A minimal representation of the orthosymplectic Lie supergroup

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Outline

Classification

Goal

Classification of all possible representations of a given group/algebra.

Classification

Goal

Classification of all irreducible representations of a given group/algebra.

Classification

Goal

Classification of all unitary irreducible representations of a given Lie group.

Connected compact groups

Figure: Élie Cartan

Figure: Hermann Weyl

Semisimple groups

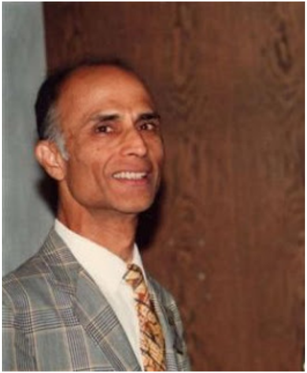

Figure: Harish-Chandra

The orbit method

Figure: Alexandre Kirillov

The orbit method (or geometric quantization)

Gives a connection between

- the unitary irreducible representations of G
- the coadjoint orbits of $\mathfrak{g}^*$.

Minimal representations

Minimal representation: hand-waving definition

The representation associated to the minimal nilpotent coadjoint orbit via the orbit method.

Special properties

- Very small: lowest possible Gelfand-Kirillov dimension.
- Difficult from orbit method point of view.

Minimal representations: technical definition

Minimal representation: technical definition

A unitary representation of a simple real Lie group G
is called *minimal* if the annihilator ideal
of the derived representation
of the universal enveloping algebra of $\text{Lie}(G)_{\mathbb{C}}$ is
the Joseph ideal.

Definition (Joseph ideal)

The Joseph ideal is the unique completely prime,
two-sided ideal in the universal enveloping algebra
such that the associated variety
is the closure of the minimal nilpotent coadjoint orbit.

W. Gan, G. Savin. On minimal representations definitions and properties. Represent. Theory **9** (2005), 46–93.

Minimal representations: an example

The metaplectic representation

Unitary irreducible representation of $Mp(2n,\mathbb{R})$, a double cover of $Sp(2n,\mathbb{R})$, on $L^2_{even}(\mathbb{R}^n)$. On algebra level it is given by

$$\mathrm{d}\mu \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} = -\pi i \sum_{i,j=1}^{n} C_{ij} y_i y_j \qquad \text{for } C \in \mathrm{Sym}(n,\mathbb{R})$$

$$\mathrm{d}\mu \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} = -\frac{1}{2}\mathrm{tr}(A) - \sum_{i,j=1}^{n} A_{ij} y_j \partial_i \qquad \text{for } A \in \mathrm{M}(n,\mathbb{R})$$

$$\mathrm{d}\mu \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \frac{1}{4\pi i} \sum_{i,j=1}^{n} B_{ij} \partial_i \partial_j \qquad \text{for } B \in \mathrm{Sym}(n,\mathbb{R}).$$

Other prominent example is given by the minimal representation of $O(p, q)$.

There exists a unified construction of minimal representation using Jordan algebras developed in [HKM].

[HKM] J. Hilgert, T. Kobayashi, J. Möllers. Minimal representations via Bessel operators. J. Math. Soc. Japan **66** (2014), no. 2, 349–414.

Supersymmetry

- Introduced in the 70s.
- Treat bosons and fermions at the same footing.
- Add ‘odd stuff’ to the ordinary (even) ‘stuff’.

Super vector space

Definition

A super vector space is a $\mathbb{Z}_2$ graded vector space, i.e.

$$V = V_{\bar{0}} \oplus V_{\bar{1}}.$$

The elements in $V_{\bar{0}} \cup V_{\bar{1}}$ are called homogeneous.

We define parity for homogeneous elements as

$$|u| = i \quad \text{if } u \in V_{\bar{i}}.$$

Definition of a Lie superalgebra

A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a super vector space with a bilinear product [,] which

- is a graded product

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \text{ for } i, j \in \mathbb{Z}_2$$

- is super anti-commutative

$$[X, Y] = -(-1)^{|X||Y|}[Y, X]$$

- satisfies the super Jacobi identity

$$(-1)^{|X||Z|}[X, [Y, Z]] + (-1)^{|Y||X|}[Y, [Z, X]] + (-1)^{|Z||Y|}[Z, [X, Y]] = 0.$$

The orthosymplectic Lie superalgebra

Consists of the $(p+q+2n) \times (p+q+2n)$ matrices for which

$$X^{st}\Omega + \Omega X = 0$$

with

$$\Omega = \begin{pmatrix} I_p & & & \\ & -I_q & & \\ & & & -I_n \\ & & I_n & \end{pmatrix}.$$

Bracket: $[X, Y] = XY - (-1)^{|X||Y|} YX$.

The orthosymplectic Lie superalgebra $\mathfrak{osp}(1,0|2)$

Defining equation

$$\begin{pmatrix} a & d & g \\ -b & e & f \\ -c & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 0.$$

So $$\mathfrak{osp}(1,0|2n) = \left\{ X = \begin{pmatrix} 0 & b & c \\ c & e & f \\ -b & h & -e \end{pmatrix} \mid \quad b, c, e, f, h \in \mathbb{R} \right\}.$$

Even part:

$$X_{\bar{0}} = \begin{pmatrix} 0 & & \\ & e & f \\ & h & -e \end{pmatrix}$$

Odd part:

$$X_{\bar{1}} = \begin{pmatrix} & b & c \\ c & & \\ -b & & \end{pmatrix}$$

Goal

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Construct minimal representations for Lie supergroups.
$\rightarrow$ Focus on the example $OSp(p, q|2n)$.

Approach

Generalize the unified construction of minimal representation using Jordan algebras developed in [HKM].

