On the Algeometry Problem

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I'll start with some personal words...

Model: Sophus Lie's theory – the <u>Lie functor</u>: $\begin{array}{c} G = G(\mathfrak{g}) \\ \downarrow\uparrow \\ \mathfrak{g} = \mathrm{Lie}(G) \end{array}$

Problem. *Given some class of algebras, is there a corresponding class of "geometric objects", in such a way that the algebra can be considered as kind of "tangent algebra" of the geomery, and the geometry as kind of "integral version" of the algebra:*

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In former papers and talks, I've called this the "generalized coquecigrue problem", in honor of Jean-Louis Loday; but Karl-Hermann convinced me that, possibly, time has come to give it a more serious name...

Plan of the talk

Geometry

$\downarrow\uparrow$

Algebra

1. Introduction
   - Status of the problem
   - What do I mean by "algebra" ?
   - What do I mean by "geometry" ?
2. Going down: "derivation" (differential calculus)
3. Going up: "integration" (no general method, for the moment) – examples:
   - Lie examples
   - associative examples
   - Jordan examples and Jordan-Lie examples

Status of the problem

It is

- wide open
- answer "known" for important cases : Lie, Jordan, associative (binary and ternary)
- work in good progress for others : alternative, Moufang, Leibniz,...
- difficult: so far each case uses its own methods and ideas
- interesting: so far, each case gave new and beautiful insights on geometry and was source of new maths
- challenging – especially for the Lie community (because *groups* are central in this problem)
- important (I'm pretty sure that theoretical physics will be very happy to use such a theory, once we have it.)

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- binary algebras may have a *unit*, or not: it's important to make precise if we talk about *unital* associative, or Jordan, algebras (unit is part of the structure), or not,
- I don't assume algebras to be finite-dimensional!
- I don't even assume algebras to be defined over a *field*: I prefer to allow them to be defined over a general (commutative, unital) *ring* $\mathbb{K}$ (general philosophy: infinite dimensional geometry over a field is best treated by working over rings, since then you are not tempted to use bases).

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- a "space" (manifold) with "structure" (typically, encoded by a "structure map", such as the group law of a Lie group, or the Loos product $(x, y) \mapsto s_x(y)$ of a symmetric space),

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- "structure" may also be a *relation*: like "incidence" in projective geometry, or "transversality" on Grassmannians;
- anyhow, "space" and "structure" should be understood in some classical sense; in this respect, I do not follow the philosophy of *Non-Commutative Geometry* (NCG)!

Going down: differentiation

Example: *"the Lie algebra of a Lie group is a kind of differentiation of the group law"*

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- as torsion tensor of the *canonical curvature free connection*: fine, too; but first you have to talk about *connections*...

What is the "best", or "most natural", method ? To give an answer, one has to speak first about *differential calculus* itself...

"Functorial": we use that the *tangent functor* T is a functor:

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Differential calculus. I: functorial approach

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In general: $(M, \ldots)$ geometry $\Rightarrow (TM, T(\ldots))$ geometry
$\Rightarrow (TTM, TT(\ldots))$ geometry, and so on...
... get higher order tangent geometry $(T^n M, T^n(\ldots))$

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Lie example: TG is a group with splitting exact sequence

$$0 \to (\mathfrak{g}, +) \to TG \to G \to 1.$$

That is, semidirect product: $TG \cong G \times \mathfrak{g}$. Repeat this. Add index 1 everywhere in the first step, index 2 in the second step, etc. Get

$$TTG \cong G \times \mathfrak{g}_1 \times \mathfrak{g}_2 \times \mathfrak{g}_{12}.$$

For $X, Y \in \mathfrak{g}$, let X_α the corresponding image in $\mathfrak{g}_\alpha$ ("lift").

Theorem. *For all $X, Y \in \mathfrak{g}$,* $\boxed{[X, Y]_{12} = [X_1, Y_2]^{\mathrm{group}}}$, *where* $[g, h]^{\mathrm{group}} = ghg^{-1}h^{-1}$ *is the* **group** *commutator.*

Jacobi: need *third order* differential calculus! Look at $TTTG$:

$$T^3G \cong (G \times \mathfrak{g}_1 \times \mathfrak{g}_2 \times \mathfrak{g}_{12}) \times (\mathfrak{g}_3 \times \mathfrak{g}_{12} \times \mathfrak{g}_{23} \times \mathfrak{g}_{123})$$

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$$(X_3 \cdot Y_2) \cdot Z_1 = X_3 \cdot (Y_2 \cdot Z_1)$$

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Theorem. *By writing elements of T^nG uniquely as "products in ascending lexicographic order", the n-fold tangent group T^nG is diffeomorphic to a product of G with $2^n - 1$ copies $\mathfrak{g}_A$ of $\mathfrak{g}$, labelled by subsets $A \subset \{1, \ldots, n\}$. There is an explicit "product formula", in terms of the Lie bracket, describing this group.*

Rk : this a "left analog" of the Campbell-Hausdorff formula... However, inversion is *not* the map $X \mapsto -X$!

