Symmetries in multisymplectic geometry

In honor of Joachim Hilgert

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July 26, 2018

Contents

(Time-dependant) classical mechanics on Q, an n-dimensional configuration space, is geometrised on T^*Q or $T^*I \times T^*Q$ for I a time intervall and $\psi : I \to T^*Q$ as follows:

Hamilton's equations

$$\frac{\partial \mathcal{H}}{\partial q^a}(t, \psi(t)) = \frac{d(p_a \circ \psi(t))}{dt}, \frac{\partial \mathcal{H}}{\partial p_a}(t, \psi(t)) = \frac{d(q^a \circ \psi(t))}{dt}.$$

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Equivalently for $\Psi = (id_I, \mathcal{H}, \psi) : I \to T^*(I \times Q)$:

$$(\Psi_*)\left(\frac{d}{dt}\right) \lrcorner\, \omega_{\Psi(t)} = -dH_{\Psi(t)} \text{ with } H = \mathcal{H} - p.$$

Note: $X_H \lrcorner\, \omega = -dH$.

Let Σ be a k-dimensional manifold with a volume form vol^{Σ} and a dual k-vector field γ^{Σ}, and $\pi : E = \Sigma \times Q \to \Sigma$.

We call $\phi : \Sigma \to Q$ a "field" and $L : J^1(\pi) \to \mathbb{R}$ a "Lagrange function".

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Lagrangean field theory

Extrema of

$$\phi \mapsto \int_{\Sigma} L(j^1(\phi)vol^{\Sigma}$$

are the solutions of the field theory, they fulfill the Euler-Lagrange equations.

If L is regular, the higher dimensional Legrendre transformation yields equations on the analogues of $T^*(I \times Q)$ and $I \times T^*Q$:

$\mathcal{M}(\pi) = \Lambda^k T^*\Sigma \oplus (T^*Q \otimes \Lambda^{k-1} T^*\Sigma)$ resp.
$\mathcal{P}(\pi) = T^*Q \otimes \Lambda^{k-1} T^*\Sigma,$

both as vector bundles over $E = \Sigma \times Q$ for maps $\Psi : \Sigma \to \mathcal{M}(\pi)$ resp. $\tilde{\Psi} : \Sigma \to \mathcal{P}(\pi)$.

Coordinates for $\mathcal{M}(\pi)$ are (x^μ, q^a, p_a^μ, p); p being absent on $\mathcal{P}(\pi)$.

Hamilton-Volterra equations

$\forall \mu \in \{1, ..., k\}$ and $\forall a \in \{1, ..., n\}$,

$$\frac{\partial \mathcal{H}}{\partial q^a}(\tilde{\Psi}(x)) = \sum_{\mu=1}^{n} \frac{\partial (p_a^\mu \circ \tilde{\Psi})(x)}{\partial x^\mu},$$

$$-\frac{\partial \mathcal{H}}{\partial p_a^\mu}(\tilde{\Psi}(x)) = \frac{\partial (q^a \circ \tilde{\Psi})(x)}{\partial x^\mu}.$$

Time derivates are replaced by partials in coordinate directions x^μ of Σ, replacing I.

With $\Psi = (\mathcal{H}, \tilde{\Psi}) : \Sigma \to \mathcal{M}(\pi)$ and $H = \mathcal{H} - p$, we have equivalently:

$$(\Psi_*)(\gamma^\Sigma) \lrcorner\, \omega_{\Psi(t)} = -dH_{\Psi(t)} \text{ with } H = \mathcal{H} - p.$$

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Hamilton-De Donder-Weyl (HDW) equations:

$$X_H \lrcorner\, \omega = -dH.$$

Solving them is not enough for finding the section Ψ or the field ϕ, since if $k > 1$ going from HDW to a map is an extra step (that is trivially assured by the existence of flows of vector fields, as opposed to multivector fields)!

Definition

A “multisymplectic” manifold (M, ω) is a pair, where M is a manifold, $k \geq 1$ and $\omega \in \Omega^{k+1}(M)$ is a closed differential form satisfying the following non-degeneracy condition: The map

$$\iota_\bullet \omega : TM \to \Lambda^k T^* M, \quad v \mapsto \iota_v \omega = v \lrcorner \omega$$

is injective. For fixed degree $k+1$ of the form such manifolds are also called “k-plectic”.

