 0, -1)$


$\alpha + \beta$
α
β

Next figure is for $n = 3$, $\alpha = (3, 0, -3)$, $\beta = (2, 0, -2)$



$\alpha + \beta$
α
β

In the first case the condition

$$\sup |\beta_i - \beta_j| < \inf_{i \neq j} |\alpha_i - \alpha_j|$$

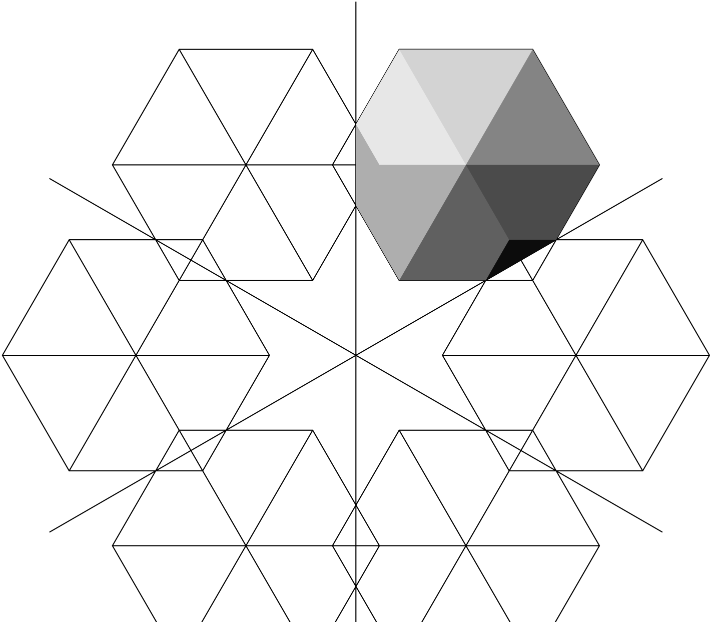
is satisfied, and, under that condition,

$$\mathcal{H}(\alpha, \beta) = S(\alpha, \beta) \cap C_n = \alpha + C(\beta)$$

where C_n is the chamber

$$C_n = \{x_1 > x_2 > \cdots > x_n\}.$$

In the second case the condition is not satisfied. There are cancellations and the situation is more complicated.



Relation to representation theory

π_λ irreducible representation of $U(n)$ with highest weight λ,
$\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ $(\lambda_i \in \mathbb{Z})$.

Littlewood-Richardson coefficients $c^\gamma_{\alpha,\beta}$:

$$\pi_\alpha \otimes \pi_\beta = \sum_\gamma c^\gamma_{\alpha,\beta} \pi_\gamma.$$

Theorem $c^\gamma_{\alpha,\beta} \neq 0$ if and only if $\gamma \in \mathcal{H}(\alpha,\beta)$;
i.e. there exist $n \times n$ Hermitian matrices A, B, C with $C = A + B$, the α_i are the eigenvalues of A, the β_i of B, the γ_i of C.

(Klyachko, 1998; Knutson, Tao, 1999)

In the case of the space of real symmetric matrices $\mathcal{H}_n(\mathbb{R})$,
with the action of the orthogonal group $O(n)$, for $n \geq 3$,
we don't know any explicit formula for Heckman's measure,
and for the measures $\nu_{\alpha,\beta}$.
This setting is natural, however the problem is more difficult
that in the case of the space of Hermitian matrices,
and one should not expect any explicit formula.
However the supports should be the same as in the case of $\mathcal{H}_n(\mathbb{C})$
with the action of the unitary group $U(n)$, according to Fulton (1998).

There should be an analogue of our results
in case of pseudo-Hermitian matrices.
In this setting, an analogue of Horn's conjecture has been established by Foth (2010).
An analogue of our result could probably be obtained by using a formula for the Laplace transform of an orbital measure
for the action of the pseudo-unitary group $U(p,q)$
on the space $\mathcal{H}_n(\mathbb{C}^n)$ $(n = p + q)$.
This formula is due Ben Saïd and Ørsted (2005).

More generally one could consider Horn's problem
for the adjoint action of a compact Lie group on its Lie algebra.
The Fourier transform of an orbital measure is explicitely given
by the Harish-Chandra integral formula [1957].
Heckman's paper [1982] is written in this framework. One can expect
that there is an analogue of our result in this setting.
In particular one can consider the action of the orthogonal group
on the space of real skew-symmetric matrices, as Zuber did (2017).

Bhatia, R. (2001). Algebra to quantum cohomology: the story of Alfred Horn's inequalities, *Amer. Math. Monthly*, **108**, 289–318.

Lidskii, V. S. (1950). The proper values of the sum and product of symmetric matrices, *Doklady Akademi Nauk SSSR*, **75**, 769–772.

Berezin, F. A. & I. M. Gelfand (1962). Some remarks on spherical functions on symmetric Riemannian manifolds, *Amer. Math. Soc. Transl., Series 2*, **21**, 193–238.

Horn, A. (1962). Eigenvalues of sums of Hermitian matrices, *Pacific J. Math.*, **12**, 225–241.

Klyachko, A. A. (1998). Stable vector bundles and Hermitian operators, *Selecta Math. (N.S.)*, **4**, 419–445.

Fulton, W. (1998). Eigenvalues of sums of Hermitian matrices(after Klyachko), Séminaire Bourbaki, exposé 845, June 1998, *Astérisque*, **252**, 255–269.

Knutson, A. & T. Tao (1999). The honeycomb model of $GL_n(mathbbC)$ tensor products, *J. Amer. Math. Soc.*, **12**, 1055–1090.

FRUMKIN, A., & A. GOLDBERGER (2006). On the distribution of the spectrum of the sum of two hermitian or real symmetric matrices, *Advances in Applied Mathematicis*, **37**, 268–286.

ZUBER, J.-B. (2017). Horn's problem and Harish-Chandra's integrals. *preprint*.

FARAUT, J. (2018). Horn's problem, and Fourier analysis. *preprint*.

HARISH-CHANDRA (1957). Differential operators on a semisimple Lie algebra, *Amer. J. Math.*, **79**, 87–120.

ITZYKSON, C., J.-B. ZUBER (1980). The planar approximation II, *J. Math. Physics*, **21**, 411–421.

HECKMAN, G. (1982). Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups, *Invent. math.*, **67**, 333–356.

FOTH, P. (2010). Eigenvalues of sums of pseudo-Hermitian matrices, *Electronic Journal of Linear Algebra (ELA)*, **20**, 115-125.