...book project and two arxiv papers; cf 50th SSL proceedings...

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cf. the *tangent groupoid* (Alain Connes):

$t = 0$ (in the limit): tangent bundle, $\mathrm{Tg}_0 = T$ tangent functor,

$t = 1$: *pair groupoid* functor $\mathrm{Tg}_1 M = \mathrm{PG}(M) = M \times M$

groupoid: $(x, y) * (y, z) = (x, z), \quad (x, y)^{-1} = (y, x),$

$t \in [0, 1]$: Tg_t "continuous interpolation" (Connes)

Differential calculus. II: Conceptual calculus

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Fundamental theorem of conceptual calculus. *Let M be a smooth Hausdorff manifold (over some topological base ring $\mathbb{K}$). Then there is a Lie groupoid $\mathrm{Tg}(M)$ over the base $M \times \mathbb{K}$, such that for each $t \in \mathbb{K}$ the groupoid $\mathrm{Tg}_t M$ over the base M satisfies:*

- $\mathrm{Tg}_0 M$ *is the tangent bundle (group bundle),*
- $\mathrm{Tg}_t M \cong \mathrm{PG}(M)$ *when t is invertible in $\mathbb{K}$,*
- *for every $t \in \mathbb{K}$, $\mathrm{Tg}_t M$ is a smooth manifold over the ring*

$$\mathbb{K}_t := \mathbb{K}[X]/(X^2 - tX) = \mathbb{K} \oplus b\mathbb{K}, \quad b^2 = tb.$$

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Theorem. *In the situation of the preceding theorem,* $\mathrm{Tg}^2 M$ *is a* (strict) double groupoid, *and so is, for each* $(t_1, t_2) \in \mathbb{K}^2$,

$$\mathrm{Tg}^2_{(t_1,t_2)} M := \mathrm{Tg}_{t_2}(\mathrm{Tg}_{t_1} M).$$

This is also the scalar extension of M *by the ring* $\mathbb{K}_{(t_1,t_2)} = (\mathbb{K}_{t_1})_{t_2}$. *And so on: we get an* n-fold groupoid $\mathrm{Tg}^{\mathrm{n}} M$ *which is "Lie" over a certain ring* $\mathbb{K}_{(t_1,\ldots,t_n)}$.

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Summary. Lie groups and Lie groupoids (even: higher ones) enter naturally into the foundations of differential calculus and of differential geometry (cf. "Ehresman's program").

Therefore Lie theory plays a central role in the whole of the "algeometry problem"!

Lie’s third theorem. *Every (finite-dimensional, real or complex) Lie algebra is associated to some Lie group.*

Recall: the whole proof is long. It can be seperated in several parts: some of them are purely algebraic, but inevitably some of them are “transcendental”: arguments of topology and “hard” analysis are used. None generalizes completely to infinite-dimensional situations...

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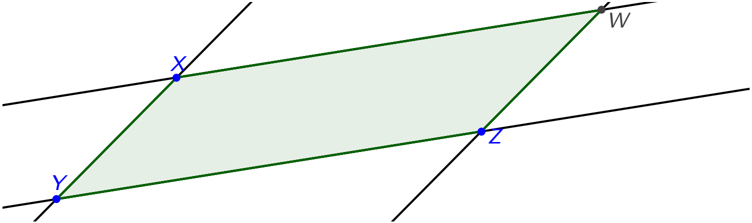
Surprise: the Lie case is the “most difficult” – in the Jordan and associative case, arguments are purely algebraic, no need for “transcendental help” ! – that’s why Jordan (and associative) theory works so nicely in infinite dimensional situations, and over general base fields and rings: call it “algebraic integration”.

Theorem. *Every associative algebra is associated to some associative geometry.* (B.-Kinyon.)
Every Jordan algebra is associated to some Jordan geometry.

What is the geometry belonging to an associative algebra? – Before giving general answers, let's look at the following four pictures:

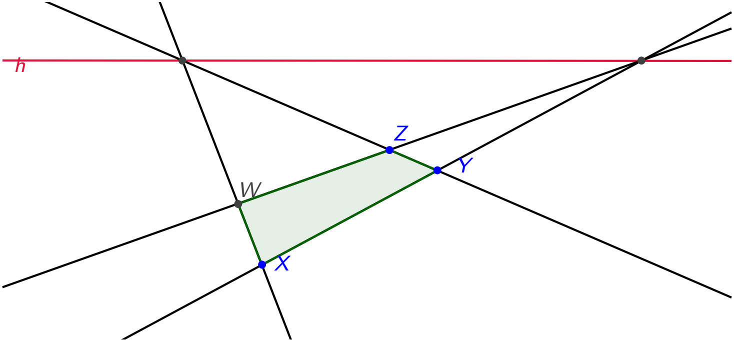


This is an ordinary parallelogram in an ordinary plane. There is a geometric formula and an algebraic one describing $(X, Y, Z) \mapsto W$:

$$W = \big(\mathrm{para}(Y \vee Z, X)\big) \wedge \big(\mathrm{para}(Y \vee X, Z)\big),$$

$$W = X - Y + Z$$

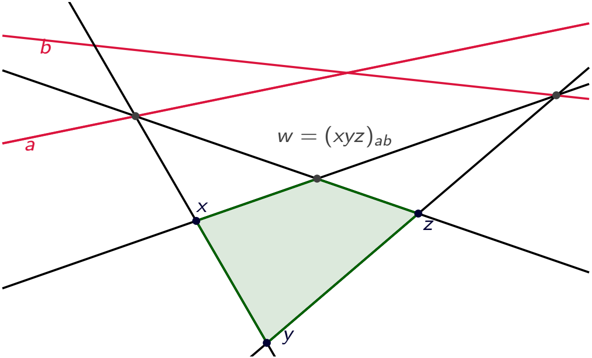
The second picture: it's the same as before,



where I have just added the drawing of the line at infinity h (horizon). We are in a projective plane.

$$w = \Big(\big((x \vee y) \wedge h \big) \vee z \Big) \wedge \Big(\big((z \vee y) \wedge h \big) \vee x \Big)$$

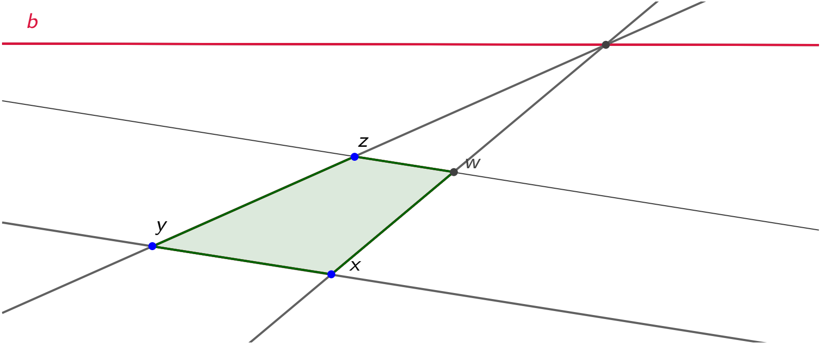
Now let's be schizophrenic and introduce *two horizons* a and b instead of h, and do the following construction:



The point w is now defined by the formula

$$w := (xyz)_{ab} := \Big(\big((x \vee y) \wedge a\big) \vee z\Big) \wedge \Big(\big((z \vee y) \wedge b\big) \vee x\Big)$$

To get another view of the preceding image, let's move back to infinity one of the two horizons, say a:



Theorem. *For any fixed a, b, y, the law $(x, z) \mapsto w$ defines a group law on U_{ab} = the projective plane with lines a, b, removed.*

The whole thing (map associating to five data: (a, b, x, y, z), the point $w = (xyz)_{ab}$) is what I call the *non-commutative plane*. It's the smallest dimensional non-trivial *associative geometry*.

Underlying set: a general Grassmannian X and its dual X'

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(Weak) geometric structure: binary relation called *transversality*, $a \top x$ iff the sum $a + x$ is direct; $U_a = \{x \mid x \top a\}$ the set of complements of a; for *every* pair $(a, b) \in X' \times X'$, let $U_{ab} = U_a \cap U_b$ (set of common complements)

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Algebraic-geometric structure: Theorem. *for any $y \in U_{ab}$, there is a group structure on U_{ab} with neutral element y.* The product xz in this group depends on a, b, y, so let us denote it by $x \cdot z = (xyz)_{ab} = \Gamma(x, a, y, b, z)$. This defines the *structure map*

$$\Gamma : X \times X' \times X \times X' \times X \to X, \quad (x, a, y, b, z) \mapsto x \cdot_y z = (xyz)_{ab}.$$

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$$\Gamma : X \times X' \times X \times X' \times X \to X, \quad (x, a, y, b, z) \mapsto x \cdot_y z = (xyz)_{ab}.$$

Algebraic Identites: the structure map Γ satisfies certain algebraic identities, all of them "nice and natural".

Definition. *Axiomatically, a structure of the type $(X, X', \top, \Gamma)$ is then called associative geometry.*

Theorem. (B-Kinyon.) *Associative geometries are the geometries belonging to* associative pairs *(have equivalence of categories...) Under this correspondence, unital associative* algebras *correspond to* self-dual *associative geometries (with base "point").*

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Example: not self-dual. (X, X') projective spaces corresponds to $(M(1, n; \mathbb{K}), M(n, 1; \mathbb{K}))$, row and column vectors in duality, $n > 1$. Special case $n = 2$: pictures shown above!