[HKM] J. Hilgert, T. Kobayashi, J. Möllers. Minimal representations via Bessel operators. J. Math. Soc. Japan **66** (2014), no. 2, 349–414.

How to construct minimal representations for simple Lie groups?

- Start from a simple Jordan algebra.
- Associate some Lie algebras/groups:
  - structure algebra/group
  - the *Tits-Kantor-Koecher Lie algebra* / conformal group.
- Construct a representation from this TKK Lie algebra on the Jordan algebra.

⟶ Representation is still too big.

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How to construct minimal representations for simple Lie groups?

- Study the orbits of the Jordan algebra under the action of the structure group.
- Show that this representation restricts to the minimal orbit.
- Infinitesimally unitary representation with respect to some L^2 measure.
- Integrate this restricted representation to a unitary representation of the conformal group.

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Minimal representations for Lie supergroups: what do we need?

- Jordan superalgebras ✓ [Ka]
- Structure algebra and TKK algebras ✓ [BC1]
- Representation on the Jordan superalgebra ✓ [BC2]

[Ka] V. G. Kac.
Classification of simple Z-graded Lie superalgebras and simple Jordan superalgebras.
Comm. Algebra **5** (1977), no. 13, 1375-1400.

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[BC1] S. Barbier, K. Coulembier.
On structure and TKK algebras for Jordan superalgebras.
Comm. Algebra 46 (2018), no 2, 684-704.

Structure algebra and TKK

The spin factor Jordan superalgebra

$$J := \mathbb{R}e \oplus \mathbb{R}^{p+q-3|2n}$$

The structure algebra

$$\mathfrak{str}(J) = \mathfrak{osp}(p-1, q-1|2n) \oplus \mathbb{R}L_e$$

The Tits-Kantor-Koecher construction

$$\mathrm{TKK}(J) = J \oplus \mathfrak{str}(J) \oplus J = \mathfrak{osp}(p, q|2n)$$

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[BC2] S. Barbier, K. Coulembier.
Polynomial Realisations of Lie (Super)Algebras and Bessel Operators.
International Mathematics Research Notices 2017, no. 10, 3148-3179.

Minimal representations for Lie supergroups: what do we need?

- Jordan superalgebras ✓ [Ka]
- Structure algebra and TKK algebras ✓ [BC1]
- Representation on the Jordan superalgebra ✓ [BC2]

These steps were done in general. For the next steps we restrict to $\mathfrak{osp}(p, q|2n)$.

Minimal representations for $\mathfrak{osp}(p, q|2n)$: what do we need?

- Jordan superalgebras ✓
- Structure algebra and TKK algebras ✓
- Representation on the Jordan superalgebra ✓
- Minimal orbit and restriction to this orbit ✓ [BF]
- Integration to group level ✓ [BF]

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[BF] S. Barbier and J. Frahm,
A minimal representation of the orthosymplectic Lie superalgebra,
45 pages, arXiv:1710.07271.

Harish-Chandra supermodules

$G = (G_0, \mathfrak{g}, \sigma)$ a Lie supergroup,
G_0 is connected and real reductive,
K_0 is a maximal compact subgroup of G_0.

Definition (Harish-Chandra supermodule)

A super vector space V is a Harish-Chandra supermodule if V

- is a locally finite K_0-representation
- it has a compatible $\mathfrak{g}$-module structure
- finitely generated over $U(\mathfrak{g})$
- K_0-multiplicity finite.

A. Alldridge. Fréchet Globalisations of Harish-Chandra Supermodules. Int Math Res Notices 2017, no. 17, 5137-5181.

A Harish-Chandra supermodule

Set $\mu = \max(p-2n, q) - 3$, and $\nu = \min(p-2n, q) - 3$

$$\mathfrak{g} = \mathfrak{osp}(p, q|2n), \qquad \mathfrak{k} = \mathfrak{osp}(p|2n) \oplus \mathfrak{so}(q).$$

Define

$$W = U(\mathfrak{g})\widetilde{K}_{\frac{\nu}{2}}(|X|)$$

with $\widetilde{K}_{\frac{\nu}{2}}(|X|)$ the modified Bessel function of the third kind.

Theorem

If $p+q$ is even and $p-2n > 0$, then W is a Harish-Chandra supermodule with $\mathfrak{k}$-decomposition $W = \bigoplus_j W_j$

$$W_j \cong \mathcal{H}^{\frac{\mu-\nu}{2}+j}(\mathbb{R}^{p|2n}) \otimes \mathcal{H}^{j}(\mathbb{R}^{q}) \qquad \text{if } p-2n \leq q,$$
$$W_j \cong \mathcal{H}^{j}(\mathbb{R}^{p|2n}) \otimes \mathcal{H}^{\frac{\mu-\nu}{2}+j}(\mathbb{R}^{q}) \qquad \text{if } p-2n \geq q.$$

Minimal representations for $\mathfrak{osp}(p, q|2n)$: what do we need?

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- Representation on the Jordan superalgebra ✓
- Minimal orbit and restriction to this orbit ✓ [BF]
- Integration to group level ✓ [BF]

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Properties of the minimal representation

- Gelfand-Kirillov dimension: $p + q - 3$.
- The annihilator ideal is the Joseph ideal constructed in constructed in [CSS] if $p + q - 2n - 2 > 0$.
- There exists non-degenerate superhermitian, sesquilinear form for which the representation is skew-symmetric.

K. Coulembier, P. Somberg, V. Souček. Joseph ideals and harmonic analysis for $\mathfrak{osp}(m|2n)$. Int. Math. Res. Not. IMRN (2014), no. 15, 4