Examples galore

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- hyperkähler manifolds
- semisimple Lie groups with the Cartan three-form

Note that for $j > 1$ typically the following map is neither injective nor surjective:

$$\iota_\bullet\omega : \Lambda^j TM \to \Lambda^{k+1-j} T^*M, \quad u \wedge v \mapsto (u \wedge v) \lrcorner \omega \,.$$

Nevertheless the **HDW equation**

$$X \lrcorner \omega = -d\alpha \,,$$

whose solutions are couples (X, α) with $X \in \mathfrak{X}^{n-k}(M)$, a multivector field, and $\alpha \in \Omega^k(M)$, a k-form, are central for multisymplectic geometry!

The linear case

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- canonical multisymplectic structure on $\mathbb{R}^6 = \Lambda^2(\mathbb{R}^3)^* \oplus \mathbb{R}^3$ and

$$\omega(\alpha \oplus u, \beta \oplus v, \gamma \oplus w) = \alpha(v, w) - \beta(u, w) + \gamma(u, v)$$

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  with a "parabolic stabilizer"
- $\mathbb{R}^6 = \mathbb{C}^3$ as a real vector space; the real part of a complex volume form on $\mathbb{C}^3$ is a nondenerate 3-form with $SL(3,\mathbb{C})$ as stabilizer.

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- $\mathbb{R}^6 = \mathbb{C}^3$ as a real manifold; the real part ω of the complex volume form $dz^1 \wedge dz^2 \wedge dz^3$ on $\mathbb{C}^3$ is a 2-plectic form with r-transitively acting symmetry group for all r.

For Joachim

Theorem

Let (G,ω) be a real semi-simple Lie group with its canonical three-form, given in the neutral element as follows:

$$\omega_e(\xi,\eta,\zeta) = B([\xi,\eta],\zeta) \text{ for all } \xi,\eta,\zeta \in \mathfrak{g} = T_eG$$

where B is the Killingform of $\mathfrak{g}$.

Then (G,ω) has constant linear type but is flat if and only if its dimension is three.

A multisymplectic manifold (M,ω) is called "flat" if it has local coordinates with the property that the multisymplectic form has constant coefficients.

Proof. Constancy of linear type follows immediately from the bi-invariance of ω. Without loss of generality, we can assume for the rest of the proof, that G is connected and simple. In the three-dimensional case the flatness is a consequence of the Darboux theorem for volume forms. For all real simple Lie groups of dimension higher than three, we have

$$Aut(\mathfrak{g}, \omega_e) = Aut(\mathfrak{g}, [\cdot, \cdot]) \subset Aut(\mathfrak{g}, \langle \cdot, \cdot \rangle),$$

where the leftmost and rightmost terms are linear automorphisms preserving the respective tensor and the middle term are the Lie algebra automorphisms of $\mathfrak{g}$. The left equality is a standard fact of Lie theory and the right inclusion follows, because the Killing form is intrinsically defined from the Lie bracket.

Let us assume that G admits a chart $\phi : U \subset G \to V \subset \mathfrak{g}$ near e, such that $(T_g\phi)^*\omega_e = \omega_g$, where ω_e should be interpreted as the constant coefficient extension of $\omega_e \in \mathfrak{g} = T_e\mathfrak{g}$. The natural left-invariant pseudo-Riemannian metric on G is defined by $h_g = -(\theta^L_g)^*\langle \cdot, \cdot \rangle$, where $\theta^L_g : T_gG \to \mathfrak{g}$ is the Maurer-Cartan one-form. By construction we have

$$(\theta^L_g) \circ (T_g\phi)^{-1} \in Aut(\mathfrak{g}, \omega_e).$$

So $(\theta^L_g) \circ (T_g\phi)^{-1}$ preserves $h_e = -\langle \cdot, \cdot \rangle$, i.e.

$$(T_g\phi)^* h_e = (T_g\phi)^*((\theta^L_g) \circ (T_g\phi)^{-1})^* h_e = (\theta^L_g)^* h_e = h_g$$

This means that ϕ is a flat chart for (G, h), where h is the canonical left-invariant metric on G. Such a chart can not exist, because real simple Lie groups with canonical left-invariant metric have non-zero curvature.