Example: self-dual. $X = X' =$ Grassmannian of n-spaces in $\mathbb{K}^{2n}$, corresponds to $(\mathbb{A}, \mathbb{A})$ with $\mathbb{A}$ full associative algebra of $n \times n$-matrices. It is the *generalized projective line over* $\mathbb{A}$:

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"Integration" revisited. Start with any unital associative algebra $\mathbb{A}$; then the space (manifold) is explicitly given by

$$X = \mathrm{Gras}_{\mathbb{A}}^{\mathbb{A}}(\mathbb{A} \oplus \mathbb{A}) = X'$$

with structure map Γ defined as above.

Recall: an associative algebra gives rise to a *special Jordan algebra* $x \bullet y = \frac{1}{2}(xy + yx)$ (better: *quadratic* algebras; McCrimmon).

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Definition. The *Jordan structure map* J of an associative geometry (X, X', Γ) is the restriction of the pentary structure map $\Gamma(x, a, y, b, z)$ to the diagonal $x = z$:

$$J_x^{ab}(y) = \Gamma(x, a, y, b, x) = (xyx)_{ab}$$

Clue: when (a, b) is fixed, the map J_x^{ab} is the *inversion map* of the group U_{ab} with origin x. Hence U_{ab} becomes a *symmetric space* with Loos' structure map $(y, x) \mapsto J_y^{ab}(x)$.

Second clue: when (x, y) is fixed, we also get an *abelian group* U_x with origin y and product $(a, b) \mapsto J_y^{ab}(x)$ (vector group, in fact)

Definition. A *special Jordan geometry* is a subspace of an associative geometry that is stable under the Jordan structure map.

Axiomatic definition. A *Jordan geometry* is given by a pair of spaces (X, X'), a *transversality relation* $\top$ and a *Jordan structure map* $(x, a, y, b) \mapsto J_y^{ab}(x)$, such that certain algebraic identities hold. These say, among other things, that J_y^{ab} is an inversion (has order 2), fixes y, and exchanges a and b, and

- for fixed (a, b), U_{ab} is a *symmetric space* (the "homotopes"),
- for fixed (x, y), (U_x, y) is an *abelian group*.

Theorem. (B.14.) *Jordan geometries are the geometries belonging to general* Jordan pairs*.* Self-dual *Jordan geometries belong to general Jordan* algebras*.*

Rk. Previous results:

- Be00 (LNM 1754), real finite dimensional case (using Loos 69)
- Be02: "generalized projective geometries", char $\neq 2$ (midpoints!)
- Be-Neeb (2005): integration: **model** via *filtered Lie algebras*
- Be14: the results above hold for characteristic 2 included!

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What is behind this? Recall the *associator* of an algebra β

$$A_\beta(x, y, z) := (xy)z - x(yz) = \beta(\beta(x, y), z) - \beta(x, \beta(y, z)).$$

Lemma. *Let* $\mathbb{A}$ *be an* associative *algebra, i.e.,* $A_\beta = 0$*. Then the associators of the symmetric part* $J(x, y) = xy + yx$ *and of the skew-symmetric part* $L(x, y) = xy - yx$ agree, *up to a sign:*

$$A_J = -A_L.$$

Proof. $L(L(x, y), z) - L(x, L(y, z)) = xzy + yzx - yxz - zxy$, and for $J(J(x, y), z) - J(x, J(y, z))$ you get its *negative*!

Definition. (Physics people...) *Let $k \in \mathbb{K}$ be a constant. A* Jordan-Lie algebra with Jordan-Lie constant k *is an algebra $\mathbb{A}$ with two products $[x, y]$ and $x \bullet y$, such that*

(JL1) $(V, [\cdot, \cdot])$ *is a Lie algebra,*

(JL2) $(V, \bullet)$ *is a Jordan algebra,*

(JL3) *the Lie algebra acts by derivations of $\bullet$, that is,*

$$[x, u \bullet v] = [x, u] \bullet v + u \bullet [x, v],$$

(JL4) *the* associator identity*: associators of both products are proportional, by a factor k, that is, $A_\bullet = kA_{[-,-]}$. Written out,*

$$(x \bullet y) \bullet z - x \bullet (y \bullet z) = k\big([[x, y], z] - [x, [y, z]]\big).$$

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Question. *What geometries correspond to Jordan-Lie algebras?*

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Question. *What geometries correspond to Jordan-Lie algebras?*

- for $k = -1$ (or $k < 0$) these are the associative geometries!
- for $k = 1$ (or $k > 0$) these are... "very special" Jordan geometries, belonging to Hermitian parts of $*$-algebras: "geometries of Quantum Mechanics" (cf. arxiv).