Multisymplectic and Hamiltonian actions

(M, ω) k-plectic manifold and X vector field preserving ω. A Lie algebra homomorphism $\tau : \mathfrak{g} \to \mathfrak{X}(M, \omega)$, the ω-preserving vector fields, is called a "multisymplectic action".

$\mathbf{k = 1}$: X Hamiltonian vector field iff $X = X_f$ with $X_f \lrcorner \omega = -df$.
Note: $\{f, g\} = X_f \lrcorner X_g \lrcorner \omega$ is a Lie bracket!

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A "co-moment" is a Lie algebra homomorphism $\lambda : \mathfrak{g} \to \Omega^0(M)$ s.th.

$$X_{\lambda(\xi)} = \tau(\xi) \text{ forall } \xi \in \mathfrak{g} .$$

$\mathbf{k > 1}$: $\mathfrak{X}_{\mathrm{Ham}}(M,\omega) = \{X = X_\alpha \mid X_\alpha \lrcorner \omega = -d\alpha\}$

The space of such forms α is noted $\Omega^{k-1}_{\mathrm{Ham}}(M,\omega) =: L_0$.

Note: $l_2(\alpha,\beta) = \{\alpha,\beta\} = X_\alpha \lrcorner X_\beta \lrcorner \omega$ is not a Lie bracket:

Jacobi identity holds up to $d \circ l_3(...)$!

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Barnich-Fulp-Lada-Stasheff: if M is contractible, given any acyclic resolution of L_0 there is a Lie ∞-structure on it. Here John Baez and Christopher Rogers give it explicitely:

$$l_1 = d_{\text{de Rham}} \quad \text{and} \quad l_n(\alpha_1, ..., \alpha_n) = \pm X_{\alpha_1} \lrcorner ... X_{\alpha_n} \lrcorner \omega \,.$$

Assume $f_1 = \lambda : \mathfrak{g} \to \Omega^{k-1}_{\mathrm{Ham}}(M, \omega)$ s.th. $X_{\lambda(\xi)} = \tau(\xi)$ forall $\xi \in \mathfrak{g}$.

This is not good enough for a co-moment, since l_2 is not a Lie bracket and f_1 cannot be a morphism of Lie algebras!!!

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Wayout: complete f_1 to a Lie ∞-morphism: $\{f_j\}_{j\geq 1}$ with

$$f_j : \Lambda^j \mathfrak{g} \to \Omega^{k-j}(M)$$

such that ($f_{k+1} = 0$): for all j

$$\partial f_j + l_1 f_{j+1} = -f_1^* l_{j+1} .$$

What happens for $k = 2$?

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- $f_2([\xi, \eta] \wedge \zeta) - f_2([\xi, \zeta] \wedge \eta) + f_2([\eta, \zeta] \wedge \xi) = l_3(f_1(\xi), f_1(\eta), f_1(\zeta))$

Theorem. (Callies-Frégier-Rogers-Zambon. resp. Ryvkin-W.)
There are cohomological classes governing existence and unicity of a (homotopy) co-moment if an infinitesimal multisymplectic action of a Lie algebra is given.

Conserved quantities

Definition. Let X be a vector field on M, then a differential form α is called "conserved (under X)" if the Lie derivative $L_X\alpha$ is exact.

(**Remark.** This definition is motivated by Lagrangean field theories!)

A multisymplectic Noether type theorem

Theorem (L.Ryvkin-T.W.-M.Zambon). Let (M, ω) be k-plectic, H a $k-1$-form and X_H a vector field s.th. $X_H \lrcorner \omega = -dH$, $\mathfrak{g}$ a Lie algebra acting with a co-moment on (M, ω) and such that $L_{\tau(\xi)} H = 0$ for all $\xi \in \mathfrak{g}$.

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Then for all $j \geq 1$ and all p in $\ker(\delta_j) \subset \Lambda^j \mathfrak{g}$, the $(k-j)$-form $f_j(p)$ is conserved under X_H.

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